ELEMENTARY CONSTRUCTION OF LUSZTIG'S CANONICAL BASIS

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Abstract. In these expository notes we present an elementary construction of Lusztig's canonical basis in type ADE. The method, which is essentially Lusztig's original approach, is to use the braid group to reduce to rank two calculations. Some of the wonderful properties of the canonical basis are already readily visible; that it descends to a basis for every highest weight integrable representation, and that it is a crystal basis.

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1. Introduction

Fix a complex simple Lie algebra $\mathfrak{g}$ and let $U_q^-(\mathfrak{g})$ be the lower triangular part of the corresponding quantized universal enveloping algebra. Lusztig's canonical basis B is a basis for $U_q^-(\mathfrak{g})$, unique once the Chevalley generators are fixed, which has remarkable properties. Perhaps the three most important are:

(i) For each irreducible representation V_λ the image of B in $V_\lambda = U_q^-(\mathfrak{g})/I_\lambda$ is a basis; equivalently, the intersection of B with every ideal I_λ is a basis for the ideal.
(ii) B is a crystal basis in the sense of Kashiwara.
(iii) In symmetric type, the structure constants of B with respect to multiplication are Laurent polynomials in q with positive coefficients.

Much has been made of (iii), and it helped give birth to a whole new field of math: categorification. While this is a wonderful fact, the association of canonical bases with categorification has, I believe, obscured the fact that Lusztig's original construction is quite elementary. Using only basic properties of the braid group action on $U_q(\mathfrak{g})$ and rank

2 calculations, one can establish the existence and uniqueness of a canonical basis, and show that it satisfies both (i) and (ii). Property (iii) is mysterious with this approach, but perhaps that is to be expected, since it does not always hold is non-symmetric type, and the arguments here essentially work in all finite types.

These notes present Lusztig's elementary construction. They are fairly self contained, the biggest exception being that we refer to Lusztig's book [Lus93] for one elementary but long calculation in type $\mathfrak{sl}_3$. The results can all be found in Lusztig's papers [Lus90a, Lus90b, Lus90c, Lus90d] and his book [Lus93, Chapters 41 and 42]. Since we are interested in the connection with crystals, conventions have been chosen to match [Kas91, Sai94].

Lusztig's canonical basis is the same as Kashiwara's global crystal basis [Kas91], and Kashiwara's construction is also "elementary," at least in the sense that it does not use categorification. However, Kashiwara's construction is quite different from that presented here, and considerably more difficult. It is based on a complicated induction known as the "grand loop argument." Of course, Kashiwara's construction has a big advantage in that it works beyond finite type.

We restrict to the ADE case for simplicity. The construction is not much harder in other finite types, but requires some more notation. The rank two calculations are also considerably more difficult in types B_2 and G_2. In fact, I'm not sure they've ever been done as described here; instead, one uses a folding argument to understand the types B_2 and G_2 in terms of the simply laced types A_3 and D_4 respectively (see [BZ01, Lus11]).

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2. Notation

Let $\mathfrak{g}$ be a complex simple Lie algebra of type ADE, $U_q(\mathfrak{g})$ its quantized universal enveloping algebra, and $E_i, F_i, K_i^{\pm 1}$ for $i \in I$ the standard generators. Here I indexes the nodes of the corresponding Dynkin diagram, so we can discuss nodes being adjacent. Conventions are chosen so that

$$K_i E_i K_i^{-1} = q^2 E_i \quad \text{and } E_i F_j - F_j E_i = \frac{K_i - K_i^{-1}}{q - q^{-1}} \delta_{ij}. \tag{1}$$

We use the standard triangular decomposition,

$$U_q(\mathfrak{g}) = U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}), \tag{2}$$

where $U_q^-(\mathfrak{g})$ is the subalgebra generated by the F_i. Bar involution is the $\mathbb{Q}$ algebra involution of $U_q(\mathfrak{g})$ defined on generators by

$$\bar{E}_i = E_i, \quad \bar{F}_i = F_i, \quad \bar{K}_i = K_i^{-1}, \quad \bar{q} = q^{-1}. \tag{3}$$

$\{\alpha_i\}$ be the set of simple roots for $\mathfrak{g}$. Let $(\cdot, \cdot)$ be the standard bilinear form on root space. For a postive root β, define its height $\mathrm{ht}(\beta)$ to be the sum of the coeficients when β is written as a sum of simple roots.

3. Braid group action and PBW bases

There is a family of algebra automorphisms T_i of $U_q(\mathfrak{g})$, one for each $i \in I$, given by

$$T_i(F_j) = \begin{cases} F_j & i \text{ not adjacent to } j \\ F_jF_i - qF_iF_j & i \text{ adjacent to } j \\ -K_j^{-1}E_j & i = j, \end{cases} \tag{4}$$

$$T_i(E_j) = \begin{cases} E_j & i \text{ not adjacent to } j \\ E_jE_i - q^{-1}E_iE_j & i \text{ adjacent to } j \\ -F_jK_j & i = j, \end{cases} \tag{5}$$

$$T_i(K_j) = \begin{cases} K_j & i \text{ not adjacent to } j \\ K_iK_j & i \text{ adjacent to } j \\ K_j^{-1} & i = j. \end{cases} \tag{6}$$

One can easily check that these respect the defining relations of $U_q(\mathfrak{g})$, and that they satisfy the braid relations (i.e. $T_iT_jT_i = T_jT_iT_j$ for i and j adjacent, and $T_iT_j = T_jT_i$ otherwise). Each T_i performs the Weyl group reflection s_i on the weight of an element, where $U_q(\mathfrak{g})$ is graded by $\mathrm{wt}(E_i) = -\mathrm{wt}(F_i) = \alpha_i$, $\mathrm{wt}(K_i) = 0$.

Fix a reduced expression $w_0 = s_{i_1} \cdots s_{i_N}$ for the longest element of the Weyl group. Let $\mathbf{i}$ denote the sequence $i_1, i_2, \ldots, i_N$. Define "root vectors"

$$\begin{aligned} F_{\mathbf{i}:\beta_1} &= F_{i_1} \\ F_{\mathbf{i}:\beta_2} &= T_{i_1}F_{i_2} \\ F_{\mathbf{i}:\beta_3} &= T_{i_1}T_{i_2}F_{i_3} \\ &\;\vdots\;. \end{aligned} \tag{7}$$

The notation β_k in the subscripts is because the weight of each root vector is a negative root, and each negative root appears this way exactly once; we think of the root vectors as indexed by the corresponding roots. When the reduced expression is clear, we leave off the subscript $\mathbf{i}$. Let

$$B_{\mathbf{i}} = \{F_{\mathbf{i}:\beta_1}^{(a_1)}F_{\mathbf{i}:\beta_2}^{(a_2)} \cdots F_{\mathbf{i}:\beta_N}^{(a_N)} : a_1, \ldots, a_N \in \mathbb{Z}_{\geq 0}\}. \tag{8}$$

Here $X^{(a)}$ is the q-divided power $X^a/([a][a-1]\cdots[2])$, and $[n] = q^{n-1} + q^{n-3} + \cdots + q^{-n+1}$.

Lemma 3.1. *Fix a reduced expression* $\mathbf{i}$*.*

(i) If i_k, i_{k+1} are not adjacent, then reversing their order gives another reduced expression, and the root vectors are unchanged (although they are reordered).

(ii) If $i_k = i_{k+2}$ and is adjacent to i_{k+1}, then $\beta_k + \beta_{k+2} = \beta_{k+1}$, and

$$F_{\beta_{k+1}} = F_{\beta_{k+2}}F_{\beta_k} - qF_{\beta_k}F_{\beta_{k+2}}.$$

Furthermore, for the new reduced expression $\mathbf{i}'$ were $i_ki_{k+1}i_k$ is replaced with $i_{k+1}i_ki_{k+1}$, $F_{\mathbf{i}',\beta} = F_{\mathbf{i},\beta}$ for all $\beta \neq \beta_{k+1}$.

(iii) For any reduced expression and any simple root α_i, $F_{\alpha_i} = F_i$. In particular, $\beta_N = F_{\sigma(i_N)}$, where σ is the Dynkin diagram automorphism given by $\alpha_{\sigma(i)} = -w_0\alpha_i$.

Proof. Part (i) and (ii) follow by applying $T_{i_{k-1}}^{-1} \cdots T_{i_1}^{-1}$ and then doing a rank two calculation. Part (iii) is an immediate consequence of (ii), since α_i is not the sum of any two positive roots, and if $i_1 = i$ then $F_{\alpha_i} = F_i$ by definition. □

Lemma 3.2. *Each root vector F_{β_k} is in $U_q^-(\mathfrak{g})$.*

Proof. Proceed by induction on the height of $\beta = \beta_k$, the case of a simple root being immediate from Lemma 3.1 (iii). So assume β is not simple. Fix i so that $(\alpha_i, \beta) > 0$. There are reduced expressions $\mathbf{i}'$ and $\mathbf{i}''$ with $i'_1 = i, i''_N = \sigma(i)$, and so $\beta'_1 = \beta''_N = \alpha_i$. One can move from $\mathbf{i}$ to either $\mathbf{i}'$ or $\mathbf{i}''$ by sequences of braid moves, and one of these sequences must move α_i past β. At that step F_β changes. The first time F_β changes Lemma 3.1 (ii) allows us to conclude by induction that $F_\beta \in U_q^-(\mathfrak{g})$. □

Lemma 3.3. *If $j \geq k$, then $T_{i_j}^{-1} \cdots T_{i_1}^{-1} F_{\beta_k} \in U_g^{\geq 0}(\mathfrak{g})$.*

Proof.

$$T_{i_k}^{-1} \cdots T_{i_1}^{-1} F_{\beta_k} = -K_i^{-1} E_i^{-1}, \tag{9}$$

and $i_{k+1} \cdots, i_N, i_1, \cdots i_k$ is another reduced expression. The claim follows from Lemma 3.2 (with E_i instead of F_i) because the T_i are algebra automorphisms and preserve $U_q^0(\mathfrak{g})$. □

Lemma 3.4. *For any $\mathbf{i}$, $B_\mathbf{i}$ is a basis for $U_q^-(\mathfrak{g})$.*

Proof. The dimension of each weight space of $U_q^-(\mathfrak{g})$ is given by Kostant's partition function, so the size of the proposed basis is correct, and hence it suffices to show that these elements are linearly independent. Proceed by induction on k, showing that the set of such elements where $a_j = 0$ for $j > k$ is linearly independent. The key is that

$$T_{i_1}^{-1} F^{\mathbf{a}} = (-K_i^{-1} E_i^{-1})^{(a_1)} \otimes F_{\mathbf{i}'}^{\mathbf{a}'} \in U_q^{\geq 0}(\mathfrak{g}) \otimes U_q^-(\mathfrak{g}), \tag{10}$$

where $\mathbf{i}' = (i_2, i_3, \cdots, i_N, i_1)$ and $\mathbf{a}' = (a_2, a_3, \ldots, a_k, 0, \ldots, 0)$. The $F_{\mathbf{i}'}^{\mathbf{a}'}$ are linearly independant by induction, so the vectors $T_{i_1}^{-1} F_\mathbf{i}^\mathbf{a}$ are linearly independant by the triangular decomposition of $U_q(\mathfrak{g})$. The result follows since $T_{i_1}^{-1}$ is an algebra automorpohism. □

Denote the basis element corresponding to exponents $\mathbf{a} = (a_1, \ldots a_N)$ by $F_\mathbf{i}^\mathbf{a}$, and call $\mathbf{a}$ its Lusztig data.

Lemma 3.5. *Fix $\mathbf{i}$ and $1 \leq j < k \leq N$. Write $F_{\beta_k} F_{\beta_j} = \sum_\mathbf{a} p_\mathbf{a} F_\mathbf{i}^\mathbf{a}$. If $p_\mathbf{a} \neq 0$ then the only factors that appear with non-zero exponent in $F^\mathbf{a}$ are F_{β_i} for $j \leq i \leq k$.*

Proof. By Lemmas 3.2 and 3.3, and the fact that the T_i are algebra automorphisms,

$$T_{i_{j-1}}^{-1} \cdots T_{i_2}^{-1} T_{i_1}^{-1} (F_{\beta_k} F_{\beta_j}) \in U_q^-(\mathfrak{g}) \quad \text{and} \quad T_{i_k}^{-1} \cdots T_{i_2}^{-1} T_{i_1}^{-1} (F_{\beta_k} F_{\beta_j}) \in U_q^{\geq 0}(\mathfrak{g}). \tag{11}$$

A linear combination of PBW basis elements can only satisfy these conditions if, in all of them, the exponents of F_β are 0 unless $j \leq i \leq k$. □

4. Equality mod q and piecewise linear functions.

The following is key. Part (i) can be found in [Lus93, Proposition 41.1.4], and (ii) is part of [Lus93, Proposition 42.1.5]. For non-simply laced types see [Sai94]. Fix $\mathbf{i}$, and let

$$\mathcal{L} = \mathrm{span}_{\mathbb{Z}[q]} B_{\mathbf{i}}. \tag{12}$$

Certainly $B_{\mathbf{i}} + q\mathcal{L}$ is a basis for $\mathcal{L}/q\mathcal{L}$.

Theorem 4.1. *(i) $\mathcal{L}$ is independant of* $\mathbf{i}$.
(ii) $B_{\mathbf{i}} + q\mathcal{L} \subset \mathcal{L}/q\mathcal{L}$ and is independant of $\mathbf{i}$.

Proof. Any two reduced expressions are related by a sequence of braid moves, so it suffices to consider reduced expressions related by a single braid move. The case of a braid move $T_iT_j = T_jT_i$ where i and j are not adjacent is trivial. For the other case, lets say the braid move starts with $i_k = i, i_{k+1} = j, i_{k+2} = i$. It suffices to check that

$$\mathrm{span}_{\mathbb{Z}[q]}\{F^{(a_k)}_{\mathbf{i}:\beta_k} F^{(a_{k+1})}_{\mathbf{i}:\beta_{k+1}} F^{(a_{k+2})}_{\mathbf{i}:\beta_{k+2}}\} = \mathrm{span}_{\mathbb{Z}[q]}\{F^{(a_k)}_{\mathbf{i}':\beta_k} F^{(a_{k+1})}_{\mathbf{i}':\beta_{k+1}} F^{(a_{k+2})}_{\mathbf{i}':\beta_{k+2}}\}, \tag{13}$$

and that these monomials conincide modulo q. Applying $T^{-1}_{i_{k-1}} \cdots T^{-1}_{i_1}$ shows that this is equivalent to the statement in the $\mathfrak{sl}_3$ case. That is an explicit (although surprisingly difficult) calculation, which can be found in [Lus93, Chapter 42]. $\square$

5. Triangularity of bar involution and existence of the canonical basis

There are two natural lexicographical orders on Lusztig data: one where $\mathbf{a} < \mathbf{b}$ if $a_1 > b_1$ or $a_1 = b_1$ and $(a_2, \dots) < (b_2, \dots)$, and the other where one starts by comparing a_N and b_N. Consider the partial order on Lusztig data where $\mathbf{a} < \mathbf{b}$ if $\mathbf{a}$ is less then $\mathbf{b}$ for both of these orders.

Theorem 5.1. *For every reduced expression* $\mathbf{i}$ *and every Lustig data* $\mathbf{a}$*,*

$$\bar{F}^{\mathbf{a}}_{\mathbf{i}} = F^{\mathbf{a}}_{\mathbf{i}} + \sum_{\mathbf{a}' < \mathbf{a}} p^{\mathbf{a}'}_{\mathbf{a}}(q) F^{\mathbf{a}'}_{\mathbf{i}},$$

where the $p^{\mathbf{a}'}_{\mathbf{a}}(q)$ are Laurent polynomials in q.

Proof. That the coefficients are Laurent polynomials follows from the form of bar and the braid group operators. The point is the unit triangularity. This is well known, but the following proof is a little non-standard.

Proceed by induction using the above partial order. If the claim is true for all $F^{(a_j)}_{\beta_j}$, then $\bar{F}^{\mathbf{a}}_{\mathbf{i}}$ would be equal to $F^{\mathbf{a}}_{\mathbf{i}}$ plus terms obtained by replacing some of the $F_{\mathbf{i}:\beta}$ with lesser monomials. Then Lemma 3.5 implies that, once this is rearranged, all terms that appear are still less then $F^{\mathbf{a}}_{\mathbf{i}}$. Hence the minimal counter-example has to be of the form $F^{(n)}_{\beta_j}$ for some $\mathbf{i}$, j and n.

By Lemma 3.1, $F^{(n)}_{\alpha_i} = F^{(n)}_i$ satisfies the condition (it is in fact bar-invariant), so assume that $\beta = \beta_j$ is not a simple root; we will use induction on the hieght of this root. Certainly

$$\bar{F}^{(n)}_{\beta} = p(q) F^{(n)}_{\beta} + \sum_{\mathbf{a}' < \mathbf{a}} p^{\mathbf{a}'}_{\mathbf{a}}(q) F^{\mathbf{a}'}, \tag{14}$$

since $F_\beta^{(n)}$ is the unique maximal element of its weight. It remains to see that $p(q)$ is 1.

Do braid moves, changing the partial order until F_β changes (this is possible as discussed in the proof of Lemma 3.2). For the braid moves where F_β does not change, terms $< (a_j)$ get sent to linear combinations of terms that are still $< (a_j)$, so $p(q)$ does not change. Thus we can assume we are in a situation where we can apply a single braid move with j in the middle. By Lemma 3.1, $F_{\beta_j} = F_{\beta_{j+1}} F_{\beta_{j-1}} - q F_{\beta_{j-1}} F_{\beta_{j+1}}$, so

$$F_{\beta_j}^{(a_j)} = (F_{\beta_{j+1}} F_{\beta_{j-1}})^{(a_j)} \quad + \quad \text{terms which are} \quad < (0, \cdots, a_j, \cdots, 0) \ .$$

β_{j+1}, β_{j-1} are of lower height then β, so by induction and convexity of multiplication, $(F_{\beta_{j+1}} F_{\beta_{j-1}})^{(n)}$ is bar invariant up to terms that involve F_{β_k} for $k \neq j-1, j, j+1$, and hence are $< (0, \cdots a_j, \cdots 0)$. This forces $p(q) = 1$. □

Theorem 5.2. *There is a unique basis B of $U_q^-(\mathfrak{g})$ such that*

(i) B is contained in $\mathcal{L}$, $B + q\mathcal{L}$ is a basis for $\mathcal{L}/q\mathcal{L}$, and this agrees with $B^{\mathbf{i}} + q\mathcal{L}$ for some (equivalently any by Theorem 4.1) $\mathbf{i}$.

(ii) B is bar invariant.

Furthermore, the change of basis from any $B^{\mathbf{i}}$ to B is unit-triangular.

Proof. This proof can be found in [DDPW08, Lemma 0.27] in a slightly different setting. Proceed by induction on the partial order $<$, proving that there is such a basis for $V = \mathrm{span}\{F^{\mathbf{a}'}\}_{\mathbf{a}' \leq \mathbf{a}}$. The base case when $\mathbf{a}$ is minimal holds since V is one dimensional and Theorem 5.1 shows that $F^{\mathbf{a}}$ is bar-invariant.

So, fix a non-minimal $\mathbf{a}$. By Theorem 5.1,

$$\bar{F}^{\mathbf{a}} = F^{\mathbf{a}} + \sum_{\mathbf{a}' < \mathbf{a}} p^{\mathbf{a}}_{\mathbf{a}'}(q) b^{\mathbf{a}'} \tag{15}$$

for various Laurent polynomials $p^{\mathbf{a}}_{\mathbf{a}'}(q)$, where the $b^{\mathbf{a}'}$ are the inductively found elements of B. But $\bar{\bar{F}}^{\mathbf{a}} = F^{\mathbf{a}}$, which implies that each of these Laurent polynomials is of the form

$$p^{\mathbf{a}}_{\mathbf{a}'}(q) = q f^{\mathbf{a}}_{\mathbf{a}'}(q) - q^{-1} f^{\mathbf{a}}_{\mathbf{a}'}(q^{-1}), \tag{16}$$

where each $f^{\mathbf{a}}_{\mathbf{a}'}(q)$ is a polynomial. Set

$$b^{\mathbf{a}} = F^{\mathbf{a}} + \sum_{\mathbf{a}' < \mathbf{a}} q f^{\mathbf{a}}_{\mathbf{a}'}(q) b^{\mathbf{a}'}. \tag{17}$$

Certainly replacing $F^{\mathbf{a}}$ with $b^{\mathbf{a}}$ does not change $\mathcal{L}$ and $b^{\mathbf{a}} = F^{\mathbf{a}}$ mod $q\mathcal{L}$. Then

$$\bar{b}^{\mathbf{a}} = F^{\mathbf{a}} + \sum_{\mathbf{a}' < \mathbf{a}} (q f^{\mathbf{a}}_{\mathbf{a}'}(q) - q^{-1} f^{\mathbf{a}}_{\mathbf{a}'}(q^{-1})) b^{\mathbf{a}'} + \sum_{\mathbf{a}' < \mathbf{a}} q^{-1} f^{\mathbf{a}}_{\mathbf{a}'}(q^{-1}) b^{\mathbf{a}'} = F^{\mathbf{a}} + \sum_{\mathbf{a}' < \mathbf{a}} q f^{\mathbf{a}}_{\mathbf{a}'}(q) b^{\mathbf{a}'} = b^{\mathbf{a}},$$

so we have found the desired element.

Uniqueness is clear, since as the induction proceeds there is never any choice. □

The basis B from Theorem 5.2 is Lusztig's canonical basis (see [Lus90b, Theorem 3.2]).

6. Properties of the canonical basis

6.1. Descent to modules.

Theorem 6.1. *Fix a dominant integral weight λ and write $V_\lambda = U_q^-(\mathfrak{g})/I_\lambda$. Then $B \cap I_\lambda$ spans I_λ. Equivalently, $\{b + I_\lambda : b \in B, b \notin I_\lambda\}$ is a basis for V_λ.*

Proof. Write λ as a sum of fundamental weights, $\lambda = \sum c_i \omega_i$. It is well known that

$$I_\lambda = \sum_{i \in I} U_q^-(\mathfrak{g}) F_i^{c_i+1}. \tag{18}$$

It suffices to show that $B \cap U_q^-(\mathfrak{g}) F_i^n$ spans $U_q^-(\mathfrak{g}) F_i^n$ for all n.

Fix a reduced expression $\mathbf{i}$ with $i_N = \sigma(i)$, so that $\beta_N = F_i$. Then it is clear that $B^{\mathbf{i}} \cap U_q^-(\mathfrak{g}) F_i^n$ spans $U_q^-(\mathfrak{g}) F_i^n$. The change of basis from $B^{\mathbf{i}}$ to B is upper triangular, so the canonical basis elements corresponding to elements in $B^{\mathbf{i}} \cap U_q^-(\mathfrak{g}) F_i^n$ are all still in $U_q^-(\mathfrak{g}) F_i^{c_i}$, giving a spanning set. $\square$

6.2. **Crystal properties.** In a sense we have already shown that the basis B defines a combinatorial object that could be called its crystal. With that point of view, the crystal is the basis $B + q\mathcal{L}$ of $\mathcal{L}/q\mathcal{L}$, and the crystal operators $\widetilde{f}_i$ are defined as follows: Choose any reduced expression $\mathbf{i}$ where $i_1 = i$. On $B^{\mathbf{i}}$, define

$$\widetilde{f}_i F_i^{(a_1)} F_{\beta_2}^{(a_2)} \cdots F_{\beta_N}^{(a_N)} = F_i^{(a_1+1)} F_{\beta_2}^{(a_2)} \cdots F_{\beta_N}^{(a_N)}. \tag{19}$$

This gives a well defined operation on $B^{\mathbf{i}} + q\mathcal{L} = B + q\mathcal{L}$. The full structure is somewhat complicated, since one must use different reduced expressions to define each $\widetilde{f}_i$. But all reduced expression are related by braid moves so one can use Lusztig's piecewise linear operations from [Lus93] to do calculations.

We now show that the structure defined above matches Kashiwara's infinity crystal from [Kas91]. This has previously been observed by Lusztig [Lus90c] (see also [GL93, Lus11]) and also by Saito [Sai94]. We give a somewhat different proof.

We first review Kashiwara's construction of the crystal $B(\infty)$, roughly following [Kas91, §3]. For each $i \in I$, elementary calculations show that

$$E_i X = A K_i^{-1} + B K_i + X E_i \tag{20}$$

for some $A, B \in U_q^-(\mathfrak{g})$. Define $e_i' : U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})$ by $e_i'(X) = A$. As a vector space,

$$U_q^-(\mathfrak{g}) = \mathbb{C}[F_i] \otimes \ker e_i'. \tag{21}$$

Define operators $\widetilde{F}_i$ (the Kashiwara operators) by, for all $Y \in \ker e_i'$ and $n \geq 0$,

$$\widetilde{F}_i(F_i^{(n)} Y) = F_i^{(n+1)} Y. \tag{22}$$

Let $\mathcal{L}(\infty)$ to be the $\mathbb{Q}[q]_0$ lattice generated by all sequences of $\widetilde{F}_i$ acting on $1 \in U_q^-(\mathfrak{g})$. There is a unique basis $B(\infty)$ for $\mathcal{L}(\infty)/q\mathcal{L}(\infty)$ such that the residues of all the $\widetilde{F}_i$ act by partial permutations. This is the infinity crystal $B(\infty)$.

Theorem 6.2. *Let B be the canonical basis from Theorem 5.2. Then $\mathcal{L}(\infty) = span_{\mathbb{Q}[q]_0} B$, and $B(\infty) = B + q\mathcal{L}(\infty)$.*

Before proving Theorem 6.2 we need some preliminary Lemmas.

Lemma 6.3. *Fix $i \in I$, a reduced expression $\mathbf{i}$, and a root β with $\langle \beta, \alpha_i \rangle \leq 0$. Then there is a sequence of braid moves, none of which affect the relative positions of α_i and β in the corresponding order on roots, with the last move being a three term braid move with β the middle root (so that F_β changes).*

Proof. Fix j, k so that $\beta_j = \alpha_i$ and $\beta_k = \beta$. Without loss of generality $j < k$. The prefix $w = s_{i_1} \cdots s_{i_j}$ satisfies $w^{-1}\alpha_i = -\alpha_j$, which is a negative root, so w has a reduced expression of the form $s_i \cdots$. One can perform a sequence of braid moves relating these two reduced expressions which do not change the position of β. Thus we may assume $i_1 = i$. $\langle \beta, \alpha_i \rangle \leq 0$ and $\langle \beta, \rho \rangle > 0$, so we must have $\langle \beta, \alpha_\ell \rangle > 0$ for some other ℓ. Consider two cases:

If $(\alpha_i, \alpha_\ell) = 0$, then there are reduced expressions for w_0 of the form

$$s_i s_\ell \cdots \quad \text{and} \quad s_i \cdots s_{\sigma(\ell)}, \tag{23}$$

and both can be reached by performing braid moves that do not change the position of α_i. Certainly the relative positions of β and α_ℓ are different in these two expressions, so one of these sequences moves β past α_ℓ. Since $\langle \beta, \alpha_\ell \rangle > 0$, at that step β is the middle root for a 3 term braid move.

If $(\alpha_i, \alpha_\ell) = -1$, then there are reduced expressions for w_0 of the form

$$s_i s_\ell s_i \cdots \quad \text{and} \quad s_i \cdots s_{\sigma(\ell)}, \tag{24}$$

and the same argument works. $\square$

Lemma 6.4. *Fix a reduced expression $\mathbf{i}$, and let j be such that $\beta_j = \alpha_i$ is a simple root. For all $k > j$,*

$$E_i F_{\beta_k} - F_{\beta_k} E_i \in U_q^-(\mathfrak{g}) K_i.$$

Proof. Proceed by induction on the height of β_k, the case where β_k is a simple root $\alpha_j \neq \alpha_i$ being trivial since $E_i F_{\beta_k} - F_{\beta_k} E_i = 0$ by the Serre's relations.

So, assume the height is at least 2. If $(\beta_k, \alpha_i) \leq 0$, then by Lemma 6.3 we can do a sequence of braid moves that don't change the relative positions of α_i and β_k and so that the last move is a three term move with β as the middle root. At that step,

$$F_{\beta_k} = F_{\beta_{k-1}} F_{\beta_{k+1}} - q F_{\beta_{k+1}} F_{\beta_{k-1}} \tag{25}$$

where β_{k-1}, β_{k+1} are both roots of smaller height. The claim holds for $F_{\beta_{k-1}}$ and $F_{\beta_{k+1}}$ by induction, and so it follows for F_{β_k} by a short calculation.

If $\langle \beta_k, \alpha_i \rangle > 0$, perform any sequence of braid moves until β_k is the middle term of a three term move. If β_k has moved past α_i in that sequence, at the step β_k is the middle term of a three term move affecting the roots $\alpha_i, \beta_k, \beta_k - \alpha_i$, so

$$F_{\beta_k} = F_i F_{\beta_k - \alpha_i} - q F_{\beta_k - \alpha_i} F_i, \tag{26}$$

and the result follows by induction. Otherwise, α_i has not moved past β_k, and the result follows as in the previous paragraph. □

Lemma 6.5. *Fix i and $\mathbf{i}$ such that $i_1 = i$. Then*

$$\ker e'_i = span\{F_{\beta_2}^{(a_2)} \cdots F_{\beta_N}^{(a_N)}\};$$

that is, the span of PBW basis elements where the exponent of F_i is 0. In particular, $\widetilde{F}_i$ acts on $B_{\mathbf{i}}$ by simply increasing the exponent on F_i.

Proof. Certainly $E_i F_{\beta_2}^{(a_2)} \cdots F_{\beta_N}^{(a_N)}$ is equal to $F_{\beta_2}^{(a_2)} \cdots F_{\beta_N}^{(a_N)} E_i$ plus a sum of terms each of which is a PBW monomial but with one root vector F_β replaced by $E_i F_\beta - F_\beta E_i$. By Lemma 6.4, along with the fact that $K_i F_\beta = q^{-\langle \beta, \alpha_i^\vee \rangle} F_\beta K_i$, each of these factors is in $U_q^-(\mathfrak{g}) K_i$. Therefore by definition each of the vectors $F_{\beta_2}^{(a_2)} \cdots F_{\beta_N}^{(a_N)}$ is in $\ker e'_i$. It follows from (21) that the span of these vectors has the correct graded dimension, so it is the whole kernel. □

Proof of Theorem 6.2. Fix i, and choose $\mathbf{i}$ such that $i_1 = i$. By Lemma 6.5, $\widetilde{F}_i$ acts by partial permutations on the basis $B_{\mathbf{i}}$. By a simple inductive argument, this implies that $\mathrm{span}_{\mathbb{Q}[q]_0} B_{\mathbf{i}} = \mathrm{span}_{\mathbb{Q}[q]_0} B$ is the lattice generated by all sequences of $\widetilde{F}_i$ acting on $1 \in U_q^-(\mathfrak{g})$. That is, it is $\mathcal{L}(\infty)$. It also shows that $\widetilde{F}_i$ acts by partial permutations on $B_{\mathbf{i}}$, and hence on $B + \mathcal{L}(\infty) = B_{\mathbf{i}} + \mathcal{L}(\infty)$. □

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