LECTURE 1: MOTIVATION

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1. Towards quantum groups

Let us begin by discussing what quantum groups are, and why we might want to study them. We will start with the related classical objects. So, let G be a complex simple Lie group and let $\mathfrak{g}$ be its Lie algebra (tangent space at the identity $e \in G$), with Lie bracket $[,]$.

A very fruitful branch of mathematics has been concerned with studying the representation theory of G. This is often accomplished by studying the representation theory of $\mathfrak{g}$. Recall that a representation of $\mathfrak{g}$ is a map $\phi\colon \mathfrak{g} \to \mathrm{End}(V)$ such that $\phi([x,y]) = \phi(x)\phi(y) - \phi(y)\phi(x)$, which is actually the same thing as a representation of an associative algebra $U(\mathfrak{g})$, called the universal enveloping algebra. Formally, $U(\mathfrak{g})$ is simply the associative algebra generated by $\mathfrak{g}$ subject to the relations $[x,y] = xy - yx$.

People sometimes say "The quantum group $U_\hbar(\mathfrak{g})$ is a deformation of $U(\mathfrak{g})$." What might this mean? One should have $U_\hbar(\mathfrak{g}) \cong U(\mathfrak{g})[[\hbar]]$ as a vector space, and the multiplication in $U_\hbar(\mathfrak{g})$ should be the same as the multiplication in $U(\mathfrak{g})$ modulo $\hbar$. This is a bit naive though, since:

Theorem 1.1. *Every deformation of $U(\mathfrak{g})$ is trivial, i.e., always isomorphic to $U(\mathfrak{g})[[\hbar]]$ as an algebra.*

Proof. This follows from the fact that the Hochschild cohomology group $\mathrm{H}^2(U(\mathfrak{g}), U(\mathfrak{g}))$ is 0. □

A better idea is to deform $U(\mathfrak{g})$ as a Hopf algebra.

1.1. Hopf algebras.

What is the Hopf algebra structure on $U(\mathfrak{g})$? It is the structure we need to define tensor product $\otimes$ and duality * of representations.

Let's first look at tensor product. We know that the action of G on $V \otimes W$ is given by $g(v \otimes w) = gv \otimes gw$. This induces an action of $\mathfrak{g}$ on $V \otimes W$, given by $X(v \otimes w) = Xv \otimes w + v \otimes Xw$ for $x \in \mathfrak{g}$. We think of this as a map

$$\begin{aligned} \Delta\colon U(\mathfrak{g}) &\to U(\mathfrak{g}) \otimes U(\mathfrak{g}) \\ X &\mapsto X \otimes 1 + 1 \otimes X \quad (X \in \mathfrak{g}) \end{aligned}$$

and extend multiplicatively. Call this **comultiplication**.

We must also consider the trivial representation V_0 where $X \in \mathfrak{g}$ acts by 0 and scalars act as usual. This gives a map

$$\varepsilon\colon U(\mathfrak{g}) \to V_0$$

that picks out the constant term. This is the **counit**.

Finally, we consider the fact that $\mathfrak{g}$ has dual representations: $X \in \mathfrak{g}$ acts on V^* by $(Xf)(v) = f(-Xv)$, so this gives a map

$$\begin{aligned} S\colon U(\mathfrak{g}) &\to U(\mathfrak{g}) \\ X &\mapsto -X, \end{aligned}$$

extended to all of $U(\mathfrak{g})$ as an algebra antiautomorphism. This gives the **antipode**.

Recall that $(V \otimes W)^* = W^* \otimes V^*$. One can show that this forces S to be a coalgebra-anti-automorphism.

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Definition 1.2. A **Hopf algebra** over a field F consists of the data $(H, m, \iota, \Delta, \varepsilon, S)$, where H is a vector space,

$$\begin{aligned} m &: H \otimes H \to H \\ \iota &: F \to H \\ \Delta &: H \to H \otimes H \\ \varepsilon &: H \to F \\ S &: H \to H \end{aligned}$$

such that

(1) (H, m, i) is an algebra,
(2) (H, Δ, ϵ) is a coalgebra,
(3) Δ and ϵ are maps of algebras (using the tensor product algebra structure on $H \otimes H$),
(4) m, ι are maps of coalgebras,
(5) S is an algebra-antiautomorphism and a coalgebra-antiautomorphism, and
(6) the diagram

$$\begin{tikzcd} H \otimes H \arrow[r, "S\otimes 1", "\text{or } 1\otimes S"'] & H \otimes H \arrow[d, "m"] \\ H \arrow[u, "\Delta"] \arrow[r, "\iota\circ\varepsilon"] & U(\mathfrak{g}) \end{tikzcd}$$

commutes.

Remark 1.3. The commutative diagram in the definition of a Hopf algebra can be explained in various intrinsic ways. For instance, David Jordan pointed out the following: Consider the convolution product $\star$ on $\mathrm{End}(H)$, where the product $\star$ is defined by $\phi \star \psi(X) := m \circ (\phi \otimes \psi) \circ \Delta(X)$. Then $\iota \circ \epsilon$ is the identity element in this convolution algebra. The above diagrams imply that the antipode S is a two sided inverse for the identity map $1 \in \mathrm{End}(H)$ (note: this is the identity map of vector spaces, not the identity in the convolution algebra). In particular, this implies that any bialgebra can have at most one antipode. Hence the antipode should not really be considered extra data in a Hopf algebra, but rather existence of an antipode is a condition on a bialgebra.

1.2. **Uniqueness of deformations.** The following uniqueness statement is a strong motivation for studying quantum groups: if something you are interested in can be deformed in a unique way, you will almost certainly learn something interesting by studying that deformation. Here rigid means there is a good notion of (both left and right) duals, and monoidal means there is a tensor product.

Theorem 1.4 (see [2, Example 2.24])**.** *There is a unique non-trivial deformation of the category of finite-dimensional modules of $U(\mathfrak{g})$ as a rigid monoidal category.*

Proof. One shows that possible deformations of a tensor category $\mathcal{C}$ are parameterized by $\mathrm{H}^3(\mathcal{C})$, and obstructions by $\mathrm{H}^4(\mathcal{C})$, where $\mathrm{H}^\bullet$ is Davydov–Yetter cohomology. Here a tensor category is a rigid monoidal category, with the extra assumptions that $\mathbf{1}$ is simple and every object has finite length, which both hold of $U(\mathfrak{g})$-rep. In the case of $U(\mathfrak{g})$-rep, Davydov–Yetter cohomology agrees with Lie algebra cohomology, so $\mathrm{H}^3(U(\mathfrak{g})\text{-rep}) = \mathbb{C}$ and $\mathrm{H}^4(U(\mathfrak{g})\text{-rep}) = 0$. See [2, 4]. ☐

In this seminar, we will more often discuss deformations of algebras than deformations of categories. Here the situation is somewhat less clear, although we do have the following result (the original deformation is due to Drinfel'd and Jimbo, and the precise statement here can be found in [4].

Theorem 1.5. *There exists a non-trivial deformation of $U(\mathfrak{g})$ (as a Hopf algebra) whose category of representations realizes the unique deformation of $\mathrm{Rep}\, U(\mathfrak{g})$.*

Unfortunately, this deformation of $U(\mathfrak{g})$ is not unique, in the sense that there are non-isomorphic Hopf algebras $U_\hbar(\mathfrak{g})$ over $\mathbb{C}[[\hbar]]$ which are deformations of $U(\mathfrak{g})$. However, there are some uniqueness statements one can make:

Theorem 1.6 (see [4, Theorem 5])**.** *The deformation of $U(\mathfrak{g})$ is unique up equivalences including isomorphism, twistings (as quasi-Hopf algebras that preserves the fact that algebras in question are Hopf algebras – this one is quite nontrivial), and change of parameter.*

The interesting type of equivalence is twisting. The idea of this operation is simple: For any invertible element $J \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$, one can try to make a new coproduct $\Delta^J := J\Delta J^{-1}$. The result is no longer a Hopf algebra, but if one is willing to work with quasi-Hopf algebras (i.e., Hopf algebras which are not strictly associative, but rather have a non-trivial associator map), one can modify the associator such that the result is a new quasi-Hopf algebra. In some cases the result is in fact still a strict Hopf algebra. We will not discuss the notion of twisting in detail; for our purposes, it is enough to know that there is a uniqueness statement, but that it involves a quite non-trivial notion of equivalence.

There is also another way to get a uniqueness claim, which was in fact the first uniqueness result of Drinfel'd. One introduces some extra structure, namely the Cartan involution $E_i \leftrightarrow F_i$ of $U(\mathfrak{g})$. The deformation is unique if it also deforms this new structure:

Theorem 1.7 ([3], see also [4, Theorem 3])**.** *$U_\hbar(\mathfrak{g})$ is the unique (up to change of deformation parameter $\hbar$) Hopf-algebra deformation of $U(\mathfrak{g})$ subject to the additional conditions*

(1) *$U_h(\mathfrak{g})$ contains a commutative sub-Hopf algebra C such that $C/\hbar C \simeq U(\mathfrak{h})$, where $U(\mathfrak{h}) = \langle H_i \rangle_{i \in I}$ is the universal enveloping algebra of the Cartan subalgebra of $\mathfrak{g}$.*
(2) *C is invariant under an algebra involution θ which is also a coalgebra anti-automorphism, and such that θ induces the Cartan involution $E_i \leftrightarrow F_i$ on $U(\mathfrak{g}) = U_\hbar(\mathfrak{g})/\hbar U_\hbar(\mathfrak{g})$.*

1.3. **How deformations can look different.** Having made some uniqueness statements, let us consider how deformations of $U(\mathfrak{g})$ can look different. The idea is that of deformation quantization: going back to the original group G, a deformation of G should be a deformation of the algebra of functions on G. The deformed product $*$ will define a Poisson bracket by $\{f, g\} := (f * g - g * f)/\hbar$ (mod $\hbar$). If two deformations lead to non-equivalent Poisson brackets, one would think they are non-equivalent.

Transferring the notion of a Poisson bracket to $U(\mathfrak{g})$, one ends up with the notion of a co-Poisson Hopf algebra. This is a Hopf algebra along with an additional map $\delta\colon U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, satisfying some compatibility (see [1]). Any deformation $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$ gives rise to a co-Poisson Hopf structure by

$$\delta(X) := \frac{\Delta(X) - \Delta^{op}(X)}{\hbar}.$$

It turns out that one can find deformations that give rise to non-isomorphic co-Poisson Hopf structures on $U(\mathfrak{g})$. So, from this deformation quantization point of view, one would conclude that the deformation is non-unique. However, all such deformations will have equivalent categories of representations.

1.4. **How is the deformed category of $U(\mathfrak{g})$-rep different?** As a category, it is not different since algebra deformations of $U(\mathfrak{g})$ are trivial. But as a $\otimes$-category (really, as a rigid monoidal category), it has changed.

To see this, look at $V \otimes W$ and $W \otimes V$. These are always isomorphic. Over $U(\mathfrak{g})$, the isomorphism is just the flip map. So, we get an action of the symmetric group S_n on tensor product $V_1 \otimes \cdots \otimes V_n$ of representations of $(\mathfrak{g})$ by switching orders. Over $U_\hbar(\mathfrak{g})$, there is an isomorphism $V \otimes W \to \sigma^{br}(W \otimes V)$, but it is not flip on the underlying vector spaces. However, one can choose a natural family of such isomorphism such that the action of the braid group Br_n on tensor products $V_1 \otimes \cdots \otimes V_n$ of representations of $U)q(\mathfrak{g})$. In fact, one just needs to check the braid relation. See Figure 1.

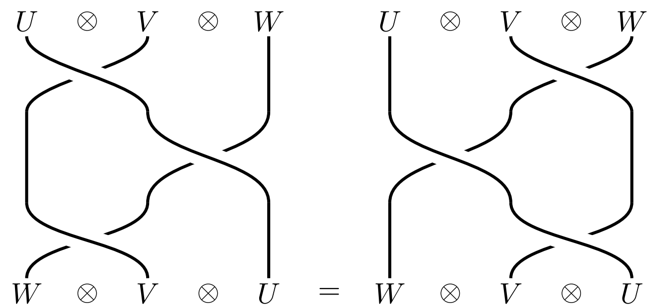

FIGURE 1. The braid relation. The crossing should be interpreted as an isomorphism from a tensor product of two representations to the tensor product in the other order.

The appearance of the braid group suggests connections to knot theory. In fact, this leads to the celebrated quantum group knot invariants, although there is still some work to do.

2. Towards crystals

2.1. **Drinfeld–Jimbo quantum groups.** Let $A = (a_{i,j})$ be the Cartan matrix for $\mathfrak{g}$. For $U(\mathfrak{g})$, have Chevalley generators E_i, F_i, and H_i with relations

$$\begin{aligned}
[H_i, E_j] &= a_{i,j} E_j \\
[H_i, F_j] &= -a_{i,j} F_j \\
[E_i, F_j] &= \delta_{i,j} H_i \\
\mathrm{ad}(E_i)^{1-a_{i,j}} E_j &= 0 \\
\mathrm{ad}(F_i)^{1-a_{i,j}} F_j &= 0.
\end{aligned}$$

The comultiplication Δ is given by

$$\begin{aligned}
H_i &\mapsto H_i \otimes 1 + 1 \otimes H_i \\
E_i &\mapsto E_i \otimes 1 + 1 \otimes E_i \\
F_i &\mapsto F_i \otimes 1 + 1 \otimes F_i.
\end{aligned}$$

Let us not consider the antipode for now, as it is in fact determined by the other data.

Let D be the diagonal matrix such that DA is symmetric, and $d_i = D_{i,i}$. Now $U_\hbar(\mathfrak{g})$ has the same generators with deformed relations:

$$\begin{aligned}
&[H_i, E_j] = a_{i,j} E_j \\
&[H_i, F_j] = -a_{i,j} F_j \\
&[E_i, F_j] = \delta_{i,j} \frac{\exp(d_i \hbar H_i) - \exp(-d_i \hbar H_i)}{\exp(d_i \hbar) - \exp(-d_i \hbar)} \\
&\sum_{k=0}^{1-a_{i,j}} \begin{bmatrix} 1-a_{i,j} \\ k \end{bmatrix}_{\exp(d_i \hbar)} E_i^k E_j E_i^{1-a_{i,j}-k} = 0 \\
&\sum_{k=0}^{1-a_{i,j}} \begin{bmatrix} 1-a_{i,j} \\ k \end{bmatrix}_{\exp(d_i \hbar)} F_i^k F_j F_i^{1-a_{i,j}-k} = 0.
\end{aligned}$$

Here $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient, which is just a polynomial in q.

The comultiplication Δ is given by

$$\begin{aligned}
H_i &\mapsto H_i \otimes 1 + 1 \otimes H_i \\
E_i &\mapsto E_i \otimes \exp(d_i \hbar H_i) + 1 \otimes E_i \\
F_i &\mapsto F_i \otimes 1 + \exp(-d_i \hbar H_i) \otimes F_i.
\end{aligned}$$

Remark 2.1. This looks non-trivial as a deformation of an algebra structure, but as discussed above it must be isomorphic to the trivial deformation. But, in most cases, no explicit isomorphism is known! Stated another way, no one knows how to write down the coproduct structure if we don't change the multiplication rules (although this can be done for $\mathfrak{sl}_2$). □

We want to be able to specialize $\hbar$ to numbers, but we can't because we have to worry about convergence. So we renormalize. Let $q = \exp(\hbar)$. For now, assume $d_i = 1$ for simplicity (simply-laced case). Also let $K_i = \exp(\hbar H_i)$, which we can think of as "q^{H_i}". We get a new algebra with generators $E_i, F_i, K_i^{\pm 1}$, and relations

$$\begin{aligned}
&K_i K_i^{-1} = 1 \\
&K_i K_j = K_j K_i \\
&[H_i, E_j] = a_{i,j} E_j \\
&[H_i, F_j] = -a_{i,j} F_j \\
&[E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}} \\
&\sum_{k=0}^{1-a_{i,j}} \begin{bmatrix} 1 - a_{i,j} \\ k \end{bmatrix}_q E_i^k E_j E_i^{1-a_{i,j}-k} = 0 \\
&\sum_{k=0}^{1-a_{i,j}} \begin{bmatrix} 1 - a_{i,j} \\ k \end{bmatrix}_q F_i^k F_j F_i^{1-a_{i,j}-k} = 0.
\end{aligned}$$

and the comultiplication becomes

$$\begin{aligned}
H_i &\mapsto H_i \otimes 1 + 1 \otimes H_i \\
E_i &\mapsto E_i \otimes K_i + 1 \otimes E_i \\
F_i &\mapsto F_i \otimes 1 + K_i^{-1} \otimes F_i.
\end{aligned}$$

Now everything is defined over $\mathbf{Z}[q, q^{-1}, (q - q^{-1})^{-1}]$. However, essentially because we were forced to introduce the generators K_i, this is no longer a deformation of $U(\mathfrak{g})$. However, it has many of the properties of a deformation.

The goal of the theory of crystals is to "draw" a highest weight representation of $U_q(\mathfrak{g})$ (in the limit at $q \to \infty$) as a colored directed graph.

Example 2.2. $\mathfrak{g} = \mathfrak{sl}_3$, $\lambda = \omega_1 + \omega_2$, so $V(\lambda)$ is the adjoint representation. Weight space decomposition:

$$
\begin{tikzcd}
 & \circ \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & \\
\circ \arrow[dr, "F_2"] & & \circ \arrow[dl, "F_1"'] \\
 & \bullet \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & \\
\circ \arrow[dr, "F_2"] & & \circ \arrow[dl, "F_1"'] \\
 & \circ &
\end{tikzcd}
$$

The vertices of the graph should correspond to a basis of $V(\lambda)$, so for instance the $\bullet$ should somehow be separated into two vertices (since this weight space is two dimensional) There are a few problems

Problem 1: In $\bullet$ the images of F_1 and F_2 and the kernels of F_1 and F_2 are distinct. So there are four distinguished 1-dimensional subspaces in $\bullet$, and it seems impossible to separate it into two vertices. Passing to the $q \to \infty$ limit will fix this, but

Problem 2: We currently cannot plug in $q = \infty$, as we are working over $\mathbb{C}[q, q^{-1}, (q - q^{-1})^{-1}]$.

Problem 3: We would really like the reverse arrows to correspond to the operators E_i. However, one can have $E_i F_i = [n] := (q^n - q^{-1})/(q - q^{-1})$, which is not 1.

All of these problems will be fixed by introducing the Kashiwara operators $\widetilde{E}_i$ and $\widetilde{F}_i$, and the notion of a crystal lattice. This will be done next week.

References

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[3] V. G. Drinfel'd, Quasi-Hopf algebras, *Algebra Analiz* **1** (1989). Translation in *Leningrad Math. J.* **1** (1990), no. 6, 1419–1457.

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