THE PETER–WEYL THEOREM FOR CLASSICAL AND QUANTUM $\mathfrak{sl}_N$

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Theorem 0.1 (Peter–Weyl)**.** *Let G be a simply-connected semisimple complex algebraic group. Then*

$$\mathcal{O}[G] = \bigoplus_V V^* \boxtimes V$$

as (G,G)-bimodules, and where the sum is over all irreducible representations of G.

Let H be a Hopf algebra. If H is finite dimensional, then H^* is also a Hopf algebra by dualizing all operations from H. We run into issues if H is infinite-dimensional, but we can find a fix. For a finite-dimensional H-module V, an element $v \in V$, and $f \in V^*$, define a linear functional on H by $c_{f,v}(u) = f(uv)$. Call these linear functionals **matrix coefficients**.

Proposition 0.2. (1) $c_{f,v}c_{g,w} = c_{g\otimes f, v\otimes w}$
(2) *For a representation V, let $\{e_1, \dots, e_n\}$ be a basis and $\{e_1^*, \dots, e_n^*\}$ be a dual basis for V^*. Then $\Delta(c_{f,v}) = \sum_i c_{f,e_i} \otimes c_{e^i,v}$*
(3) *H^0 has an antipode $\overline{S} = S^*$.*
(4) *Suppose $\varphi\colon V \to W$ is a map of H-modules. Then $c_{f,\varphi v} = c_{\varphi^* f, v}$.*

The **algebra of matrix coefficients** is the subalgebra H^0 of H^* spanned by all matrix coefficients.

Recall that H^* is an $H - H$-bimodule as follows. For $f \in H^*$, $a, b \in H$, $(a \otimes b)f$ is the function satisfying:

$$(a \otimes b)f(u) = f(S(a)ub).$$

This equips H^0 with a bimodule structure by restriction.

Proposition 0.3 (Peter–Weyl for semi-simple Hopf algebras)**.** *Suppose that H is semisimple, i.e., every finite-dimensional representation is completely reducible. Then we have the following decomposition of H^0 as a $H - H$-bimodule.*

$$H^0 \cong \bigoplus_X X^* \boxtimes X$$

where the sum is over all irreducible representations X.

Proof. Define

$$c_{V^*,V} = \mathbb{C}\{c_{f,v} \mid f \in V^*,\ v \in V\} \subset H^0.$$

We have a map

$$\iota_X \colon X^* \boxtimes X \to c_{X^*,X}$$
$$f \boxtimes v \mapsto c_{f,v}$$

We claim that $\bigoplus_X \iota_X$ is an isomorphism. Clearly each ι_X is an inclusion. This map $\oplus_X \iota_X$ is $H \times H$-linear, and so $Im(\iota_X) \cap Im(\iota_Y) = 0$ for all distinct X and Y appearing in the sum, since they are non-isomorphic simple subrepresentaitons.

To show surjectivity, let V be a finite-dimensional H-module. We have a natural map $\iota_V \colon V^* \boxtimes V \to H^0$, and we need to show that $Im(\iota_V) \subset \oplus_X Im(\iota_X)$ over all simple X's.

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Decompose $V = \bigoplus_i X_i$ into its irreducible components. Let $j_{X_i} \colon X_i \to V$ be the inclusion given by the direct sum, and let $\pi_{X_i} \colon V^* \to X_i^*$ be the dual map. Similarly, let $j_{X_i^*} \colon X_i^* \to V^*$ and $\pi_{X_i^*} \colon V \to X_i$ be the maps given by the decomposition $V^* = \oplus X_i^*$ We have two maps $1 \boxtimes j_{X_i} \colon V^* \boxtimes X_i \to V^* \boxtimes V$ and $\pi_{X_i} \boxtimes 1 \colon V^* \boxtimes X_i \to X_i^* \boxtimes X_i$.

Write any pure tensor $f \boxtimes v \in V^* \boxtimes V$ in terms of its irreducible components:

$$f = \sum_i j_{X_i^*}(\pi_{X_i}(f)), \quad v = \sum_i j_{X_i}(\pi_{X_i^*}(v)).$$

Then, we have:

$$\begin{aligned} \iota_V(f \boxtimes v) &= \sum_{m,n} j_{X_m^*}(\pi_{X_m}(f)) \boxtimes j_{X_n}(\pi_{X_n^*}(v)) \\ &= \sum_{m,n} (\pi_{X_m}(f)) \boxtimes \pi_{X_m^*} j_{X_n}(\pi_{X_n^*}(v)), \end{aligned}$$

applying Proposition 0.2(4). Each term in the sum for which $X_m \not\cong X_n$ will be zero, since $\pi_{X_m^*} j_{X_n} = 0$, in that case. Thus, we have:

$$Im(\iota_V) \subset \bigoplus Im(\iota_{X_i}),$$

as desired. □

Recall that finite-dimensional representations of $\mathbf{SL}_N$ are tensor generated by the defining representation $V_{\omega_1} = \mathbb{C}^N$. Let $\{e_1, \dots, e_N\}$ be the standard basis. This implies that $U(\mathfrak{sl}_N)^0$ is generated by $V_{\omega_1}^* \boxtimes V_{\omega_1}$. In other words, $U(\mathfrak{sl}_N)^0$ is generated by $a^i_j = c_{e^i, e_j}$.

Proposition 0.4.

$$U(\mathfrak{sl}_N) = \mathcal{O}(\mathbf{SL}_N) = \mathbb{C}[a^i_j \mid 1 \le i, j \le N]/(\det - 1).$$

Proof. The fact that the a^i_j commute follows from the fact that the tensor product of representations is symmetric, i.e., we have an isomorphism of G-modules:

$$\begin{aligned} \tau \colon V \otimes V &\to V \otimes V, \\ a \otimes b &\mapsto b \otimes a. \end{aligned}$$

We consider the image of a vector $v^i \otimes v^j \boxtimes v_k \boxtimes v_l$ under the maps $\tau^* \boxtimes \mathrm{id}$ and $\mathrm{id} \boxtimes \tau$. We compute:

(0.5)
$$\begin{array}{ccc} & v^i \otimes v^j \boxtimes v_k \otimes v_l & \\ \swarrow^{\tau^* \boxtimes \mathrm{id}} & & \searrow^{\mathrm{id} \boxtimes \tau} \\ a^i_k a^j_l = v^j \otimes v^i \boxtimes v_k \otimes v_l & & v^i \otimes v^j \boxtimes v_l \otimes v_k = a^j_l a^i_k \end{array}$$

The two images are equal in $U(\mathfrak{sl}_N)^0$ by Proposition 0.2(4). Thus, we have $a^i_k a^j_l = a^j_l a^i_k$ for all i, j, k, l, so the algebra is commutative. We have a surjection $\mathbb{C}[a^i_j \mid 1 \le i, j \le N] \to U(\mathfrak{sl}_N)^0$.

Define

$$\begin{aligned} \chi \colon 1 &\to V^{\otimes N} \\ z &\mapsto z \sum_{w \in \Sigma_N} (-1)^{\ell(w)} e_{w(1)} \otimes \cdots \otimes e_{w(N)}. \end{aligned}$$

We get maps $\chi^* \boxtimes \mathbf{1} \colon (V^*)^{\otimes N} \boxtimes \mathbf{1} \to \mathbf{1}^* \boxtimes \mathbf{1}$ and $\mathbf{1} \boxtimes \chi \colon (V^*)^{\otimes N} \boxtimes \mathbf{1} \to (V^*)^{\otimes N} \boxtimes V^{\otimes N}$. We apply these to an element $v^N \otimes \cdots v^1 \boxtimes 1 \in (V^*)^{\otimes N} \boxtimes \mathbf{1}$.

(0.6)

$$\begin{array}{ccc} & v^n \otimes \cdots v^1 \boxtimes 1 & \\ \swarrow^{\chi^*\boxtimes \mathrm{id}} & & \searrow^{\mathrm{id}\boxtimes\chi} \\ 1 = 1 \boxtimes 1 & & \sum_w (-1)^l(w) v^N \otimes \cdots \otimes v^1 \boxtimes v_{w(1)} \otimes \cdots \otimes v_{w(N)} \\ & & \| \\ & & \sum_w (-1)^{l(w)} a^1_{w(1)} \cdots a^N_{w(N)} = \det \end{array}$$

Thus, applying Proposition 0.2(4), we get the equation $\det = 1$, and this gives a surjection $\mathbb{C}[a^i_j \mid 1 \leq i, j \leq N]/(\det -1) \to U(\mathfrak{sl}_N)^0$. To prove injectivity, one passes to an associated graded (we omit the details). □

This is most useful when we want to compute $\mathcal{O}_q(\mathbf{SL}_N) := U_q(\mathfrak{sl}_N)^0$. Again, $\mathcal{O}_q(\mathbf{SL}_N)$ is generated by the $a^i_j = c_{e^i, e_j}$. However, the a^i_j do not commute.

We have a braiding σ instead of τ. We introduce notation:

$$\begin{aligned} \sigma \colon V \otimes V &\to V \otimes V \\ v_i \otimes v_j &\mapsto \sum_{k,\ell} R^{k,\ell}_{i,j} v_\ell \otimes v_k. \end{aligned}$$

We can write R explicitly as

$$R^{k,\ell}_{i,j} = q^{\delta_{i,j}} \delta_{i,k}\delta_{j,\ell} + (q - q^{-1})\theta(i-j)\delta_{i,\ell}\delta_{j,k}$$

where $\theta(t)$ is 1 if $t > 0$ and 0 otherwise. If we try to find the relations among the a^i_j using the braiding instead of τ in (0.5), we consider instead:

(0.7)

$$\begin{array}{ccc} & v^i \otimes v^j \boxtimes v_k \otimes v_l & \\ \swarrow^{\sigma^*\boxtimes \mathrm{id}} & & \searrow^{\mathrm{id}\boxtimes\sigma} \\ \sum_{m,n} R^{ij}_{mn} v^n \otimes v^m \boxtimes v_k \otimes v_l & & \sum_{o,p} R^{op}_{lk} v^i \otimes v^j \boxtimes v_p \otimes v_o \\ \| & & \| \\ \sum_{m,n} R^{ij}_{mn} a^m_k a^n_l & & \sum_{o,p} R^{op}_{lk} a^j_p a^i_o \end{array}$$

Thus, applying Propositon 0.2(4), we conclude:

$$\sum_{n,m} R^{ij}_{nm} a^n_k a^m_\ell = \sum_{o,p} a^j_o a^i_p R^{po}_{k\ell}.$$

We also have a map

$$\begin{aligned} \chi_q \colon 1 &\to V^{\otimes N} \\ 1 &\mapsto \sum_{\sigma \in \Sigma_N} (-q)^{-\ell(\sigma)} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(N)} \end{aligned}$$

and a q-determinant:

$$\det_q = \sum_{\sigma \in \Sigma_N} (-q)^{-\ell(\sigma)} a^1_{\sigma(1)} \cdots a^N_{\sigma(N)}.$$

Using χ_q in place of χ in diagram (0.6), we can present $\mathcal{O}_q(\mathbf{SL}_N)$ as the non-commutative polynomial ring generated by the a^i_j subject to the above relations and $\det_q = 1$.

Example 0.8. $\mathcal{O}_q(\mathbf{SL}_2)$ is generated by a, b, c, d subject to the relations

$$ac = qca, \quad ab = qba, \quad cd = qda, \quad bd = qdb, \quad bc = cb, \quad ad = da + (q - q^{-1})bc, \quad ad - q^{-1}bc = 1$$

This is a flat deformation of $\mathcal{O}(\mathbf{SL}_N)$. $\square$