QUANTUM GROUPS AND KNOT INVARIANTS

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1. Knots

We discuss the Jones-Conway polynomial, also known as Homfly polynomial. It is a knot invariant, and we prove its existence and uniqueness given some simple axioms (value on the unknot and the so-called *skein relations*). The proof is following [2].

Definition 1.1. Informally, a **knot** is obtained by gluing together the endpoints of a shoelace in $\mathbb{R}^3$, considered up to isotopy (but remembering orientation). A **link** is the same as a knot, but multiple shoelaces are allowed. A **tangle** consists of (oriented) shoelaces and two parallel sticks, an upper and a lower, which we may glue some endpoints to. In addition, we assign $+$ and $-$ signs to the points where the end of a shoelace is attached to the stick, dependent on the orientation of the shoelaces, in the following way: on the upper stick, assign $+$ to all the shoelaces leaving it, and $-$ to all the shoelaces arriving to it, and on the lower stick, assign $-$ to all the shoelaces leaving it, and $+$ to all the shoelaces arriving to it. □

The meaning of the sticks here is that the order of $+$ and $-$ is fixed. The shoelaces have ends attached to them, but they do not wind around the sticks or interact with them in any other way. In fact, we can assume that the entire tangle is in R^3 between planes $z = 1$ and $z = 0$, and that the sticks are parallel lines at heights 0 and 1.

We are originally interested in distinguishing knots, and partially achieve it by constructing a strong invariant of oriented links. The main theorem we prove, using quantum groups and R-matrices, is the following.

Theorem 1.2 (The Jones–Conway Polynomial)**.** *There is a unique map from the set of oriented links in $\mathbb{R}^3$ to the polynomial ring $\mathbb{Z}[x, x^{-1}, y, y^{-1}]$, denoted by $L \mapsto P_L(x, y)$, such that*

(1) *If L and L' are isotopic, then $P_L = P_{L'}$.*
(2) *$P_{\circlearrowleft}(x, y) = 1$ where $\circlearrowleft$ is the unknot.*
(3) *(Skein relations) If (X_+, X_-, X_0) is a* **Conway triple** *of oriented links, meaning a triple where all their links have one distinguished crossing, with all orientations pointing down, such that on that crossing X_+ contains X with / on top of \, X_- contains X with \ on top of /, and X_0 contains ||, and the rest of the links are the same; then polynomials associated to them satisfy the* **skein relation** $xP_{X_+} - x^{-1}P_{X_-} = yP_{X_0}$.

Example 1.3. Let $\circlearrowleft^n$ be n disjoint copies of the unknot. Then

$$P_{\circlearrowleft^n}(x, y) = (\frac{x - x^{-1}}{y})^{n-1} P_{\circlearrowleft}(x, y).$$

Draw a picture with figure eights This diagram shows that $xP_{\circlearrowleft^n} - x^{-1}P_{\circlearrowleft^n} = yP_{\circlearrowleft^{n+1}}$. □

We will first prove Theorem 1.2 assuming the following theorem, and then prove that theorem in the second half of the talk. This theorem constructs a large family of numerical invariants of knots.

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Theorem 1.4. *Let $q \in \mathbb{C}^*$, not a root of unity, and $m > 1$ a positive integer. There is a unique map $\Phi = \Phi_{m,q}$ from the set of oriented links in $\mathbb{R}^3$ to $\mathbb{C}$ such that*

(1) *If L and L' are isotopic, then $\Phi(L) = \Phi(L')$,*

(2) $\Phi(\circlearrowleft) = \dfrac{q^m - q^{-m}}{q - q^{-1}} \neq 0$,

(3) *Given a Conway triple (L_+, L_-, L_0), $q^m\Phi(L_+) - q^{-m}\Phi(L_-) = (q - q^{-1})\Phi(L_0)$.*

Let $R = \mathbb{Z}[x, x^{-1}, y, y^{-1}]$ and $\mathcal{K}$ be the set of isotopy classes of links. Set $S = R[\mathcal{K}]/(\text{skein relations})$, and consider it as an R-module.

Proposition 1.5. *The map $Q\colon R \to S$ defined by $Q(1) = [\circlearrowleft]$ is an isomorphism of R-modules.*

Note that this implies Theorem 1.2 by setting $P_L(x, y) = Q^{-1}([L])$. All three axioms of theorem 1.2 are satisfied by definition of Q, which proves existence of the map $L \mapsto P_L$ from the Theorem 1.2. The uniqueness also follows directly: if there was another map $L \mapsto P'_L$, it would induce a different map Q'; but then Q and Q' would be two different maps of free R-modules of rank 1, with the same image of the generator, so they can't be different.

Let us now prove Proposition 1.5, assuming Theorem 1.4.

Proof. Surjectivity: We will prove surjectivity by induction on the number of crossings of a link. Let L be a link with m crossings. Pick one of them and let (X_+, X_-, X_0) be the corresponding Conway triple, with $L = X_+$ or $L = X_-$. Both X_+ and X_- have at most m crossings, and X_0 has $< m$ crossings. Using skein relations, we conclude that, modulo elements with $< m$ crossings, $[L]$ is equal to $[L$ with one crossing switched$]$ multiplied by some power of x. In other words, by induction assumption and after finitely many such steps, the class of L is equivalent, modulo image(Q), to the class of L with finitely many crossings switched. It is possible to turn any knot into $\circlearrowleft^m$ by a finite number of switches of the form $X_+ \leftrightarrow X_-$. Furthermore, $[\circlearrowleft^n]$ is in the image of Q by the argument in Example 1.3. So, $[L]$ is in the image of Q.

Note we used very little about the ring here, just that x and x^{-1} are invertible (not even that they multiply to 1). This is because this part of the proof is essentially topological; it proves that any link can be unknotted using Conway triples, in the sense that one can always find a Conway triple consisting the desired link and two simpler links. This gives the algorithm for computing the map $L \mapsto P_L$, by successive applications of the skein relations. The next part of the proof, the injectivity of Q, shows that $L \mapsto P_L$ is well defined by such an algorithm. It is algebraic and uses the properties of the ring R in a significant way.

Injectivity: Define $\Phi'_{m,q}\colon R[\mathcal{K}] \to R \to \mathbb{C}$ where the first map is $\Phi_{m,q} \otimes R$ from Proposition 1.4 and the second map is the evaluation $x \mapsto q^m$ and $y \mapsto q - q^{-1}$. This factors through the skein relations, so we get a map $\Phi''_{m,q}\colon S \to \mathbb{C}$. So if $f \in \ker Q$, then $f(q^m, q - q^{-1}) = 0$ for all q, m with q not a root of unity, which implies $f = 0$. ☐

This proves Theorem 1.2 assuming Theorem 1.4. The next section proves Theorem 1.4 using representations of quantum groups. Let us show an example of an application of Theorem 1.2.

Example 1.6. Using the Jones–Conway polynomial, we can show that the trefoil knot L maybe insert a picture of a trefoil knot? is not isotopic to its mirror image. Namely, calculating P_L by using skein relations, we get it is equal to

$$P_L(x, y) = 2x^{-2} - x^{-4} + x^{-2}y^2.$$

To calculate the Jones–Conway polynomial of its mirror image $\widetilde{L}$, we can either use the same procedure or first prove that the polynomial of the mirror image $\widetilde{P}$ of a link is always obtained from the polynomial P of the link by substitution

$$\widetilde{P}(x, y) = P(x^{-1}, -y).$$

Either way, we get

$$P_{\widetilde{L}}(x,y) = 2x^2 - x^4 + x^2y^2 \neq P_L(x,y). \qquad \square$$

2. Tensor categories

We will be using two tensor categories. One is $\mathscr{V}$, the category of finite dimensional vector spaces (secretly, we will a define structure on it that will reflect the structure of finite dimensional representations of quantum groups on it). The other one is the category of tangles, defined in Definition 1.1.

Definition 2.1. The **tangle category** $\mathscr{T}$ is defined as follows. The objects of $\mathscr{T}$ are finite sequences of pluses and minuses, including the empty sequence. The space of morphisms from the sequence S_1 to S_2 is the $\mathbb{C}$-span of all possible tangles up to isotopy whose endpoints are labeled by S_1 on the lower wooden stick and S_2 on the upper one. The tensor product on $\mathscr{T}$ is given by concatenation of sequences. The composition of two maps is obtained by identifying wooden sticks in the obvious way. The unit object is the empty sequence. The identity morphism on some sequence of n signs is the tangle made of n parallel shoelaces, oriented to make attaching them at the bottom and the top possible (not that the $+-$ orientation was made consistently on top and bottom stick in this way). $\square$

Remember that in tensor categories, for objects U, V, one might want to look for an isomorphism between tensor products $U \otimes V$ and $V \otimes U$. In some examples this isomorphism is given by a simple flip, and in some examples it is more complicated, or might not exist.

Let us consider some examples.

Example 2.2. First, let us think of the category $\mathscr{V}$. There is an obvious isomorphism $U \otimes V$ and $V \otimes U$, given by $u \otimes v \mapsto v \otimes u$. Let us denote it by (12), as it permutes the first and the second factor. It is then easy to see that, as operators from triple tensor product $U \otimes V \otimes W$, they satisfy $(12)(23)(12) = (23)(12)(23)$ (as both are equal to the permutation (13)). In other words, the flip satisfies the Yang–Baxter equation (we'll call this the trivial solution). Note that the flip squares to the identity. This induces a representation of the symmetric group on the n-tuple tensor products $U^{\otimes n}$. (For more information, look up *symmetric tensor categories*). $\square$

Example 2.3. Next, consider the category of finite dimensional representations of a quantum group, for example $A = U_q(\mathfrak{sl}_2)$ or more generally $U_q(\mathfrak{g})$. It is clear that the simple flip (12) isn't an isomorphism in this category, as the action on the tensor product of two representations is given by the coproduct, and the coproduct is not symmetric. However, in this case there is an invertible element $R \in A \otimes A$, called the universal R-matrix, such that conjugation by it interchanges the coproduct with the opposite coproduct, and such that $R^{12} = R \circ (12)$ (the flip composed with the action of R on the tensor product) is the isomorphism we require. One of the requirements for this element is that $R^{12}R^{23}R^{12} = R^{23}R^{12}R^{23}$ (in other words, we have a well-defined isomorphism $U \otimes V \otimes W \to W \otimes V \otimes U$). However, this R^{12} doesn't square to 1. This doesn't define a representation of the symmetric group, but of its cover, the braid group. (For more information, look up *braided tensor categories*).

Let us just mention that there is a choice involved here; we could also be looking for R such that the role of (12) is played by $(12) \circ R$ instead of $R \circ (12)$. $\square$

Example 2.4. Finally, consider the category of tangles. For S_1 and S_2 two sequences of $+-$, we can find an isomorphism $S_1 \otimes S_2 \to S_2 \otimes S_1$; namely, the tangle where all the strings starting at S_1 are parallel to each other, all the strings starting at S_2 are parallel to each other, and strings from S_1 all pass above all strings of S_2. Notice that this indeed satisfies the Yang–Baxter equation, defines the representation of the braid group on the n-tuple tensor products, and gives $\mathscr{T}$ the structure of a braided tensor category. Notice also that this operator does *not* square to the identity. In fact, its inverse, the tangle where all strings from S_1 pass below all strings of S_2, is another solution of the same problem, defining another structure of a braided tensor category on $\mathscr{T}$. $\square$

Let us also talk about the question of duals in tensor categories, on an example of representations of a Hopf algebra A.

Example 2.5. Let V be a finite dimensional representation of a Hopf algebra A. Let S be the antipode on A. On a vector space V^*, one can define two structures of A-representation: for $a \in A, v \in V, \varphi \in V^*$, one can define $(a.\varphi)(v) = \varphi(S(a).v)$ or $(a.\varphi)(v) = \varphi(S^{-1}(a).v)$. The dual object should satisfy some compatibility axioms with respect to evaluation and coevaluation morphisms. One of the two above definitions will satisfy:

$$\mathrm{ev}_V \colon V^* \otimes V \to \mathbb{C} \text{ and } \mathrm{coev}_V \colon \mathbb{C} \to V \otimes V^*$$
$$\text{are morphisms in the category}$$

and the other one will satisfy:

$$\mathrm{ev}_V \colon V \otimes V^* \to \mathbb{C} \text{ and } \mathrm{coev}_V \colon \mathbb{C} \to V^* \otimes V$$
$$\text{are morphisms in the category.}$$

In general, it is impossible to get all four to be morphisms (except in symmetric cases).

Remember that as vector spaces, V and V^{**} are naturally identified. However, as representations of A, if we use S to define the structure of the representation on V^*, and then use S again to define the structure on V as the double dual, we end up with two different actions of A on V: one given, and the other one twisted by S^2, which is an automorphism of A but might not be the identity. $\square$

Although the category $\mathscr{V}$ can be equipped with the structure of a symmetric tensor category by the usual flip and the identification $V \cong V^{**}$, nothing is stopping us from defining a more interesting structure on it (as if we would in the above examples, when the vector spaces had the additional structure of a representation of A).

Definition 2.6. An **enhanced R-matrix** on a finite-dimensional space V is an element $c \in \mathrm{Aut}(V \otimes V)$ satisfying the Yang–Baxter equation and an element $\mu \in \mathrm{Aut}(V)$ satisfying some compatibility conditions with c. Morally, we think of c as a nontrivial replacement for the flip (12), and of μ as the nontrivial replacement for the isomorphism $V \to V^{**}$. Note that c is called an R-matrix, but is in fact more like a braiding: it only gives an isomorphism $V \otimes V \to V \otimes V$ for a specific vector space V, while an actual R-matrix should give an isomorphism $U \otimes V \to V \otimes U$ for all pairs of objects U, V in the category. $\square$

Lemma 2.7. *Let $q \in \mathbb{C}$, m a positive integer. On an m-dimensional vector space V_m with a fixed basis v_i, define $c \in \mathrm{Aut}(V \otimes V)$ and $\mu \in \mathrm{Aut}(V)$ as follows ($0 \le i < j \le m$):*

$$\begin{aligned}
\mu(v_i) &= q^{m-2i+1} v_i \\
c(v_i \otimes v_i) &= q^{-m+1} v_i \otimes v_i \\
c(v_i \otimes v_j) &= q^{-m} v_j \otimes v_i \\
c(v_j \otimes v_i) &= q^{-m} v_j \otimes v_i + q^{-m}(q - q^{-1}) v_j \otimes v_i.
\end{aligned}$$

This is an enhanced R-matrix on V_m.

Secretly, c is the action of the universal R-matrix and μ is the identification $V \cong V^{**}$ of V_m, both defined for a vector representation V_m of $U_q(\mathfrak{sl}_m)$. We will not prove the lemma as we didn't define the enhanced R-matrix with all the axioms, but it is possible to prove that it really comes from the $U_q(\mathfrak{sl}_m)$ structure, and that it fits the intuition given above. For details, see [2].

Lemma 2.8. *For every finite dimensional space V with an enhanced R-matrix on it, there exists a unique tensor functor $F \colon \mathscr{T} \to \mathscr{V}$ determined by the following properties:*

- $F(+) = V$, $F(-) = V^*$
- *F sends the tangle which is a crossing X with \ under / and both strings them oriented down to the morphism c, and the X with / under \ maps to c^{-1}.*
- *F sends a tangle shaped like $\cap$ and going to the right, i.e., mapping the $-+$ sequence to the empty sequence, to a morphism $\mathrm{ev}_V \colon V^* \otimes V \to \mathbb{C}$.*
- *F sends a tangle shaped like U and going to the right, i.e., mapping the empty sequence to $+-$, to a morphism $\mathrm{coev}_V \colon \mathbb{C} \to V \otimes V^*$*
- *F sends a tangle shaped like $\cap$ and going to the left, i.e., mapping the $+-$ sequence to the empty sequence, to a morphism $\mathrm{ev}_{V^*} \circ (\mu \otimes \mathrm{id}_{V^*}) \colon V \otimes V^* \to \mathbb{C}$.*
- *F sends a tangle shaped like U and going to the left, i.e., mapping the empty sequence to $-+$, to a morphism $(\mathrm{id}_{V^*} \otimes \mu^{-1}) \circ \mathrm{coev}_{V^*} \colon \mathbb{C} \to V^* \otimes V$.*

To prove the lemma, one needs to show that the morphisms in $\mathscr{T}$ given above, i.e., two crossings X with both strings going down, two U and two $\cap$, generate the category. Then one needs to find explicit relations they satisfy, and prove they are compatible with the axioms for the enhanced R-matrix. Instead, let us just note this fits the intuition given above for μ and c: if $\cap$ represents evaluation and $\cap$ going $-+$ to the empty sequence is given by $\mathrm{ev}_V \colon V^* \otimes V \to \mathbb{C}$, then $\cap$ going $+-$ to the empty sequence should be given by the evaluation $V \otimes V^* \to \mathbb{C}$. However, this evaluation is not a morphism, so we need to first use $\mu \otimes \mathrm{id}_{V^*} \colon V \otimes V^* \to V^{**} \otimes V^*$, and then $\mathrm{ev}_{V^*} \colon V^{**} \otimes V^* \to \mathbb{C}$.

Lemma 2.9. *In particular, the enhanced R-matrix on V_m from Lemma 2.7 gives us, by Lemma 2.8 a functor $F_{m,q} \colon \mathscr{T} \to \mathscr{V}$. This functor satisfies:*

(1) *For X_+ the crossing with \ under / and both strings oriented down, X_- the crossing with \ over / and both strings oriented down, and X_0 two parallel strings oriented down,*
$$q^m F(X_+) - q^{-m} F(X_-) = (q - q^{-1}) F(X_0).$$
(2) $F(\circlearrowleft) = \dfrac{q^m - q^{-m}}{q - q^{-1}} = [m]_q$.

Proof. The first claim is proved by explicit calculation on a tensor product of two basis vectors. The second claim is obtained by decomposing $\circlearrowleft$ as U going right and $\cap$ going left, and then calculating that $\mathrm{ev}_{V^*} \circ (\mu \otimes \mathrm{id}_{V^*}) \circ \mathrm{coev}_V = \mathrm{tr}(\mu) = \frac{q^m - q^{-m}}{q - q^{-1}}$. □

End of proof of Proposition 1.4. In 2.7 we have constructed explicit enhanced R-matrices for every q and m. In 2.8 we have shown how to turn an explicit enhanced R-matrix to a functor $\mathscr{T} \to \mathscr{V}$. We are trying to prove proposition 1.4, which constructs invariants of links. Now do the following.

For any link L, first interpret it as a tangle (by just putting a sticks above it and a stick below it, it's not tied to any of them). Then interpret this tangle as a morphism in the tangle category, from an empty sequence to an empty sequence. Then apply the functor $F_{m,q}$ to it, obtaining a morphism $F(L) \colon \mathbb{C} \to \mathbb{C}$. In $\mathscr{V}$, the only such morphisms (linear maps) are multiplications by a constant. Call this constant $\Phi_{m,q}$. We claim this is the required map $\Phi_{m,q}$ from Proposition 1.4.

It is clearly isotopy invariant because of the definition of the tangle category. We saw in lemma 2.9 that it maps $\circlearrowleft$ to $[m]_q$. Finally, the skein relations for Conway triples follow from part 2) of lemma 2.9 by tensoring left and right, and then composing above and below, with an arbitrary tangle. □

References

[1] Peter Tingley, A minus sign that used to annoy me but now I know why it is there, `arXiv:1002.0555v1`.

[2] Kassel, Christian (1995), Quantum groups, Graduate Texts in Mathematics, 155, Berlin, New York: Springer-Verlag, MR1321145, ISBN 978-0-387-94370-1