EXPLICIT $\widehat{\mathfrak{sl}}_n$ CRYSTAL MAPS BETWEEN CYLINDRIC PLANE PARTITIONS, MULTI-PARTITIONS AND MULTI-SEGMENTS

PETER TINGLEY

Abstract. There are many combinatorial realizations of the crystals B_Λ for the integral highest weight representations of $\widehat{\mathfrak{sl}}_n$. One of the earliest and perhaps best known has as it's underlying set certain ℓ-tuples of partitions, where ℓ is the level of the representation. Another more recent model has as it's underlying set certain cylindric plane partitions. Here we present a simple crystal isomorphism between these two realizations. We then consider the infinity crystal B_∞. This has a combinatorial realization where the underlying set consists of acyclic multi-segments. We give a simple description of the of the e_i-equivariant injection of B_Λ into B_∞ using the cylindric plane partition model. This has been done previously using ℓ-tuples of partitions, but using cylindric plane partitions appears to simplify things. We also note that this map to aperiodic multi-segments extends to an e_i-equivariant injection from all cylindric plane partitions into all (possibly reducible) multi-segments. As discussed in [9], this means one can think of the set of all multi-segments as the infinity crystal for $\widehat{\mathfrak{gl}}_n$ (as opposed to $\widehat{\mathfrak{sl}}_n$).

Contents

Warning! These notes are a draft, and may contain minor errors. I am confident that the main points are correct. If you do happen to find errors, I would be very grateful if you let me know

1. Introduction

In [9] we introduced a model for integrable highest weight $\widehat{\mathfrak{sl}}_n$ crystals where the underlying set consists of cylindric partitions. Here we give an explicit isomorphism between between this realization and an older realization where the underlying set consists of ℓ-tuples of partitions. We also consider the realization of the direct limit crystal B_∞ where the underlying set consists of aperiodic multi-segments. It turns out that the natural imbedding $B_\Lambda \hookrightarrow B_\infty$ is extremely easy to describe in terms of cylindric partitions.

These bijections allow us to build on the results of [9] in two ways. First, we obtain a slightly simpler description of the crystal structure on cylindric partitions (see Corollary 7.5). Second, we are able to give a simple condition on a cylindric partition to ensure that it is in the connected component generated by the highest weight element (see Theorem 4.11).

We note that our embedding of B_Λ into B_∞ extends to an embedding of all cylindric partitions on a fixed cylinder into all multi-segments (not just the acyclic ones). In fact, the set of all multi-segments along with their structure as a (reducible) $\widehat{\mathfrak{sl}}_n$ crystal is the direct limit of the crystals constructed on all cylindric partitions, as the size of the cylinder to becomes large appropriately. In [9] we noted that cylindric partitions can be thought of in a natural way as the underlying set for the crystal of an irreducible $\widehat{\mathfrak{gl}}_n$ (as opposed to $\widehat{\mathfrak{sl}}_n$) crystal. Thus the set of all multi-segments should perhaps be thought of as forming the underlying set of the $\widehat{\mathfrak{gl}}_n$ infinity crystal.

These notes consist mainly of bijections, which are explained via examples. We have tried to justify why the bijections are well defined and preserve the appropriate structure, but where we feel it is clear from the example we have not always included formal proofs.

1.1. **Acknowledgments.** I would like to thank Alistair Savage and Jae-Hoon Kwon who explained to me what "acyclic" should mean for cylindric partitions. I would also like to thank Seok-Jin Kang and the Korean math society for inviting me to the 2008 meetings in Jeju island, where that discussion took place.

2. Crystals

We use notation as in [3], and refer the reader to that book for more detail. For us, a crystal is a set B associated to a representation V of a symmetrizable Kac-Moody algebra $\mathfrak{g}$, along with operators $e_i : B \to B \cup \{0\}$ and $f_i : B \to B \cup \{0\}$, which satisfy some conditions. The set B records certain combinatorial data associated to V, and the operators e_i and f_i correspond to the Chevalley generators E_i and F_i of $\mathfrak{g}$.

Often B will be represented as an edge colored directed graph whose vertices are the elements of B, and where for $x, y \in B$, there is a c_i colored edge from x to y if and only if $f_i(x) = y$. This records all the information about B, since $e_i(y) = x$ if and only if $f_i(x) = y$. For instance, the crystal of the adjoint representation for $\mathfrak{sl}_3$ is shown in Figure 1. The graph B is connected if and only if the corresponding representation V is irreducible. In this paper we are interested in the crystals of integrable highest weight representations of $\widehat{\mathfrak{sl}}_n$. These are always infinite graphs. However, they can be understood in terms of crystal graphs for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ by the following, which follows immediately from, for example, [7, Proposition 2.4.4].

As usual, if there is an arrow a from x to y, we write $t(a) = x$ and $h(a) = y$.

Proposition 2.1. *Fix $n \geq 3$. An n-colored directed graph $G = (V, E)$ is the crystal graph of an irreducible integral highest weight representation of $\widehat{\mathfrak{sl}}_n$ if and only if all of the following hold:*

(i) *There is a "source" $v^{high} \in V$ for which there is no arrow with $h(a) = v^{high}$.*
(ii) *G is connected.*
(iii) *For any pair of colors i and j, every connected component of the graph obtained by only considering edges of colors c_i and c_j is finite. Furthermore, each such connected component is*

$$\begin{cases} An\, sl_3 \text{ crystal if} |i-j| = 1 \, mod(n) \\ An\ sl_2 \times sl_2 \text{ crystal otherwise.} \end{cases}$$

We also need to use the notion of the infinity crystal B_∞, which is defined by giving the set of irreducible $\mathfrak{g}$ crystals the structure of a directed system, and then taking the limit of that system.

Definition 2.2. *Let B, C be $\mathfrak{g}$-crystals. We say a map $s : B \to C$ is e-**equivariant** if, for all $i \in I$ and all $b \in B$, $s(e_i b) = e_i(s(b))$.*

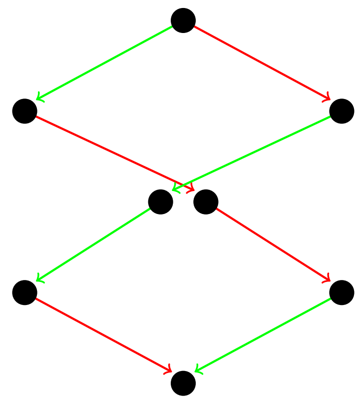

FIGURE 1. The crystal for the adjoint representation of $\mathfrak{sl}_3$. The highest weight element is the node at the top. The operator f_1 moves one step following the red arrows, if possible, and sends the element to 0 otherwise. f_2 acts in the same way, using the green arrows. The operators e_i act in the same way, but moving backwards along arrows.

Denote by B_Λ the crystal of the irreducible representation V_Λ of highest weight Λ. The following is well known (see for example [6, Section 2.2]) for a simple proof):

Proposition 2.3. *Let Λ and Λ' be dominant integral weights for $\mathfrak{g}$, and assume $\Lambda' - \Lambda$ is also dominant. Then there is a unique e-equivariant injection $\iota_\Lambda^{\Lambda'} : B_\Lambda \hookrightarrow B_{\Lambda'}$.* $\square$

Definition 2.4. *Let B_∞ be the direct limit of all B_Λ with respect to the injections in Proposition 2.3. Let $\iota_\Lambda : B_\lambda \hookrightarrow B_\infty$ be the resulting injection.*

For the remainder, we will consider crystals which correspond to representations V of $\widehat{\mathfrak{sl}}_n$. We will use Λ to denote a dominant integral weight and Λ_i to denote one of the fundamental weights, where i is a residue mod n.

3. The multi-partition realization

In this section we review the combinatorial realization of integrable highest weight $\widehat{\mathfrak{sl}}_n$ crystals where the underlying set consists of ℓ-tuples of charged partitions. This realization is essentially due to Jimbo, Misra, Miwa and Okado [5], although we more closely follow the presentation in [2, Section 2].

Definition 3.1. *A level ℓ multi-charge $\mathbf{v}$ is a set of ℓ residues in $\mathbb{Z}/n\mathbb{Z}$. This is written $\mathbf{v} = (v_0, v_1, \dots v_{\ell-1})$, where $0 \le v_0 \le v_1 \le \cdots \le v_{\ell-1} < n$.*

Definition 3.2. *A $\mathbf{v}$-cylindric multi-partition is an ordered ℓ-tuple of partitions $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(\ell-1)})$ which satisfy the conditions*

$$\begin{aligned} \lambda_i^{(k)} &\ge \lambda_{i+v_{k+1}-v_k}^{(k+1)} \qquad && 0 \le k \le \ell-2, \quad i \in \mathbb{Z}_{>0} \\ \lambda_i^{(\ell-1)} &\ge \lambda_{i+n+v_0-v_{\ell-1}}^{(0)} && i \in \mathbb{Z}_{>0}. \end{aligned}$$

Comment 3.3. *Given a $\mathbf{v}$-cylindric multi-partition $\boldsymbol{\lambda}$ one can in fact define $\lambda^{(k)}$ for all $k \in \mathbb{Z}$ by the rule $v_{k+\ell} = v_k + n$, and $\lambda_i^{(k)} = \lambda_i^{(k+\ell)}$. Then the conditions of Definition 3.2 can be expressed as*

$$\lambda_i^{(k)} \ge \lambda_{i+v_{k+1}-v_k}^{(k+1)} \quad \text{for all} \quad k \in \mathbb{Z}, i \in \mathbb{Z}_{>0}.$$

There no longer appear to be two separate conditions. In this form, cylindric multi-partitions look more like the cylindric plane partitions we use later on.

Definition 3.4. *Let* $\mathbf{v}$ *be a multi-charge. The associated dominant weight is* $\Lambda(\mathbf{v}) = \sum_{i=0}^{\ell-1} \Lambda_{v_i}$.

Definition 3.5. *Fix a* $\mathbf{v}$*-cylindric multi-partition* $\boldsymbol{\lambda}$*. Let* b *be a square in the young diagram of one of the* $\lambda^{(k)}$*. Find* (s,t) *so that* b *is the* t^{th} *square in row* s *of* $\lambda^{(k)}$*. We say* b *lies on diagonal* $d(b) = t-s+v_k$*, and color* b *with the reside of* $d(b)$ *modulo* n*, which we denote* $c_{d(b)}$*. See Figure 2.*

Definition 3.6. *A* $\mathbf{v}$*-cylindric multi-partition* $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(\ell-1)})$ *is called a* **highest lift** *if, for all* $m \in \mathbb{Z}_{>0}$*, among the colors appearing at the upper left ends of the length* m *rows of the various* $\lambda^{(k)}$*, at least one of* $\{c_0, c_1, \ldots c_{n-1}\}$ *is missing. See Figure 2.*

Definition 3.7. *Fix a* $\mathbf{v}$ *cylindric multi-partition* $\boldsymbol{\lambda}$*.*

(i) $A_i^{MP}(\boldsymbol{\lambda})$ *is the set of all* c_i *colored squares that can be removed from the Young diagrams of any of the* $\lambda^{(k)}$*, so that the result is still a partition.*
(ii) $R_i^{MP}(\boldsymbol{\lambda})$ *is the set of all* c_i *colored squares that can be removed from the Young diagram of any of the* $\lambda^{(k)}$*, so that the result is still a partition.*

Note that adding a box in A_i^{MP} *or removing a box in* R_i^{MP} *can result in a multi-partition which is no longer* $\mathbf{v}$*-cylindric.*

Definition 3.8. *Define a total order on* $A_i^{MP}(\boldsymbol{\lambda}) \sqcup R_i^{MP}(\boldsymbol{\lambda})$ *as follows: For* $b, b' \in A_i^{MP} \sqcup R_i^{MP}$*, let* $0 \le k, k' < \ell$ *be such that* $b \in \lambda^{(k)}$ *and* $b' \in \lambda^{(k')}$*. Then*

$$b \le b' \text{ if } \begin{cases} d(b) < d(b') & \text{or} \\ d(b) = d(b') \text{ and } k \le k'. \end{cases}$$

It is clear by doing any reasonable example that this is in fact a total order on $A_i^{MP}(\boldsymbol{\lambda}) \sqcup R_i^{MP}(\boldsymbol{\lambda})$.

To calculate f_i^{MP}, construct a string of brackets $S_i^{MP}(\boldsymbol{\lambda})$ be placing a '(' for every $b \in A_i^{MP}(\boldsymbol{\lambda})$ and a ')' for every $b \in R_i^{MP}(\boldsymbol{\lambda})$, ordered according to the total order in Definition 3.8. Cancel all pairs (). f_i^{MP} adds the square b corresponding to the first uncanceled '(' for the left, if there is one, and sends $\boldsymbol{\lambda}$ to 0 otherwise. Similarly, e_i^{MP} removes the square b corresponding to the first uncanceled ')' from the right, if there is one, and sends $\boldsymbol{\lambda}$ to 0 otherwise. See Figure 2.

Definition 3.9. *For any multi-charge* $\mathbf{v}$*, let* $B^{\mathbf{v}}$ *be the set of* $\mathbf{v}$ *cylindric multi-partitions.*

Theorem 3.10. *(see* [2, Section 2]*) Fix* n *and a multi-charge* $\mathbf{v}$*.*

(i) *The operators* e_i^{MP} *and* f_i^{MP} *preserve* $B^{\mathbf{v}} \cup \{0\}$*.*
(ii) $B^{\mathbf{v}}$ *along with the operators* e_i^{MP} *and* F_i^{MP} *is an* $\widehat{\mathfrak{sl}}_n$ *crystal.*
(iii) *Every connected component of the resulting crystal graph* $B^{\mathbf{v}}$ *is isomorphic to* $B_{\Lambda(\mathbf{v})}$*.*
(iv) *The operators* e_i^{MP} *and* f_i^{MP} *preserve the subset* $B^{\mathbf{v},\mathbf{highest}} \cup \{0\}$ *of those* $\mathbf{v}$*-cylindric multi-partitions which are highest lifts. The resulting subcrystal is connected, so is isomorphic to a single copy of* $B_{\Lambda(\mathbf{v})}$*.* □

4. The cylindric plane partition realization

Here we review the combinatorial realization of the crystals $B(\Lambda)$ where the underlying set consists of cylindric plane partitions, as described in [9]. This is quite similar to the realization using multi-partitions, and the exact relationship is described explicitly in the Section 5. We have set up conventions here slightly differently that in [9]. In particular, boxes in the cylindric partition are colored in a different way. The relationship is that a box colored c_i in [9] in now colored c_{-i}. This amounts to twisting by a diagram automorphism. We have done this so that later on we match the conventions for multi-segments exactly.

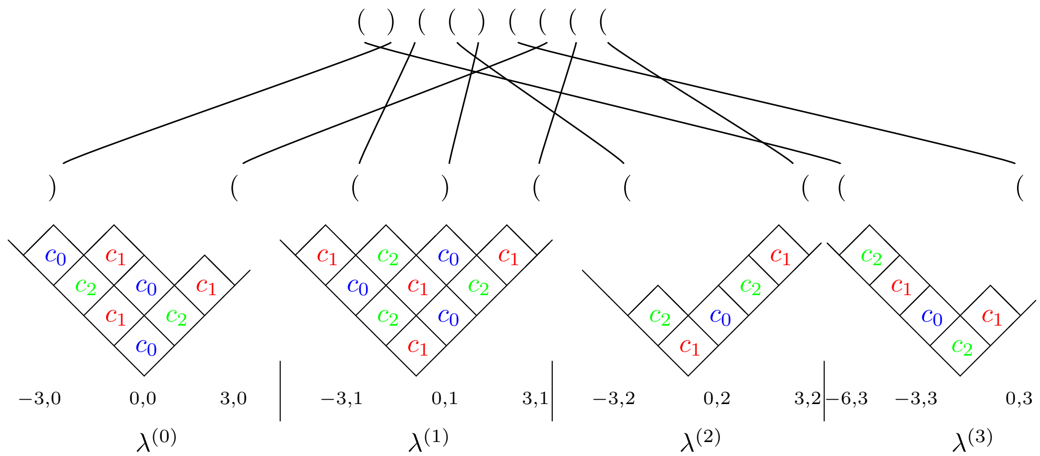

FIGURE 2. A multi-partition for $\widehat{\mathfrak{sl}}_3$ with multi-charge $\mathbf{v} = (0, 1, 1, 2)$. We draw partitions in Russian notation, with the convention that the rows of the partition are drawn sloping up and to the left. Here $\boldsymbol{\lambda} = (4.3.1, 4.3.2.1, 2.1.1.1, 4.1)$. One can check that this example is $\mathbf{v}$-cylindric. This example is not a highest lift because there is are three rows of length 4 whose upper-left boxes contain the three colors c_0, c_1 and c_2. Above each $b \in A_0^{MP}$ we have placed a '(', and above each $b \in R_0^{MP}$ we have placed a ')'. Below each c_0 diagonal, we have written the pair (d, k) from Definition 3.8. The brackets are reordered lexicographically in d and k. f_0 acts by adding the box corresponding to the first uncanceled '(' from the left in this reordered bracket string, which would end up in $\lambda^{(1)}$. Since there is no uncanceled ")", e_0 would send this multi-partition to 0.

Definition 4.1. *By a cylinder $\mathcal{C}$ we mean a square grid, drawn on a semi-infinite cylinder, with a chosen boundary, as in Figure 3. we require that when drawn as in Figure 3, the boundary intersect any vertical line only once.*

Definition 4.2. *A $\mathcal{C}$-cylindric partition π is a filling of the squares inside the grid $\mathcal{C}$ with non-negative integers, all but finitely many of which are zero, such that the result is weakly decreasing as you move away from the boundary along the grid. As shown in Figure 4, π can be represented by a 3-dimensional picture.*

Definition 4.3. *Let $B^{\mathcal{C}}$ denote the set of $\mathcal{C}$ cylindric plane partitions.*

Definition 4.4. *Draw $\mathcal{C}$ as in Figure 3. Let $\ell_{\mathcal{C}}$ be the number up steps on the boundary going up and to the right over one period, and $n_{\mathcal{C}}$ the number of steps going down and to the right. When $\mathcal{C}$ is understood, we will leave off the subscripts.*

Choose once and for all a single period of the boundary of $\mathcal{C}$, which breaks the cylinder at a local minimum of the boundary, as shown in Figure 4. Different choices of boundary give crystals of representations which are related by a Dynkin diagram automorphism. This choice is not very important, but must be made to fix notation.

Fix $\pi \in B^{\mathcal{C}}$. As in Figure 4, draw a 3-dimensional picture, where π_{ij} is the height of a pile of boxes placed at position (i, j). Label each $1 \times 1 \times 1$ box with coordinates (x, y, z) as shown in Figure 4, with the origin placed so that the center of the k^{th} box up in $\pi_{s,t}$ is given coordinates $(s, t, k - 1/2)$. Note that due to the periodicity, (x, y, z) labels the same box as $(x + \ell, y - n, z)$, so the coordinates are only well defined up to this type of transformation. Color each box with a residue modulo n by coloring the box at position (x, y, z) with c_{y-z} (note that this is well defined since $y + n - z \simeq y - z \bmod n$).

Definition 4.5. *$A_i^{CP}(\pi)$ is the set of c_i colored boxes that can be added to π so that each slice π_i (see Figure 4) is still weakly decreasing.*

$R_i^{CP}(\pi)$ is the set of c_i colored boxes that can be removed from π so that each slice π_i is still weakly decreasing.

Note: *Adding a box in $A_i^{CP}(\pi)$ or removing a box in $R_i^{CP}(\pi)$ can result in something which is no longer a cylindric partition, since the slices c_i (see Figure 4) may no longer be weakly decreasing. However it should be clear that if ALL the boxes in A_i^{CP} are added to π (or all boxes in R_i^{CP} are removed) the result does remain a cylindric partition.*

Definition 4.6. *Define $t(x,y,z) = nx/\ell + y - z$. Note that $t(x,y,z) = t(x+\ell, y-n, z)$, so t is well defined as a function on boxes in a cylindric plane partition. For a box b as in Figure 4, define $t(b)$ to be t calculated on the coordinates of the center of b.*

Definition 4.7. *Color the grid $\mathcal{C}$ as in Figure 3. Let $\mathbf{v}(\mathcal{C})$ be the ℓ-tuple of residues mod n which records the colors of the upper-left squares in the grid $\mathcal{C}$, as shown in Figure 3. Choose the representative $\mathbf{v} = (v_0, v_1, \ldots, v_n)$ with $0 \le v_0 \le v_1 \le \cdot \le v_{\ell-1} < n$. Let $\boldsymbol{\lambda}(\pi)$ be the ordered ℓ-tuple of partitions where, for $0 \le i < \ell$, $\lambda^{(i)}$ is the partition which reads the numbers in the slice of $\mathcal{C}$ labeled π_i.*

Similarly, let $\mathbf{v}'(\mathcal{C})$ be the ordered n-tuple of residues mod ℓ which record the label π_i of the upper-right most squares in the grid $\mathcal{C}$, and choose representative $\mathbf{v}' = (v'_0, v'_1, \ldots v'_{n-1})$ with $0 \le v'_0 \le v'_1 \le \cdots \le v'_{n-1} < \ell$. Let $\boldsymbol{\lambda}'$ be the ordered n-tuples of partition which read down the slices of the grid labeled c_k.

Definition 4.8. *For a cylinder $\mathcal{C}$ with $\mathbf{v}(\mathcal{C}) = (v_0, v_1, \ldots v_{\ell-1})$, define $\Lambda(\mathcal{C}) := \sum_{k=0}^{\ell} \Lambda_{v_k}$.*

Similarly, define $\Lambda'(\mathcal{C}) = \sum_{k=0}^{n} \Lambda_{v'_k}$.

Lemma 4.9. *Fix $\pi \in B^{\mathcal{C}}$. Let $b_1, b_2 \in A_i^{CP}(\pi) \bigcup R_i^{CP}(\pi)$. Then $t(b_1) = t(b_2)$ implies $b_1 = b_2$.*

Proof. We may assume $A_i^{CP}(\pi) = 0$, since $\pi' = \pi \cup A_i^{CP}(\pi)$ is still $\mathcal{C}$-cylindric, $A_i^{CP}(\pi') = 0$, and $R_i(\pi') = A_i^{CP}(\pi) \bigcup R_i^{CP}(\pi)$. It is reasonable clear for Figure 4 that any plane $t(b) =$ constant intersects the center of at most one box in $A_i^{CP}(\pi)$ over each period, from which the result follows. Alternatively, here is an algebraic proof.

Let $b_1 b_2 \in R_i(\pi)$, and assume that $t(b_1) = t(b_2)$. Use the periodicity to choose coordinate $b_1 = (x_1, y_1, z_1)$, and $b_2 = (x_2, y_2, z_2)$ so that $0 \le x_2 - x_1 < \ell$. By definition $t(b_1) = t(b_2)$ means

$$\frac{nx_1}{\ell} + y_1 - z_1 = \frac{nx_2}{\ell} + y_2 - z_2. \tag{1}$$

Since both b_1 and b_2 are colored c_i, we have

$$z_1 - y_1 \simeq z_2 - y_2 \quad \text{modulo } n. \tag{2}$$

Together with the fact that $|x_2 - x_1| < \ell$, equations (1) and (2) imply that $z_1 - y_1 = z_2 - y_2$. But then from (1) we see that $x_1 = x_2$. Hence, b_1 and b_2 belong to the same slice $\lambda'^{(r)}$ for some residue r modulo ℓ (see Definition 4.7). But $\lambda'^{(r)}$ is a young diagram, which implies that $z - y$ is strictly decreasing as one moves along its boundary. Thus there can be at most one box on the boundary of this slice with a given value of t. The result follows since any $b \in R_i^{CP}(\pi)$ must be one the boundary of π. ☐

Definition 4.10. *A $\mathcal{C}$-cylindric partition π is called* **left acyclic** *if there is no integer $m > 0$ such that m is a part of $\lambda'^{(k)}$ for all residues k mod n.*

We are now ready to present the crystal structure on $\mathcal{C}$-cylindric partitions from [9]. Let π be a $\mathcal{C}$-cylindric partition. Define $S_i^{CP}(\pi)$ to be the string of brackets formed by placing a "(" for every box in $A_i^{CP}(\pi)$, and a ")" for every box in $R_i^{CP}(\pi)$. These are ordered with the bracket corresponding to b_1 coming before the bracket corresponding to b_2 if and only if $t(b_1) < t(b_2)$ (this is a total order by Lemma 4.9). Then $f_i^{CP}(\pi)$ is the cylindric plane partition obtained by adding the box corresponding

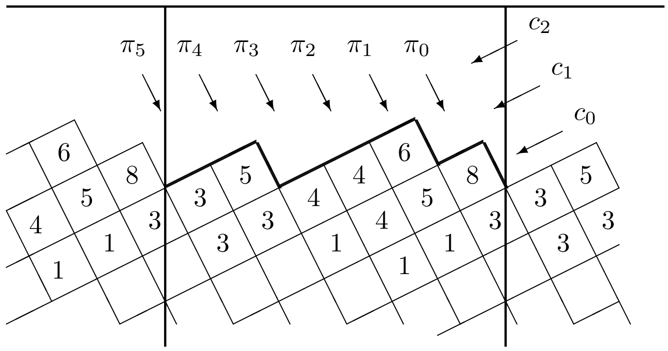

FIGURE 3. A cylinder $\mathcal{C}$ refers to a semi-infinite grid drawn on a cylinder as shown, along with the choice of boundary. This picture is periodic with one period shown between the two dark lines. To get a cylinder, one should cut the picture along these two lines, then glue the edges. We require that the grid lines be such that the boundary intersects any vertical line exactly once. Here $\ell_{\mathcal{C}} = 6$ and $n_{\mathcal{C}} = 3$. Note that these two values do not depend on the boundary, but only on the grid. That is, they could be determined locally far away fro the boundary. A $\mathcal{C}$-cylindric partition π is a filling of the squares in the grid with non-negative integers so that the result is weakly decreasing as you follow the grid down, in either of the two possible directions. We also insist that all but finitely many of the entries be zero (and record zeros as empty squares). Here the associated multi-charge and multi-partition are $\mathbf{v}(\mathcal{C}) = (0, 1, 1, 1, 2, 2)$, and $\boldsymbol{\lambda}(\psi) = (8.3; 6.5.1; 4.4.1; 4.1; 5.3; 3.3)$. The associated dual multi-charge and dual multi-partition are obtained by interchanging the roles of c_i and π_i. That is, $\mathbf{v}'(\mathcal{C}) = (0, 1, 4)$, and $\boldsymbol{\lambda}'(\psi) = (8.5.4.1; 6.4.4.3.3; 5.3.3.1.1)$ This example is left acyclic, because no $k > 0$ occurs in all three of the diagonals labeled c_0, c_1 and c_2.

to the first uncanceled "(", if there is one, and is 0 otherwise. Similarly, $e_i^{CP}(\pi)$ is the cylindric plane partition obtained by removing the box corresponding to the first uncanceled ")", if there is one, and is 0 otherwise. See Figure 4.

Theorem 4.11. *(see [9]) Fix a cylinder $\mathcal{C}$. Then:*

(i) *e_i^{CP} and f_i^{CP} preserve $B^{\mathcal{C}} \cup \{0\}$.*
(ii) *$B^{\mathcal{C}}$ along with the operators e_i^{CP} and f_i^{CP} is an $\widehat{\mathfrak{sl}}_n$ crystal.*
(iii) *Each irreducible component of $B^{\mathcal{C}}$ is isomorphic to $B_{\Lambda(\mathcal{C})}$.*
(iv) *The set of left acyclic $\mathcal{C}$ cylindric partitions forms a single connected component of $B^{\mathcal{C}}$, and thus is isomorphic to $B_{\lambda(\mathcal{C})}$.*

Parts (i), (ii) and (iii) of Theorem 4.11 are proven in [9]. Part (iv) can also be proven directly by examining the crystal structure on $\mathcal{C}$ cylindric partitions. However, we delay the proof since it is immediate from the bijection developed in Section 5.

5. THE CRYSTAL ISOMORPHISM BETWEEN ℓ-PARTITIONS AND CYLINDRIC PLANE PARTITIONS

Theorem 5.1. *The map $\pi \to (\boldsymbol{\lambda}(\pi))$ from definition 4.7 is an isomorphism of crystals.*

Proof. It is clear that $\pi \to \boldsymbol{\lambda}(\pi)$ induces a bijection between $A_i^{CP}(\pi)$ and $A_i^{MP}(\boldsymbol{\lambda}(\pi))$ (respectively $R_i^{CP}(\pi)$ and $R_i^{MP}(\boldsymbol{\lambda}(\pi))$. So we must establish that the strings of brackets S^{CP} and S^{MP} read the boxes in $A_i \sqcup R_i$ in exactly the same order. Consider the cylindric partition in Figure 4. $\boldsymbol{\lambda}(\pi)$ consists of

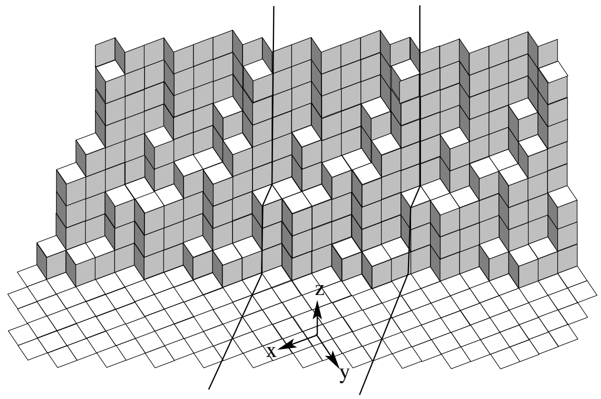

FIGURE 4. The three dimensional representation of the cylindric plane partition shown in Figure 3. The picture is periodic, with one period shown between the dark lines. The first layer of boxes should be colored with c_0, c_1 and c_2 according to which of these diagonals they lie on (see Figure 3). Higher levels are colored according to the rule that if a box b' is immediately above b, and b is colored c_i, then b' is colored c_{i+1}. The planes $t(x, y, z) = C$ intersect the "floor" of the picture in a line which is horizontal in the projection shown, and are angled so that (x, y, z) and $(x, y + 1, z + 1)$ are always on the same plane. For any $i \in I$, each such plane intersects the center of at most one c_i colored box that could be added or removed to/from π in each period. The crystal operator f_i^{CP} acts on π by placing a "(" for each box in $A_i^{CP}(\pi)$ and a ")" for each box in $R_i^{CP}(\pi)$, ordered by t calculated on the coordinates of the center of the box. f_i^{CP} adds a box corresponding to the first uncanceled "(", if there is one, and sends the element to 0 otherwise. Note that adding a box from $R_i^{CP}(\pi)$ to π need not result in a cylindric plane partition. However, it turns out that $f_i^{CP}(\pi)$ is always a cylindric plane partition.

a 6-tuple of partitions, corresponding to the six slices π_k. These six partitions, along with the coloring of their squares inherited from the coloring of the boxes of π, are shown in Figure 5. We have placed this 6-tuple of partitions such that the function t evaluated on the center of a box is equal to the horizontal position of the center of the corresponding square in the diagram. Thus, S^{CP} is calculated using the string of brackets as shown. One should then convince oneself that this is equivalent to the construction of S^{MP} shown in Figure 2. The best way is to redraw this example with the partitions placed as in Example 2, and see that S^{MP} and S^{CP} agree. $\square$

This bijection allows us to finish the proof of Theorem 4.11.

Proof of Theorem 4.11 Part (iv). By Theorem 3.10 Part (iv), it suffices to check that the condition of π being left acyclic is equivalent to $\boldsymbol{\lambda}(\pi)$being a highest lift. This is immediate from definitions. $\square$

6. THE MULTI-SEGMENT REALIZATION

Here we review a realization of the direct limit crystal B_∞ where the underlying set consists of multi-segments. We mostly follow the conventions of [8].

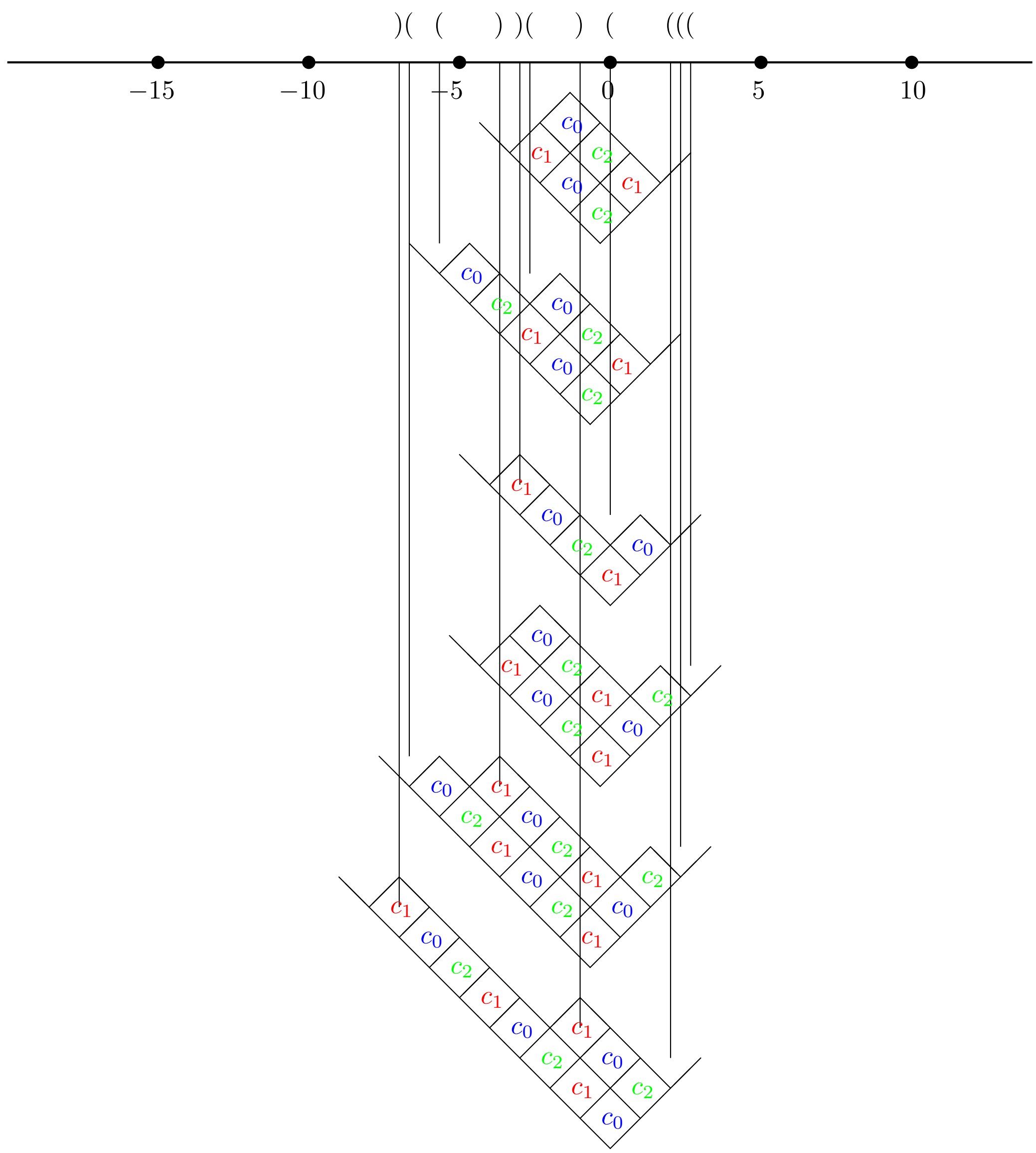

FIGURE 5. Calculation of f_1^{CP} for the cylindric partition shown in Figures 3 and 4. The six partitions here are the partitions π_i from Figure 4. They are placed so that $t(b)$, as used in calculating $f_i^{CP}(\pi)$, is the horizontal location of the b. The string of brackets $S^{CP}(\pi)$ is shown above. These 6 partitions form a $\mathbf{v}$-cylindric multi-partition, where $\mathbf{v} = (0, 1, 1, 1, 2, 2)$. If one then calculates S^{MP}, one sees that the same brackets appear in exactly the same order. This is essentially the proof that this bijection between cylindric partitions and multi-partitions preserves the crystal structure.

Definition 6.1. *An n-**segment** is an interval $[a, b]$ on the real line, with endpoints $a, b \in \mathbb{Z} + 1/2$, and considered up to shifting in either direction by integer multiples of n. In particular, $[a, b]$ and $[a + n, b + n]$ are considered the same.*

For any reside $i \in \mathbb{Z}/n\mathbb{Z}$ and any $k \in \mathbb{Z}$, we use the notation $[i; k)$ to mean the segment $[i - 1/2, i + k - 1/2]$. Similarly, we use the notation $(k; i]$ to mean the segment $[i - k + 1/2, i + 1/2]$.

Definition 6.2. *A **type** n **multi-segment** ψ is a finite collection of n-segments. Note that the empty collection is allowed, and is denoted $\emptyset$. Note also that ψ can contain several copies of the same segment. We often just say "multi-segment", where n is understood.*

Definition 6.3. *A multi-segment ψ is called **cyclic** if there is a positive integer k, such that, for all i mod n, there is a copy of $(k, i]$ in ψ. Otherwise, ψ is called **acyclic**.*

Example 6.4. For $n = 3$, a possible multi-segment would be $\{(1; 2], (1; 2], (2; 3], (0; 5], (1; 7]\}$. This example is acyclic because, for any fixed k, it is missing at least one of the possible segments $(k; i]$. An example of a cyclic multi-segment would be $\{(2; 0], (2; 1], (2; 2]\}$.

We now put a crystal structure on the set of multi-segments. We follow [8], although we have reworded the rule using string of brackets instead of the function $S_{k,i}$ used there. This is done in order to more closely match the conventions of [9].

Let ψ be a multi-segment. For each residue $i \in \mathbb{Z}/n\mathbb{Z}$, construct a string of brackets S_i^{MS} as follows: For each $k \in \mathbb{Z}_{>0}$, let $S_{i,k}^{MS}$ be the string of brackets $((\cdots(())\cdots))$, where the number of '(' is the number of copies of $(k; i-1]$ in ψ, and the number of ')' is the number of copies of $(k; i]$ in ψ. Let $S_i^{MS} = \cdots S_{i,3}^{MS} S_{i,2}^{MS} S_{i,1}^{MS}$. Then $f_i^{MS}(\psi)$ is obtained form ψ by:

$$\begin{cases} \text{changing one } (k; i-1] \text{ to } (k+1, i] & \text{if the first uncanceled `(' from the left corresponds to a } (k; i-1]. \\ \text{changing } \psi \text{ to } \psi \sqcup \{(1; i]\} & \text{if there is no uncanceled `('} \end{cases}$$

$e_i^{MS}(\psi)$ is obtained from ψ by

$$\begin{cases} \text{changing one } (k; i] \text{ to } (k-1, i-1] & \text{if the first uncanceled `)' from the right corresponds to a } (k; i]. \\ e_i(\psi) = 0 & \text{if there is no uncanceled `)'} \end{cases}$$

Definition 6.5. *Let B^{MS} be the edge colored directed graph with a vertex for every multi-segment ψ and a c_i-colored edge from ψ to ψ' if and only if $f_i^{MS}(\psi) = \psi'$.*

Theorem 6.6. *(see* [8, Section 4.4]*) Each connected component of B^{MS} is a copy of the direct limit crystal B_∞. The set of aperiodic multi-segments forms the vertices of a single copy of B_∞.*

Example 6.7. Consider $\widehat{\mathfrak{sl}}_3$ and the multi-segment

$$\begin{aligned} \psi &= \{[0; 8), [1; 6), [0; 5), [2; 5), [0; 4), [1; 4), [1; 4), [1; 3), [1; 3), [2; 3), [2; 3), [0; 1), [2; 1), [2; 1)\} \\ &= \{(8; 1], (6; 0], (5; 1], (5; 0], (4; 0], (4; 1], (4; 1], (3; 0], (3; 0], (3; 1], (3; 1], (1; 0], (1; 2], (1; 2]\}. \end{aligned}$$

Let us calculate $f_1^{MS}(\psi)$. We need to consider only those segments $(\cdot; 0]$ and $(\cdot; 1]$. These are shown below, ordered from largest to smallest, with the appropriate bracket drawn above:

$$\begin{array}{cccccccccccc}) & (& (&) & (&) &) & (& (&) &) & (\\ (8;1] & (6;0] & (5;0] & (5;1] & (4;0] & (4;1] & (4;1] & (3;0] & (3;0] & (3;1] & (3;1] & (1;0] \end{array}$$

Then

$f_1^{MS}(\psi) = \{(8;1],(6;0],(5;1],(5;0],(4;0],(4;1],(4;1],(3;0],(3;0],(3;1],(3;1],(2;1],(1;2],(1;2]\}.$

$(f_1^{MS})^2(\psi) = \{(8;1],(6;0],(5;1],(5;0],(4;0],(4;1],(4;1],(3;0],(3;0],(3;1],(3;1],(2;1],(1;2],(1;2],(1;1]\}.$

$e_1^{MS}(\psi) = \{(7;0],(6;0],(5;1],(5;0],(4;0],(4;1],(4;1],(3;0],(3;0],(3;1],(3;1],(1;0],(1;2],(1;2]\}.$

$(e_1^{MS})^2(\psi) = 0.$

Comment 6.8. *ψ in the above example is acyclic. If one takes the union of ψ with, for example $\{(2;0],(2;1],(2;2]\}$, then this would insert a pair of canceling brackets into each string S_i^{MS}. One can see that this operation commutes with all e_i^{MS} and f_i^{MS}. From this one can show that the subset of acyclic multi-segments is closed under the crystal operations.*

7. The relationship between the cylindric plane partitions and the multi-segments

As discussed in Section 2, for any highest weight Λ there is an e-equivariant injection $\iota_\Lambda : B_\Lambda \to B_\infty$. Left acyclic $\mathcal{C}$ cylindric partitions give a realization of $B_{\Lambda(\mathcal{C})}$ and acyclic multisegments give a realization of B_∞. In this section we explicitly describe ι_Λ using these realizations. In fact, we construct an e-equivariant injection $\iota_\mathcal{C} : B^\mathcal{C} \to B^{MS}$, which agrees with ι_Λ when restricted to the left-acyclic $\mathcal{C}$ cylindric partitions. We then study the inverse to $\iota_\mathcal{C}$. That is, for any given multi-segment ψ, we determine for which $\mathcal{C}$ there is a $\mathcal{C}$-cylindric partition π_ψ such that $\iota_\mathcal{C}(\pi_\psi) = \psi$. In the case that it exists, we describe π_ψ.

The map ι_Λ was described in [1, Theorem 5.11] as a map from highest lift multi-partitions to multi-segments. Our results on the inverse of $\iota_\mathcal{C}$ correspond to results on multi-partitions studied in [4]. So the construction is not new, but it does seem to be cleaner in the language of cylindric partitions.

Another advantage of our approach is that we are considering all cylindric partitions and all multi-segments. For this reason our construction can be interpreted as saying that the crystal B^{MS} consisting of all multisegments is a direct limit of the crystals of irreducible $\widehat{\mathfrak{gl}}_n$ representations.

Lemma 7.1. *Fix $\pi \in B^\mathcal{C}$. Let $b_1, b_2 \in A_i^{CP}(\pi) \sqcup R_i^{CP}(\pi)$, and assume b_1 and b_2 have coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Then $t(b_1) < t(b_2)$ implies $z_1 \geq z_2$.*

Comment 7.2. *Note that the coordinates (x, y, z) are not uniquely defined due to the periodicity. The z coordinate however is well defined, so the statement of Lemma 7.1 is precise.*

Proof of Lemma 7.1. If one adds all boxes in $A_i^{CP}(\pi)$ to π, the result is a new $\mathcal{C}$-cylindric partition π' with $A_i(\pi') = \emptyset$ and $R_i(\pi') = A_i(\pi) \cup R_i(\pi)$. Thus without loss of generality, we may assume that $A_i^{CP}(\pi) = \emptyset$. Choose $b_1, b_2 \in R_i^{CP}(\pi)$. Use the periodicity to choose coordinates (x_1, y_1, z_1) for b_1 and (x_2, y_2, z_2) for b_2 with $0 \leq x_2 - x_1 < \ell$ (see Figure 3 for the coordinate axes). Assume for a contradiction that $t(b_1) < t(b_2)$ and $z_1 < z_2$.

The boxes b_1, b_2 are both colored c_i, so

$$z_1 - y_1 \cong z_2 - y_2 \text{ modulo } n. \tag{3}$$

By definition, $t(b_1) < t(b_2)$ is equivalent to

$$\frac{nx_1}{\ell} + y_1 - z_1 < \frac{nx_2}{\ell} + y_2 - z_2. \tag{4}$$

We chose coordinates so that $|x_1 - x_2| < \ell$. Hence (3) and (4) imply $y_1 - z_1 = y_2 - z_2$. Since we assumed $z_1 < z_2$, this implies that $y_1 < y_2$.

But now $y_1 < y_2$ and $x_1 < x_2$. Since π is a cylindric partition the coordinate z of a box at the top edge of π is weakly decreasing in both the x and y direction. This implies that $z_1 > z_2$, contradicting our assumption. $\square$

Lemma 7.3. *If $b_1 \in A_i^{CP}(\pi)$ and $b_2 \in R_i^{CP}(\pi)$ have the same height z, then $t(b) > t(b')$.*

Proof. Assume $b_1 \in A_i^{CP}(\pi)$ and $b_2 \in R_i^{CP}(\pi)$ have the same height z. Since b_1 and b_2 are the same height and the same color, the must both be in $\lambda'^{(k)}$ for the same residue k mod n, and thus they can be give coordinates $b_1 = (x_1, y, z)$, $b_2 = (x_2, y, z)$. Then $\pi_{x_1,y} = z - 1$ and $\pi_{x_2,y} = z$. Since $\lambda'^{(k)}$ is a partition, this implies $x_1 > x_2$, which in turn implies $t(b_1) > t(b_2)$. $\square$

Lemma 7.4. *Fix $z_0 \in \mathbb{Z}_{>0}$. There is at most one $b \in A_i(\pi)$ of height z_0 such that $\pi \cup b$ is still $\mathcal{C}$-cylindric.*

Proof. Every $b \in A_i(\pi)$ of height z_0 lies in $\lambda'^{(k)}$ for the same residue k mod n. The lemma follows because there is at most one box that can be added to $\lambda'^{(k)}$ at height z_0 so that the result is still a partition. $\square$

The above Lemmas allow a new description of the crystal structure on cylindric partitions. This description is slightly easier to remember in that it does require introducing the function t.

Corollary 7.5. *The crystal operators e_i^{CP} and f_i^{CP} on $B^{\mathcal{C}}$ can be calculated as follows:*

- *$A_i^{CP}(\pi)$ and $R_i^{CP}(\pi)$ are as in Definition 4.5.*
- *Construct a string of brackets $S_i^{CP'}$ by placing a '(' for every $b \in A_i^{CP}(\pi)$ and a ')' for every $b \in R_i^{CP}(\pi)$. These are ordered so that, if $z_1 > z_2$, then all brackets corresponding to boxes of height z_1 are to the left of all brackets corresponding to boxes of height z_2, and all ')' corresponding to boxes of height z_i come to the left of all '(' corresponding to boxes of height z_i.*
- *To calculate $f_i(\pi)$, find the first uncanceled '(' from the left. If this corresponds to a box of height z, then by Lemma 7.4 there will be a unique $b \in A_i^{CP}(\pi)$ of height z such that $\pi \cup b$ is still $\mathcal{C}$-cylindric. Then $f_i(\pi) = \pi \cup b$. If there is no uncanceled '(', then $f_i(\pi) = 0$.*
- *To calculate $e_i(\pi)$, find the first uncanceled ')' from the right. If this corresponds to a box of height k, then there will be a unique $b \in R_i^{CP}(\pi)$ such that $\pi \backslash b$ is still $\mathcal{C}$-cylindric. Then $e_i(\pi) = \pi \backslash b$. If there is no uncanceled ')', then $e_i(\pi) = 0$.*

Proof. Let $e^{CP'}$ and $f^{CP'}$ be the operators calculated as described above. By Lemmas 7.1 and 7.3, $S_i^{CP'}$ is equal to S_i^{CP}, and corresponding brackets in $S_i^{CP'}$ and S_i^{CP} correspond to boxes of the same height. Thus both $f_i^{CP'}$ and f_i^{CP} add a box b at the same height, or else both send π to 0. By Lemma 7.4 this implies $f_i^{CP'} = f_i^{CP}$. The argument for e_i is completely analogous. $\square$

Theorem 7.6. *Fix a cylinder $\mathcal{C}$ whose diagonals have been labeled π_i and c_j for $i, j \in \mathbb{Z}$, as in Figure 3. Recall that π_{ij} is the number in the box at the intersection of π_i and c_j. There is an e-equivariant injection $\iota_{\mathcal{C}} : B^{\mathcal{C}} \hookrightarrow B^{MS}$ given by*

$$
\begin{aligned}
\iota_{\mathcal{C}} B^{\mathcal{C}} &\longrightarrow B^{MS} \\
\pi \qquad &\longrightarrow \iota_{\mathcal{C}}(\pi) := \{[-j; \pi_{i,j}) : \pi_{i,j} \neq 0\}.
\end{aligned} \tag{5}
$$

Comment 7.7. *Note that finding $\iota_{\mathcal{C}}(\pi)$ takes essentially no calculation. However, the segments in $\iota_{\mathcal{C}}(\pi)$ are written in the form $[i; z)$ for residues i modulo n. To calculate e_i^{MS} and f_i^{MS}, one would need to rewrite these in the form $(z, i']$.*

Proof of Theorem 7.6. Let $S_i^{CP'}$ be the string of brackets introduced in Corollary 7.5, and S_i^{MS} the string of brackets from Section 6. By Corollary 7.5, it suffices to show that

(i) $S_i^{CP'}(\pi)$ has an uncanceled ')' if an only if $S^{MS}(\iota_{\mathcal{C}}(\pi))$ has an uncanceled ')'.

(ii) Assume $S_i^{CP'}(\pi)$ has an uncanceled ')", and let z be the height of the box $b \in R_i^{CP}(\pi)$ corresponding to the first uncanceled ')' from the right in $S_i^{CP'}$. Then the first uncanceled ')' in $S^{MS}(\iota_{\mathcal{C}}(\pi))$ corresponds to a segment $(z; i]$.

In order to show this, introduce another string of brackets $S_i^{\widetilde{CP}}(\pi)$ as follows. There is a '(' for every pair of coordinates (x, y) such that the top box in the stack of boxes $\pi_{x,y}$ is colored c_{i-1}, and a ')' for every pair of coordinates (x, y) such that the top box in the stack of boxes $\pi_{x,y}$ is colored c_i. These are ordered so that brackets corresponding to higher stacks always come before brackets corresponding to lower stacks, and '(' corresponding to stack of a given height z come to the left of all ')' corresponding to parts of height z.

It should be clear that $S^{\widetilde{CP}}$ and $S^{CP'}$ are the same except for the insertion of canceling pairs of brackets '()' Thus the uncanceled brackets of $S_i^{\widetilde{CP}}(\pi)$ and $S_i^{CP'}(\pi)$ correspond to boxes of exactly the same heights.

It is also clear from the definitions that $S_i^{\widetilde{CP}}(\pi) = S_i^{MS}(\iota_\mathcal{C}(\pi))$, and that any bracket in $S_i^{\widetilde{CP}}(\pi)$ coming from a box of height z corresponds to a bracket in $S_i^{MS}(\iota_\mathcal{C}(\pi))$ coming from a segment $(z, i]$. The result follows. □

Corollary 7.8. *Fix $\mathcal{C}$ with $n_\mathcal{C} = n$ and $\ell_\mathcal{C} = \ell$. Let ψ be any multi-segment. If there exists a $\mathcal{C}$ cylindric partition π_ψ with $\psi = \iota_\mathcal{C}(\pi_\psi)$, then π is given as follows:*

- *write the multi-segment ψ in the form*

$$\psi = \{[0; z_{0,1}), [0; z_{0,2}), \dots, [0; z_{0,k_0}), [1; z_{1,1}), \dots, [1, z_{1,k_1}), \dots, [n-1, z_{n-1,1}), \dots, [n-1, z_{n-1,k_{n-1}})\}.$$

- *π is the $\mathcal{C}$ cylindric partition such that, for each $j \in \mathbb{Z}$ and $s > 0$,*

$$\pi_{v'_j(\mathcal{C})+s,j} = \begin{cases} z_{j,s} & \text{if } s \le k_j \\ 0 & \text{otherwise,} \end{cases}$$

where $v'_j(\mathcal{C})$ is as in Definition 4.7, but extended to be well defined for all $j \in \mathbb{Z}$ using the rule $v'_{j+n} = v'_j + \ell$.

Proof of Corollary 7.8. This is immediate from Theorem 7.6 and the definition of $\iota_\mathcal{C}$. □

Comment 7.9. *Corollary 7.8 also gives an effective way to check if ψ is in im $\iota_\mathcal{C}$. One simply writes down the corresponding π_ψ according to the above rule, and $\psi \in$ im $\iota_\mathcal{C}$ exactly if the result is $\mathcal{C}$ cylindric.*

Comment 7.10. *Fix cylinders $\mathcal{C}_1$ and $\mathcal{C}_2$ with $\mathbf{v}'(\mathcal{C}_1) = ({v'}^1_0, {v'}^1_1, \dots, {v'}^1_{n-1})$ and $\mathbf{v}'(\mathcal{C}_2) = ({v'}^2_0, {v'}^2_1, \dots {v'}^2_{n-1})$ (see Definition 4.7). Assume that, for all $0 \le j < n$, ${v'}^2_j \ge {v'}^1_j$. Then Corollary 7.8 can be modified to give an injection $\iota^{\mathcal{C}_2}_{\mathcal{C}_1} : B^{\mathcal{C}_1} \hookrightarrow B^{\mathcal{C}_2}$ by, for all $\pi \in B^{\mathcal{C}_1}$, $0 \le k < n$ and $s > 0$,*

$$\iota(\pi)_{{v'}^2_k+s,k} = \pi_{{v'}^1_k+s,k}. \tag{6}$$

That is, $\iota(\pi)$ is the unique $\mathcal{C}_2$ cylindric partition that has the same dual multi-partition $({\lambda'}^{(1)}, \dots, {\lambda'}^{(n-1)})$ as π. It follows from Theorem 7.6 that this map is e-equivariant, since $\iota_{\mathcal{C}_2} \circ \iota^{\mathcal{C}_2}_{\mathcal{C}_1}(\pi) = \iota_{\mathcal{C}_1}(\pi)$.

In this way, $\{B^\mathcal{C}\}$ becomes a directed system. It is straightforward to see that the limit of this system is naturally identified with B^{MS}. As discussed in [9], $B^\mathcal{C}$ can be thought of as the crystal of an irreducible $\widehat{\mathfrak{gl}}_n$ (as opposed to $\widehat{\mathfrak{sl}}_n$) representation. Thus is some sense one can think of B^{MS} as the infinity crystal for $\widehat{\mathfrak{gl}}_n$.

References

[1] Susumu Ariki, Nicolas Jacon, and Céderic Lecouvey. The modular branching rule for affine Hecke algebras of type A. *Preprint.* arXiv:0808.3915v2.

[2] Omar Foda, Bernard Leclerc, Masato Okado, Jean-Yves Thibon, and Trevor A. Welsh. Branching functions of $A^{(1)}_{n-1}$ and Jantzen-Seitz problem for Ariki-Koike algebras. *Adv. Math.*, 141(2):322–365, 1999.

[3] Jin Hong and Seok-Jin Kang. *Introduction to quantum groups and crystal bases*, volume 42 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[4] Nicolas Jacon and Céderic Lecouvey. Kashiwara and Zelevinski involutions in affine type A. *Preprint.* arXiv:0901.0443v1.

[5] Michio Jimbo, Kailash C. Misra, Tetsuji Miwa, and Masato Okado. Combinatorics of representations of $\mathrm{U}_q(\widehat{\mathfrak{sl}}(n))$ at $q = 0$. *Comm. Math. Phys.*, 136(3):543–566, 1991.

[6] Joel Kamnitzer and Peter Tingley. A definition of the crystal commutor using Kashiwara's involution. *Journal of Algebraic Combinatorics*, 29 Issue 2:261, 2009.

[7] Seok-Jin Kang, Masaki Kashiwara, Kailash C. Misra, Tetsuji Miwa, Toshiki Nakashima, and Atsushi Nakayashiki. Affine crystals and vertex models. In *Infinite analysis, Part A, B (Kyoto, 1991)*, volume 16 of *Adv. Ser. Math. Phys.*, pages 449–484. World Sci. Publ., River Edge, NJ, 1992.

[8] Bernard Leclerc, Jean-Yves Thibon, and Eric Vasserot. Zelevinsky's involution at roots of unity. *J. Reine Angew. Math.*, 513:33–51, 1999.

[9] Peter Tingley. Three combinatorial models for $\widehat{\mathfrak{sl}}_n$ crystals, with applications to cylindric plane partitions. *International Mathematics Research Notices.*, 2007:Article ID rnm143, 40 pages, 2007. Errata: arXiv:math/0702062.

E-mail address: `P.Tingley@ms.unimelb.edu.au`

Department of Math and Stats, University of Melbourne, Melbourne AUSTRALIA