Various constructions of (affine) MV polytopes [1]

Peter Tingley (Loyola-Chicago)

Includes work with T. Dunlap, P. Baumann, J. Kamnitzer, D. Muthiah, B. Webster

Montreal Aug 21, 2012

[1] Slides available at http://webpages.math.luc.edu/∼ptingley/

Outline

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1. Background
   - Crystals (by example)
   - The infinity crystal
2. Three constructions of MV polytopes
   - From PBW bases
   - From quiver varieties
   - From categorification (KLR algebras)
   - Sketch of a proof
3. Affine MV polytopes
   - Definition
   - Construction from KLR algebras
   - Rank 2 combinatorics

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- Any finite dimensional representation decomposes (as a vector space) into the direct sum of weight spaces.
- The standard generators move you from one weight space to another in a predictable way.

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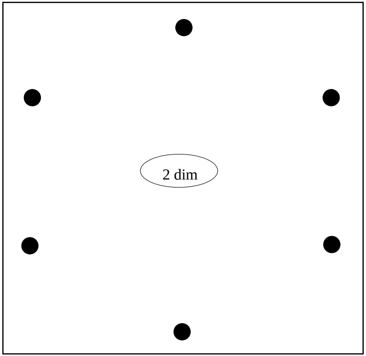

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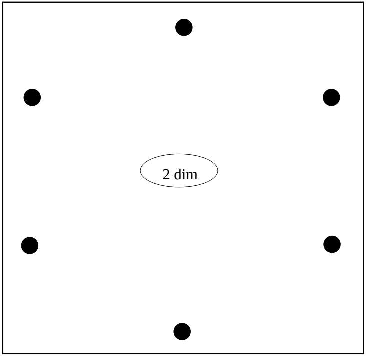

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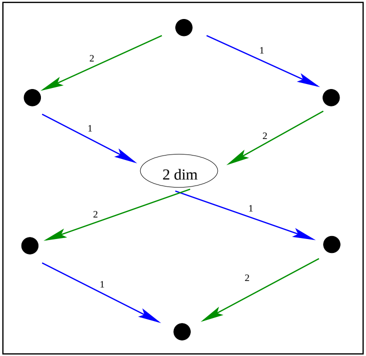

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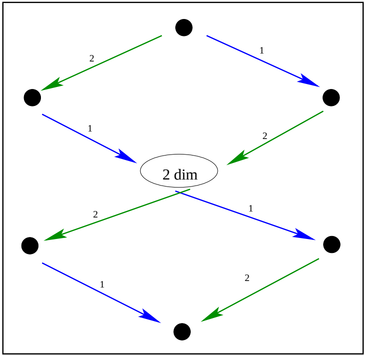

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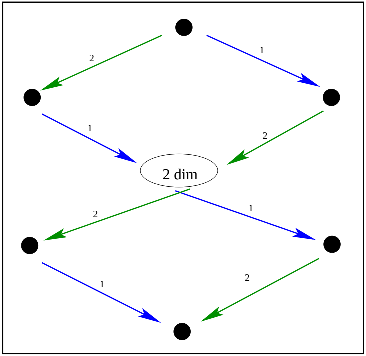

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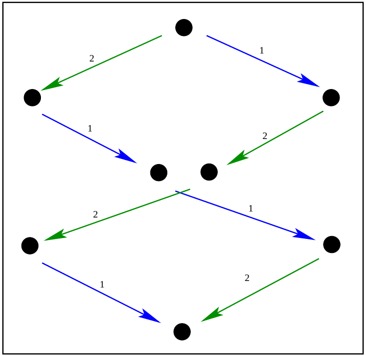

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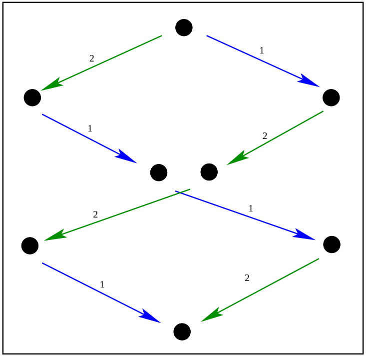

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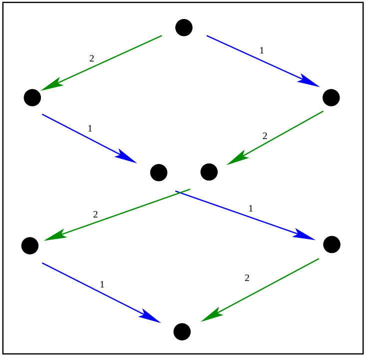

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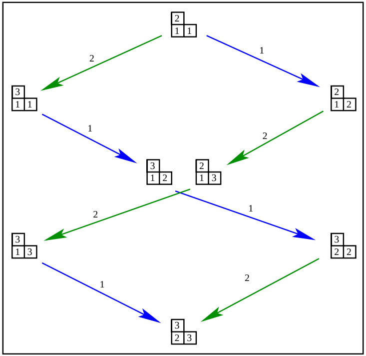

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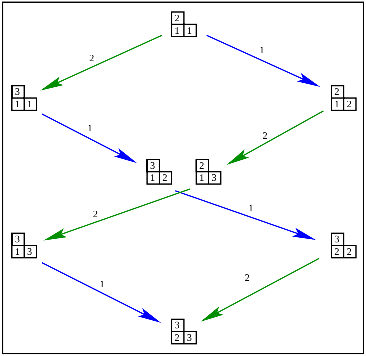

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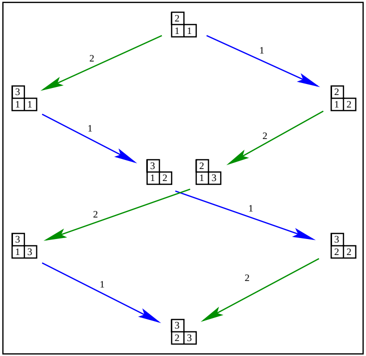

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- Here you see that the graded dimension of the representation is the generating function for semi-standard Young tableaux.

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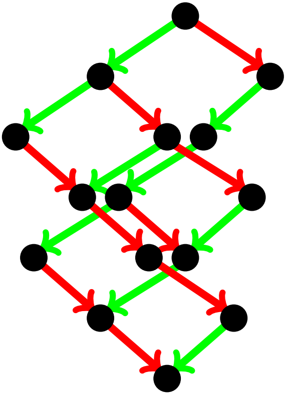

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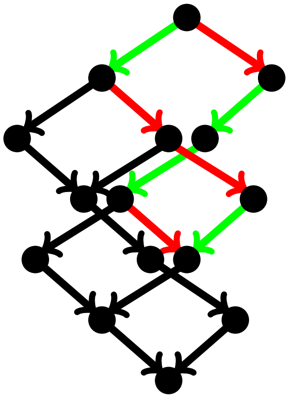

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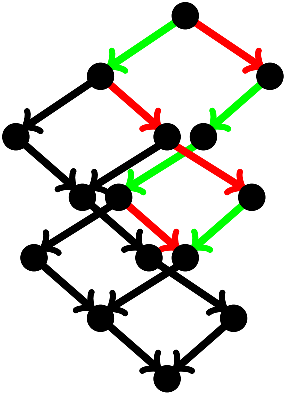

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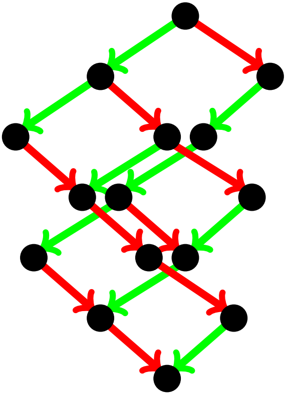

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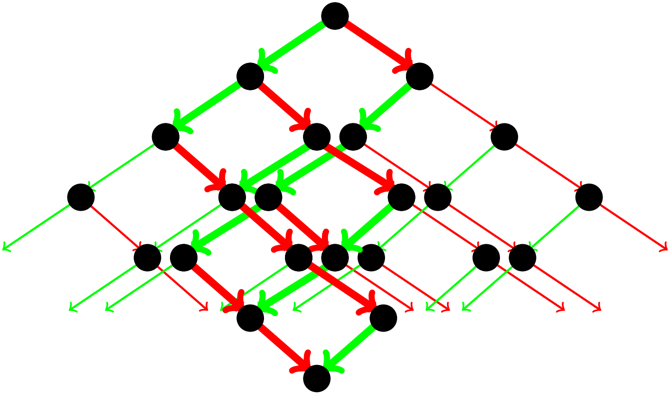

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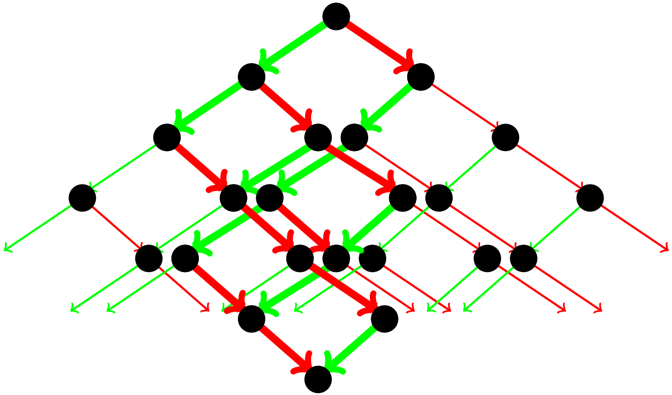

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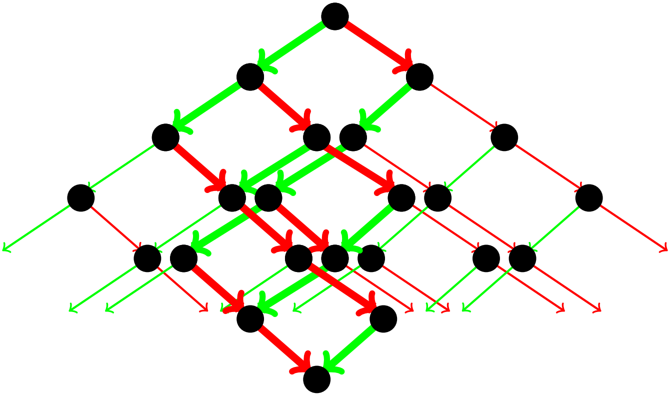

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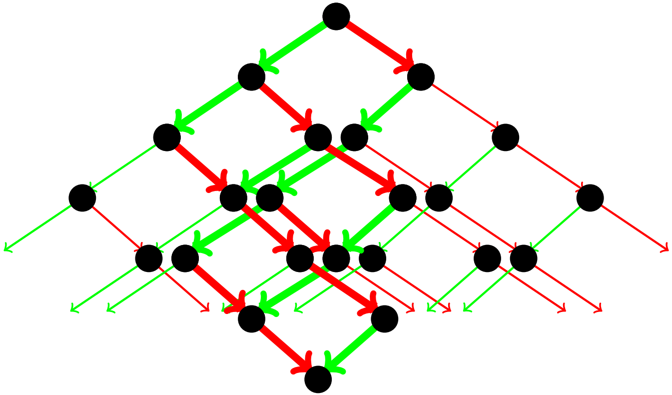

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- MV polytopes are one way to do this.

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- There is one parameterization of $B(\infty)$ for each expression for w_0....it is natural to ask how they are related.
- In a sense MV polytopes give an answer: We record each monomial as a path in weight space, and this is the 1-skeleton of the MV polytope.

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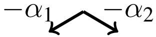

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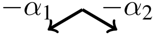

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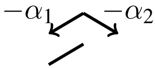

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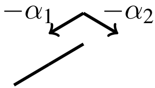

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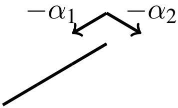

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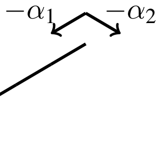

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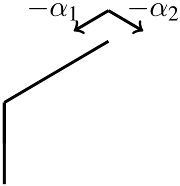

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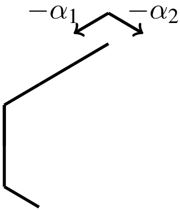

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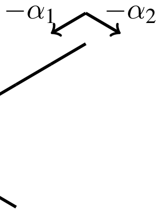

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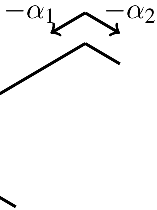

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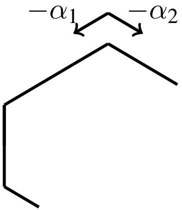

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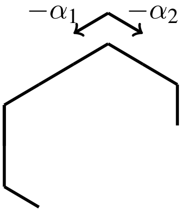

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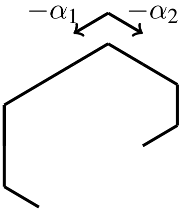

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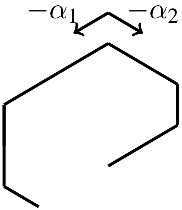

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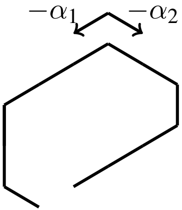

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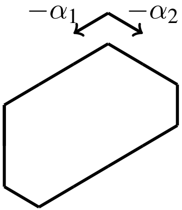

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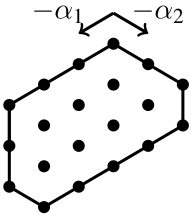

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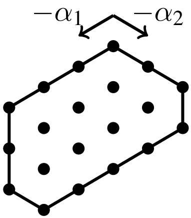

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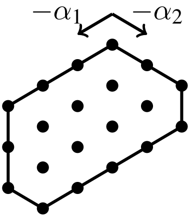

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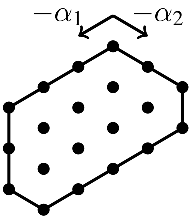

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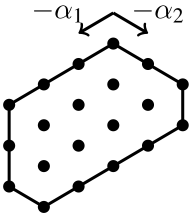

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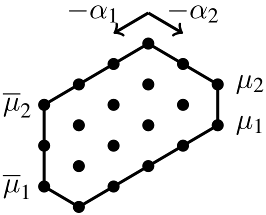

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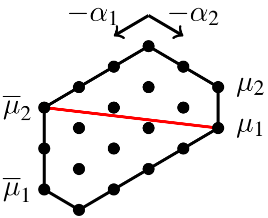

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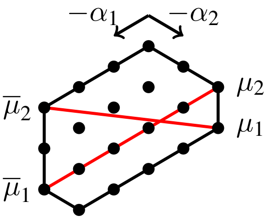

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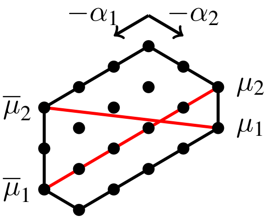

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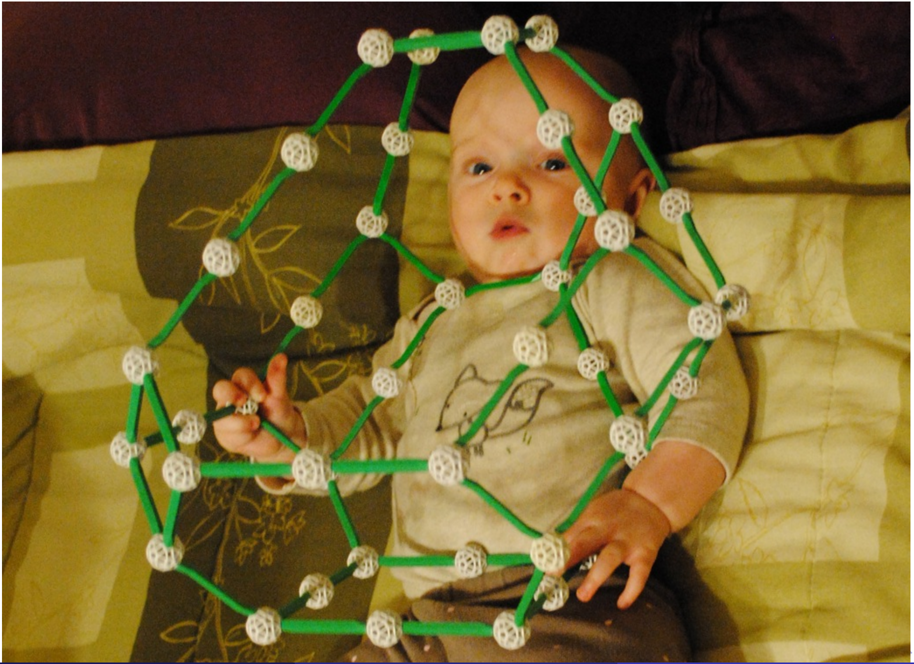
  - Both have slope at most the corresponding simple root.
  - For one of the diagonals, the slope is equal to the simple root.
- Remarkably, understanding rank 2 is enough! A polytope is MV exactly if all its rank 2 faces are MV polytopes of the right types.



Properties of *MV* polytopes

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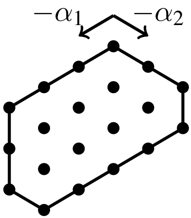

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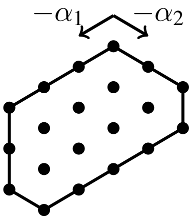

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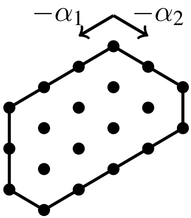

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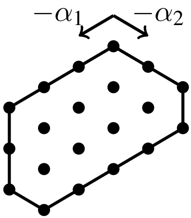

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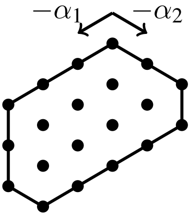

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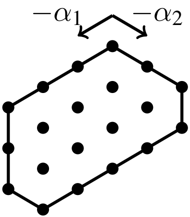

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- Tensor product multiplicities are given by counting MV polytopes subject to conditions on top and bottom edge lengths.

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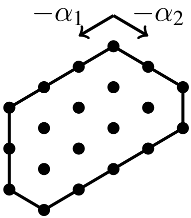

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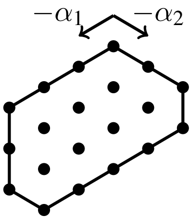

- We'll now look at several places these polytopes arise (quiver varieties and KLR algebras, as well as PBW bases). These constructions make sense in affine type, and are used to figure out what affine MV polytopes should be.

Quiver varieties (for $\mathfrak{sl}_5$)

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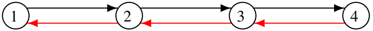
$Q =$  .

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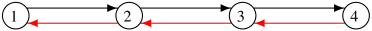

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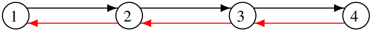
$Q =$  .

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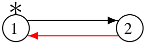 $= 0,$

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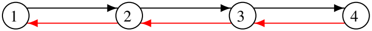
$Q =$  .

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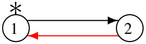
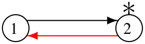
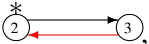


$= 0,$ 

$=$ 

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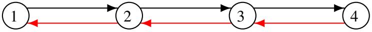
$Q =$  .

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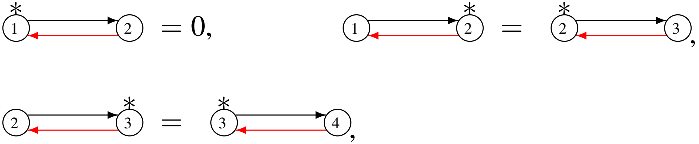

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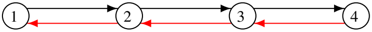
$Q =$  .

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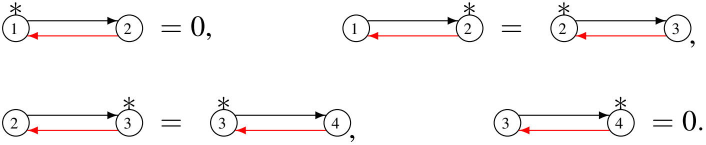

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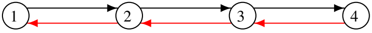
$Q =$  .

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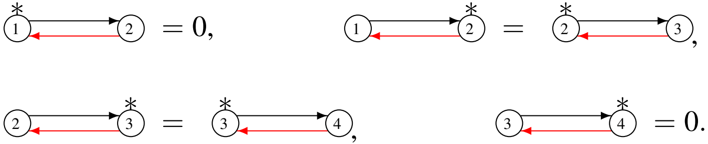

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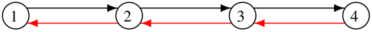
$Q =$  .

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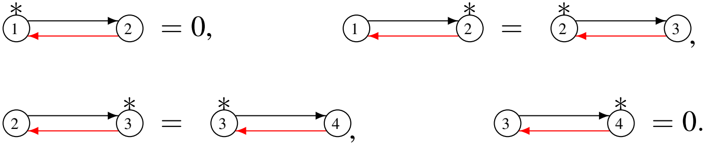

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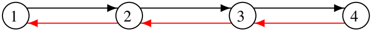
$Q =$  .

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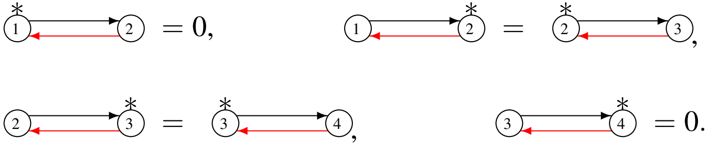

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- We often identify $\mathbf{v}$ with the element $\sum_I v_i \alpha_i$ in the root lattice.

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Theorem (basically due to Baumann-Kamnitzer)

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Fix $b \in B(\infty)$ and let Z_b be the corresponding component in some $\Lambda(\mathbf{v})$.

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Fix $b \in B(\infty)$ *and let* Z_b *be the corresponding component in some* $\Lambda(\mathbf{v})$*.*
Fix $\pi \in Z_b$ *generic, and let* $T = (\pi, V)$ *be the corresponding representation.*

MV polytopes from quiver varieties

Theorem (basically due to Baumann-Kamnitzer)

Fix $b \in B(\infty)$ and let Z_b be the corresponding component in some $\Lambda(\mathbf{v})$. Fix $\pi \in Z_b$ generic, and let $T = (\pi, V)$ be the corresponding representation. Then the MV polytope MV_b is the convex hull of the dimension vectors of all subrepresentations of T.

Harder-Narasimhan filtrations and Lusztig data

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- Fix a representation T of Λ. For each such biconvex order, there is a unique filtration

$$T = T_N \supseteq T_{N-1} \supseteq \cdots \supseteq T_1 \supseteq T_0 = 0$$

of T such that
• $\dim T_k/T_{k-1}$ is a multiple of β_k.
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- The Lusztig data a_k with respect to this expression (i.e. the exponents in Lusztig's monomial) for $Z \in B(\infty)$ are defined by $\dim T_k/T_{k-1} = a_k\beta_k$ for $T \in Z$ generic.

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- When the convex order is by argument for some linear function $c : \mathfrak{h}^* \to \mathbb{C}$ taking all positive roots to the upper half plan, this is the usual Harder-Narasimhan filtrations.

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- The modules R_k can be defined recursively as the unique modules of dimension β_k which admit no Homs from R_j for $j < k$. They can also be explicitly defined in terms of reflection functors.

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- The indecomposable representations $R^{\mathbf{i}}_\beta$ are

$$R^{\mathbf{i}}_{\alpha_2} = 2$$

$$R^{\mathbf{i}}_{\alpha_2+\alpha_3} = 2 \longleftarrow 3$$

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- The simple modules (up to grading shift) index the crystal $B(\infty)$ (and actually better, they are a canonical basis).

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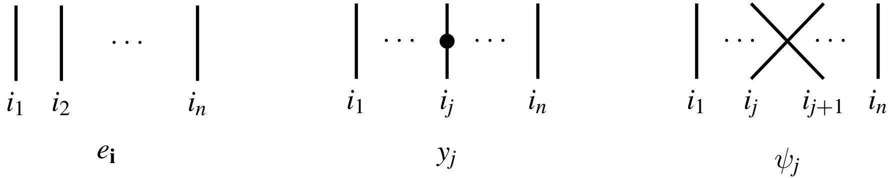

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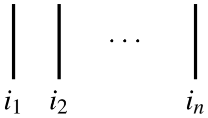
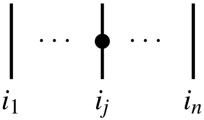
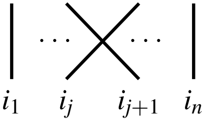


$e_{\mathbf{i}}$



y_j



ψ_j

And there are some relations:

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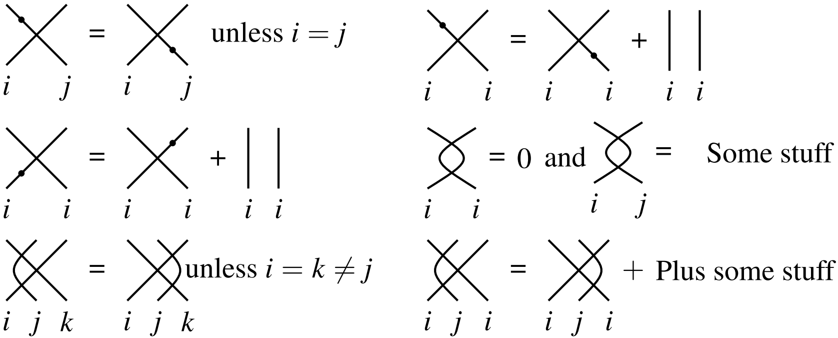

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The MV polytope MV_b is the convex hull of the weights γ such that

$$R(\gamma) \times R(\nu - \gamma) \subset R(\nu)$$

acts non-trivially on L_b.

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For each convex order $\beta_1 \prec \cdots \prec \beta_N$ there are distinguished irreducibles L_k of $R(\beta_k)$ such that: For each irreducible L there is a unique $(a_k) \in \mathbb{Z}^N_{\geq 0}$ such that L is the head of

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The collection (a_k) is the Lusztig data for L.

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- There are really 3 steps: defining the polytopes, showing they exist, and characterizing 2 faces (of types $A_1 \times A_1, A_2, B_2, G_2, A_1^{(1)}, A_2^{(2)}$).

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We have given a combinatorial description of MV polytopes for both rank 2 affine cases, so this is a complete definition (I'll show the $\widehat{\mathfrak{sl}}_2$ case at the end).

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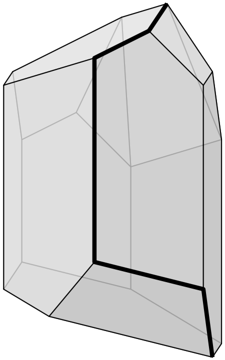

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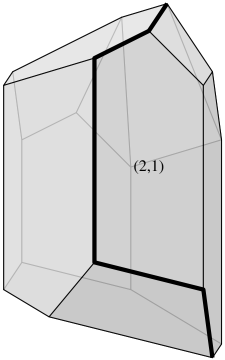

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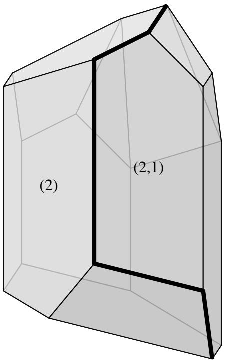

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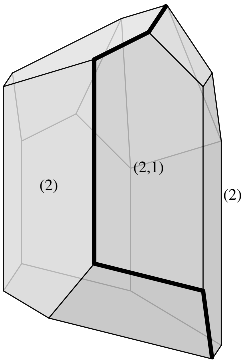

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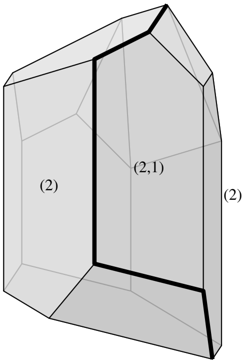

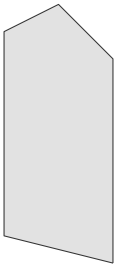

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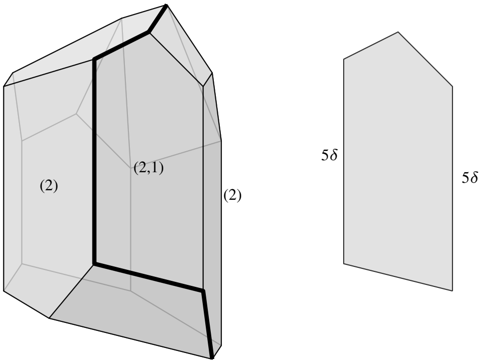

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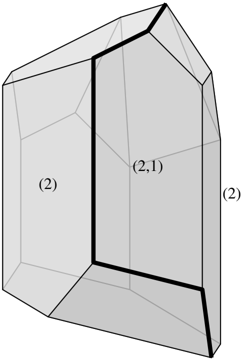

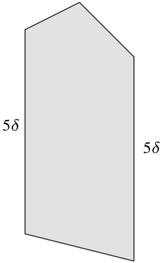

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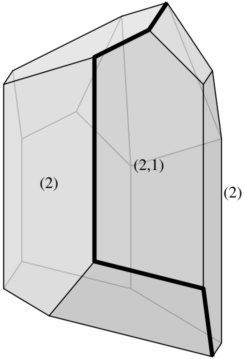

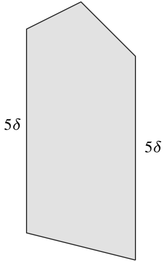

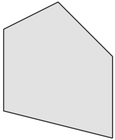

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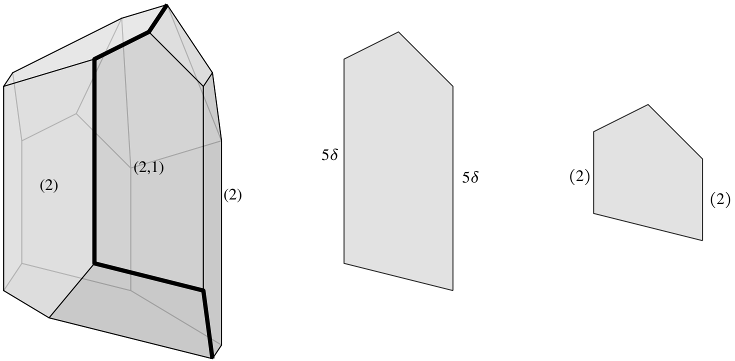

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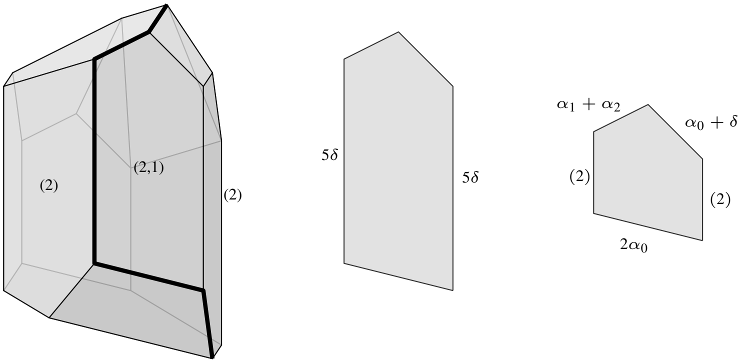

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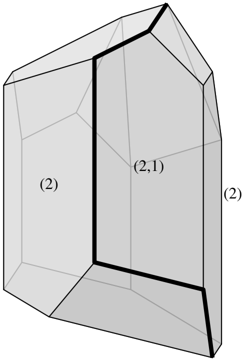

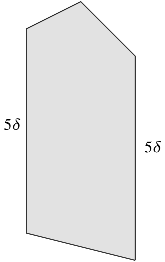

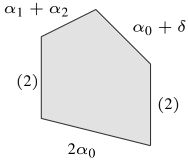

MV for copy of $\widehat{\mathfrak{sl}}_2$ with simple roots $\alpha_0, \alpha_1 + \alpha_2$.

Statement of affine MV polytope theorem

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- The most important idea is a generalized notion of Lusztig data.

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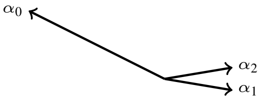

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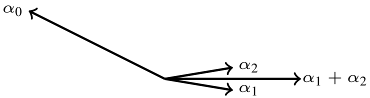

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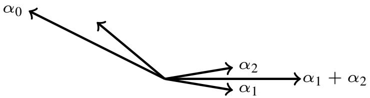

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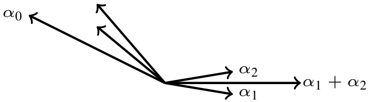

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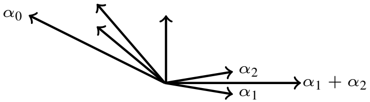

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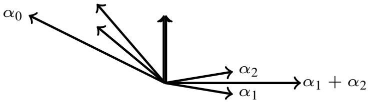

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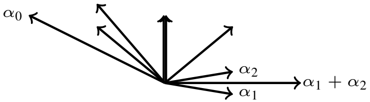

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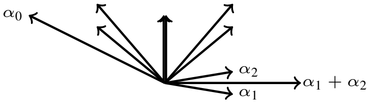

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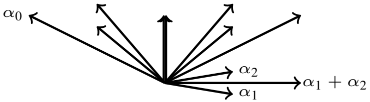

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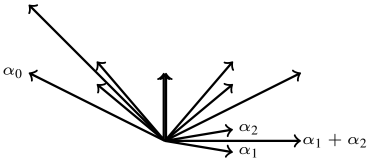

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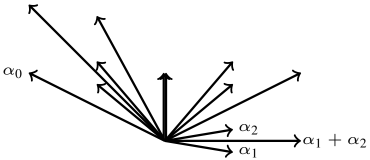

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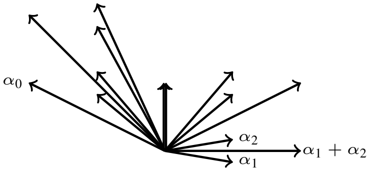

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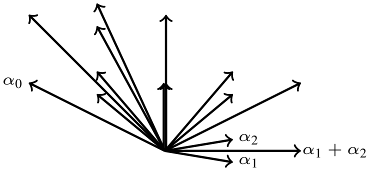

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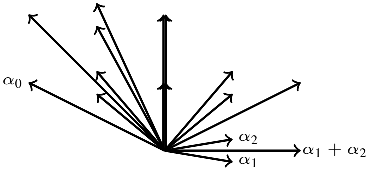

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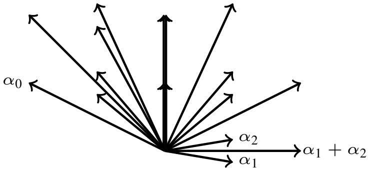

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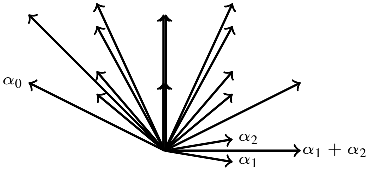

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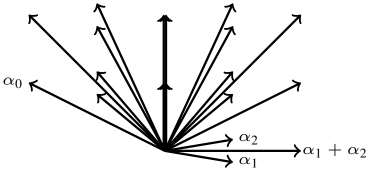

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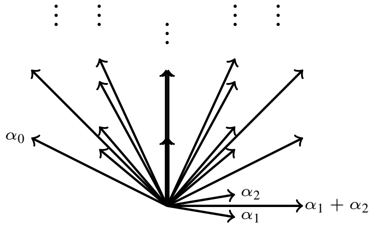

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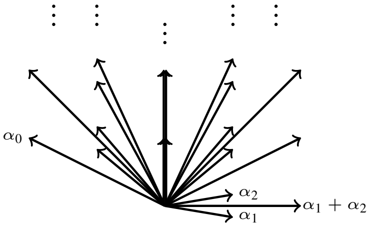

$$\alpha_1 \prec \alpha_1 + \alpha_2 \prec \alpha_2 \prec \alpha_1 + \alpha_2 + \delta \prec \alpha_1 + \delta \prec \alpha_1 + \alpha_2 + 2\delta \prec \alpha_2 + \delta \prec \ldots \delta \cdots \prec \alpha_0 + \alpha_2 \prec \alpha_0 + \alpha_1 \prec \alpha_0$$

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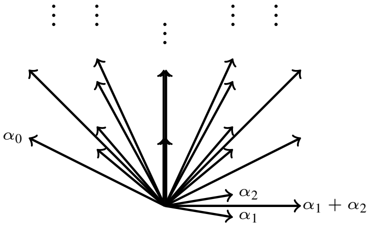

$\alpha_1 \prec \alpha_1 + \alpha_2 \prec \alpha_2 \prec \alpha_1 + \alpha_2 + \delta \prec \alpha_1 + \delta \prec \alpha_1 + \alpha_2 + 2\delta \prec \alpha_2 + \delta \prec \ldots \delta \cdots \prec \alpha_0 + \alpha_2 \prec \alpha_0 + \alpha_1 \prec \alpha_0$

- For every such order, and every irrep L, we get a unique expression $L = A\,(L(\theta_1), L(\theta_2), \cdots, L(\theta_k))$ with θ increasing.

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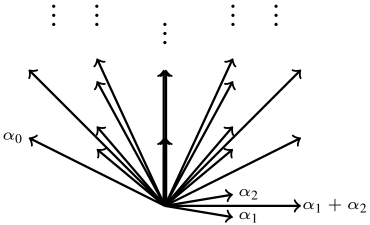

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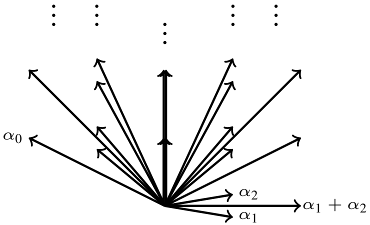

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- Here $L(\pi/2)$ is associated to an edge parallel to δ. Using Kostant partition function, the number of possible such L in $R(k\theta)$ is the number of pairs of partitions of total size k.

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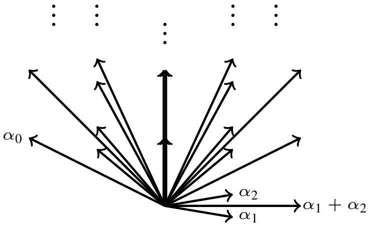

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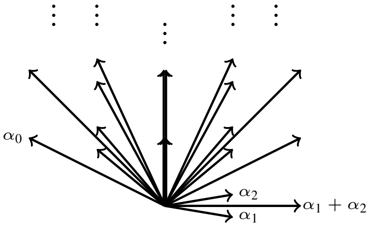

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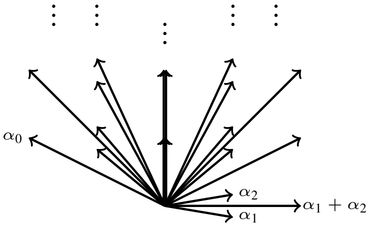
Generalized Lusztig data (from KLR algebras).



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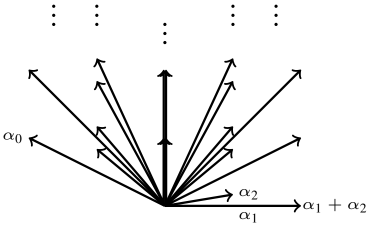

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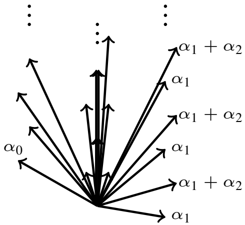

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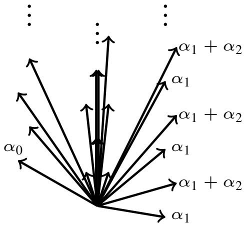

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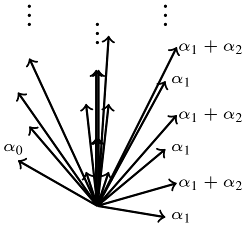

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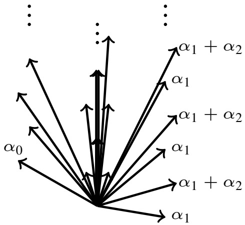

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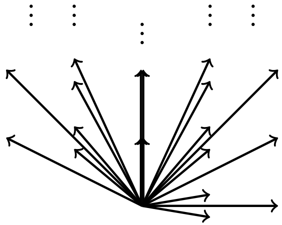

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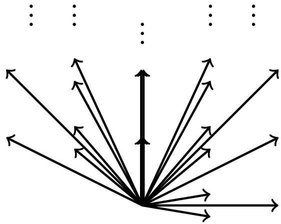

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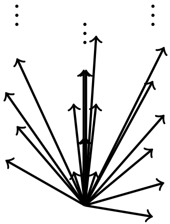

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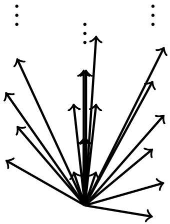

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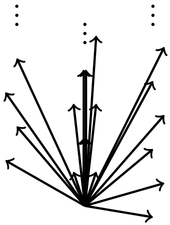

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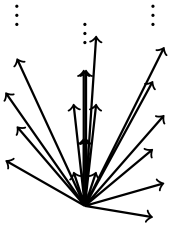

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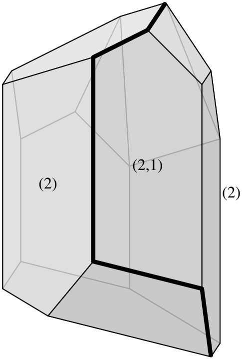
- This is because, if L^h is highest weight and L' is in the component of the trivial, $L^h \circ L'$ is irreducible.

$\widehat{\mathfrak{sl}}_3$ example again



The rank 2 polytopes

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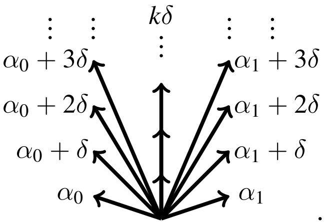

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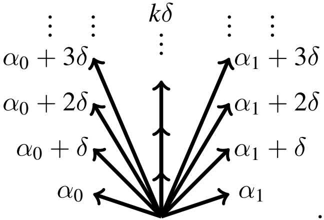

- So the underling polytope should have all edges parallel to these roots.

$\widehat{\mathfrak{sl}}_2$ MV polytopes

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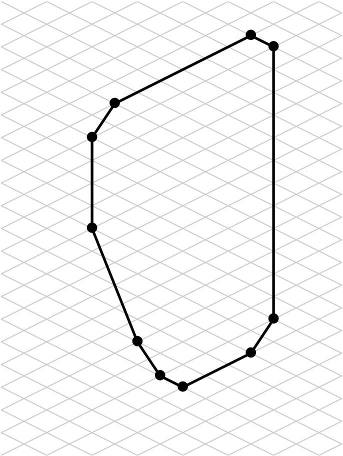

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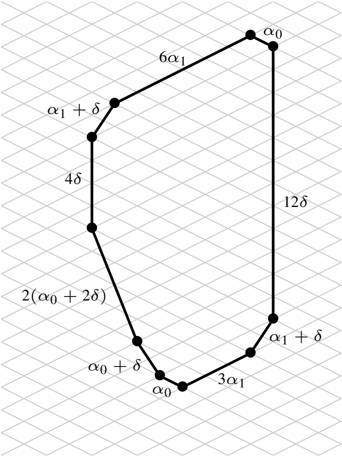

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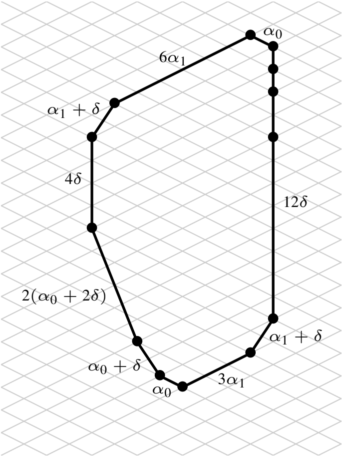

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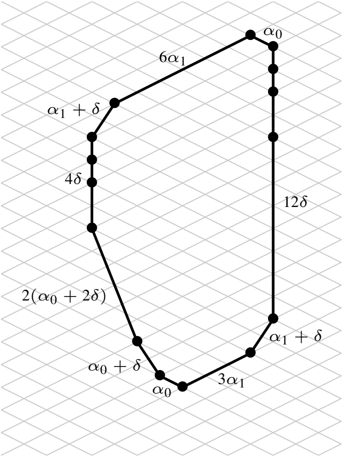

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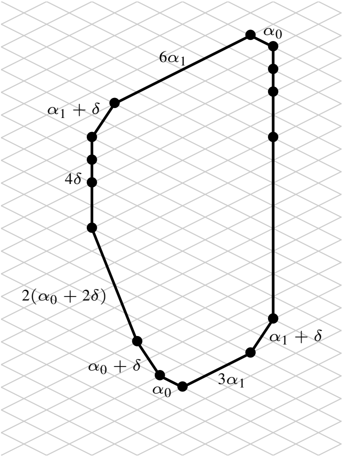

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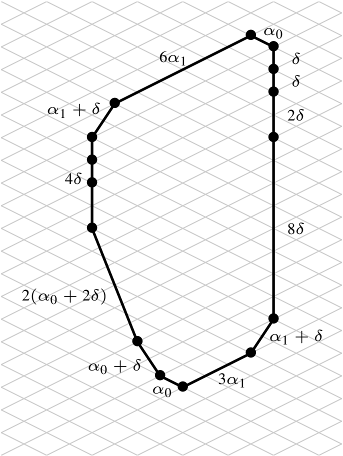

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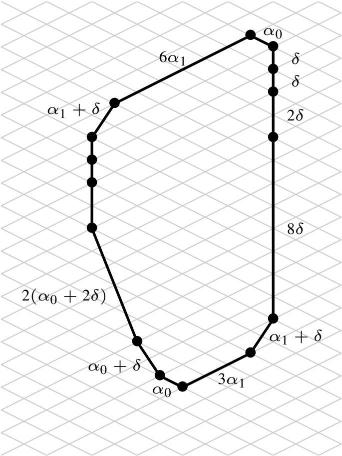

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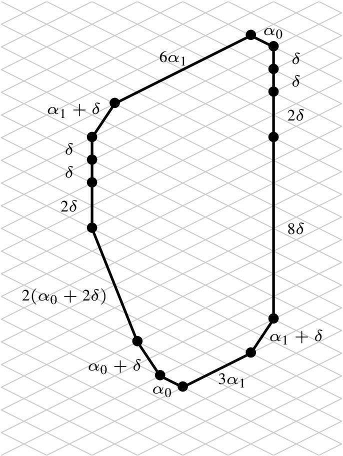

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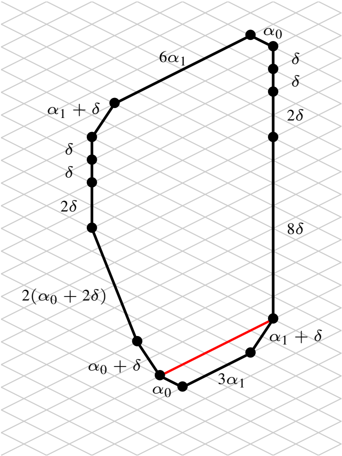

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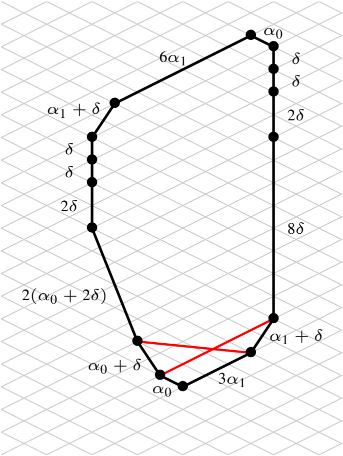

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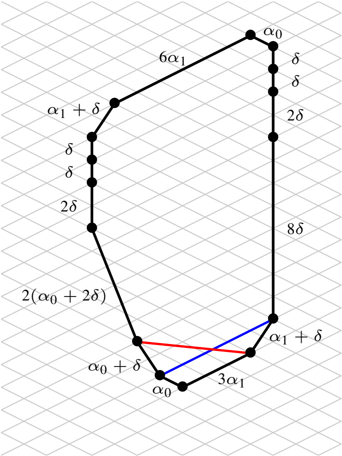

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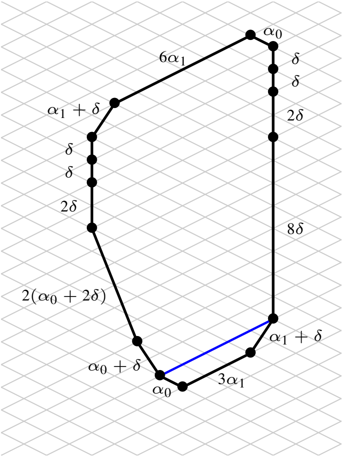

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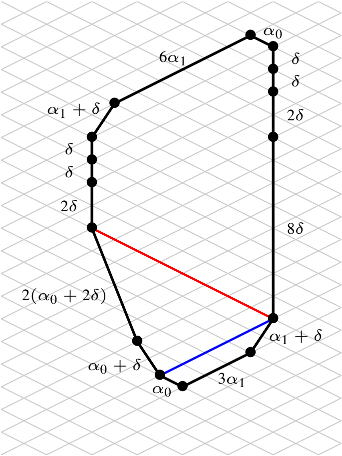

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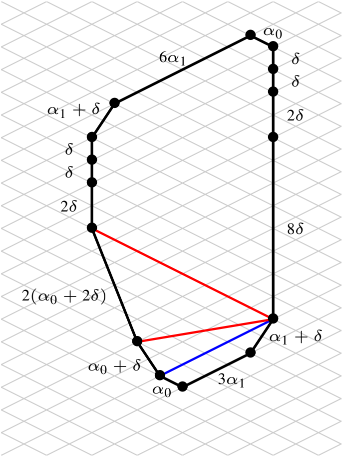

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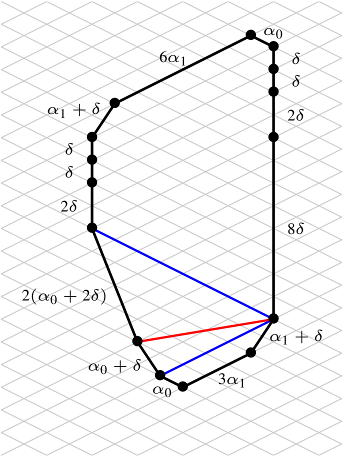

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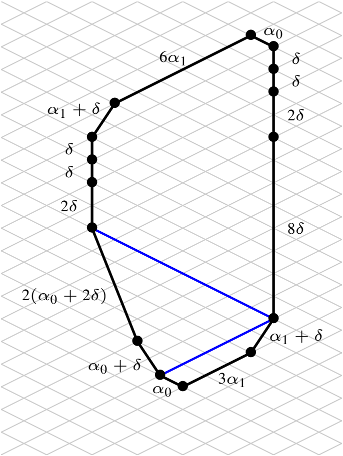

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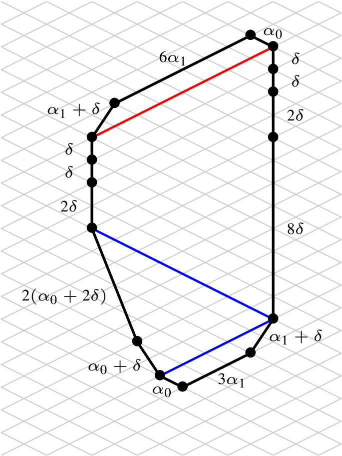

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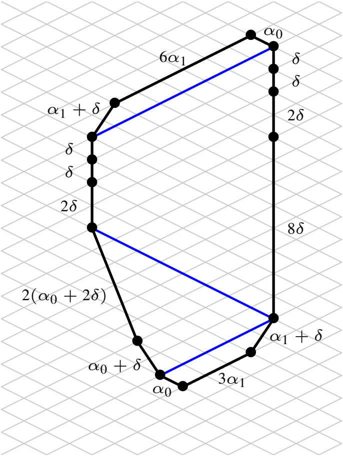

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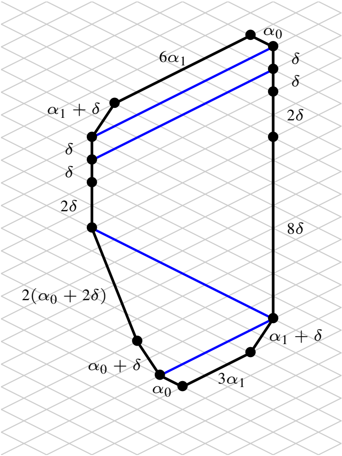

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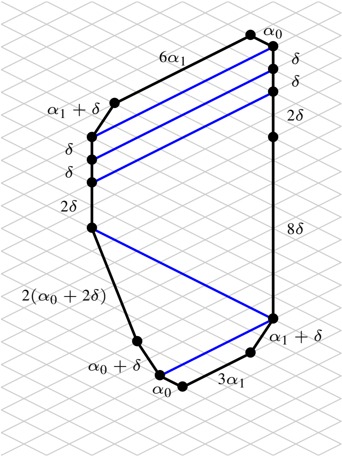

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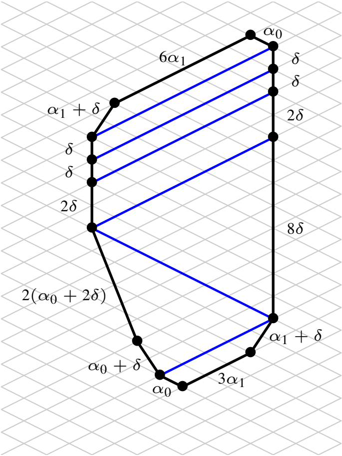

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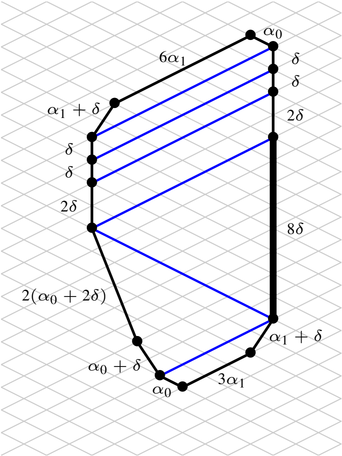

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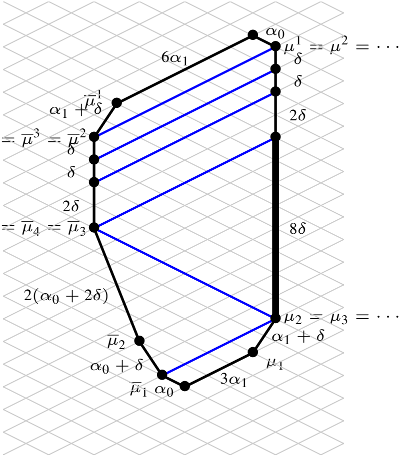

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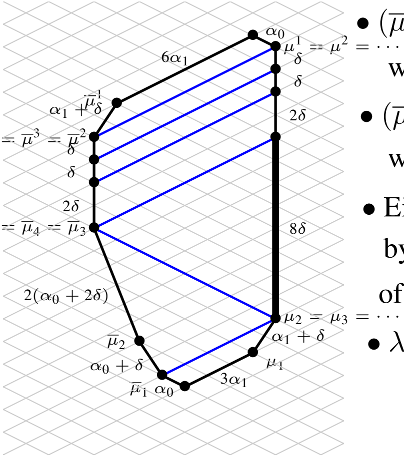

- $(\overline{\mu}_k - \mu_{k-1}, \omega_1) \leq 0$ and $(\mu_k - \overline{\mu}_{k-1}, \omega_0) \leq 0$, with at least one of these being an equality.
- $(\overline{\mu}^k - \mu^{k-1}, \omega_0) \geq 0$ and $(\mu^k - \overline{\mu}^{k-1}, \omega_1) \geq 0$ with at least one of these being an equality.
- Either $\lambda = \overline{\lambda}$, or λ is obtained from $\overline{\lambda}$ by adding or removing a single part of size the width of the polytope.
- $\lambda_1, \overline{\lambda}_1$ are at most the width of the polytope.

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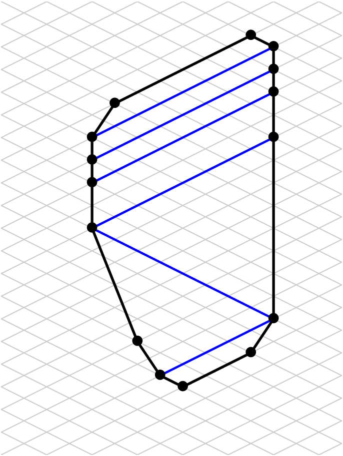

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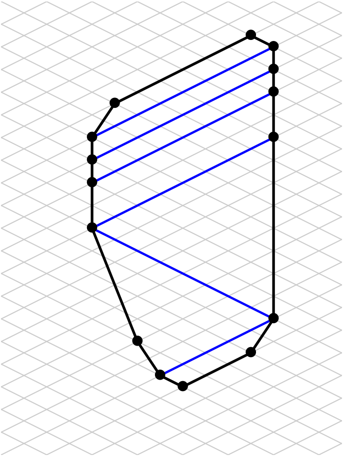

Theorem (B-D-K-T)

There is a unique decorated polytope of this type for any choice of edge lengths on the right side.

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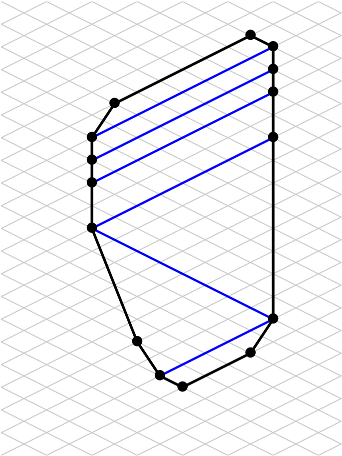

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There is a unique decorated polytope of this type for any choice of edge lengths on the right side.

These, along with natural crystal operators, realize $B(\infty)$.

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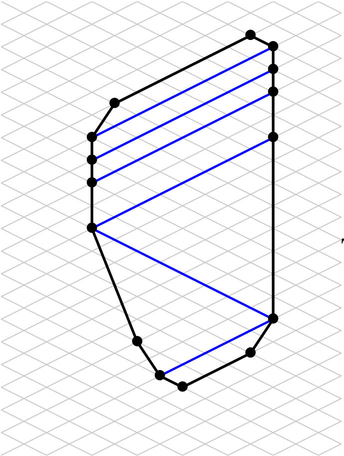

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They have all the properties we want.

Thanks!