LECTURE 7: REALIZTING $U^{-}(\mathfrak{g})$ USING LUSZTIG'S NILPOTENT VARIETY

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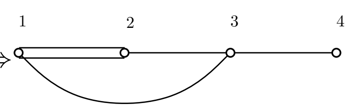
This will be the first in a series of lectures on a geometric way of realizing the algebra $U^{-}(\mathfrak{g})$, the crystal $B(\infty)$, highest weight representations of $\mathfrak{g}$, and crystals of these highest weight modules. Note that, although we realize both the reprsentation of $\mathfrak{g}$ and the crsytal of this representation, we do not realize it as a representation of $U_q(]\mathfrak{g})$. This can be done (see), but is much more difficult. The geometric spaces we use will be Lusztig's varieties $\Lambda(V)$ from [L] (sometimes called Lusztig's nilpotent variety), and later on Nakajima's varieties $\mathfrak{L}(v,w)$ from [N]. Note that throughout this story we assume that $\mathfrak{g}$ is a *symmetric* Kac-Moody algebra. Some constructions can be extended to the symmetrizable case by "folding" arguments based on the observation that $U(\mathfrak{g})$ for symmetrizible $\mathfrak{g}$ can be embedded into $U(\mathfrak{g}')$ for a related symmetric $\mathfrak{g}'$.

Today we define $\Lambda(V)$. Then we construct a product on $\oplus_v \mathcal{M}(\Lambda(V)/\operatorname{GL}(V))$ (i.e. on the sum of the spaces of invariant constructible functions on $\Lambda(V)$ as the dimension vector $\dim V$ varies) so that $\mathcal{U}(\mathfrak{g}) \hookrightarrow \oplus_v \mathcal{M}(\Lambda(v)/\operatorname{GL}(V))$.

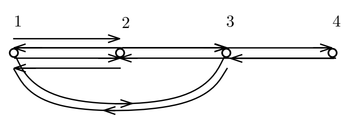
1. Quivers and path algebras

Fix a *symmetric* (generalized) Cartan matrix and its associated Dynkin diagram. For example,

$$\begin{bmatrix} 2 & -2 & -1 & 0 \\ -2 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \Longleftrightarrow$$



Let Q be the diagram with two directed arrows for each edge (note that there are no loops). So the above example would become



We denote by I the set of vertices, and by H the set of arrows. The *path algebra* $\mathbb{C}Q$ of Q is the algebra which as a vector space is just the $\mathbb{C}$ span of the paths in Q, and where multiplication is given by concatination. So for example

$$(\,2 \longrightarrow 3\,)(\,1 \longrightarrow 2\,) = \;1 \longrightarrow 2 \longrightarrow 3 \quad \text{and}$$
$$(\,1 \longrightarrow 2\,)(\,2 \longrightarrow 3\,) = 0.$$

Let π_i be the path of length 0 at vertex i, which is a projection in $\mathbb{C}Q$.

Fix an I graded vector space $V = V_1 + \cdots V_n$. Let

$$E(V) := \{\text{ representations of } \mathbb{C}Q \text{ on } V \mid V_i = \pi_i V\} \tag{1}$$

Explicitly $E(V) := \bigoplus_{a:i\to j} Hom(V_i, V_j)$, where the sum is over paths a from i to j.

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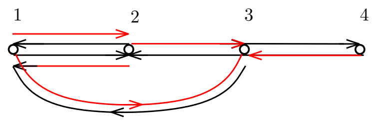
2. The Nilpotent variety $\Lambda(V)$

The variety $E(V)$ is a bit too simple for our purposes. To define the nilpotenet variety we need, we color one arrow corresponding to each edge of the underlying graph.



For each arrows $a \in H$, let $\epsilon(a) = 1$ if a is black and -1 if a is red. Since the arrows come in pairs corresponding the the edges of the underlying graph, we can define on involuton $\bar{\cdot}$ on H which interchanges the two edges in each pair (i.e. takes an arrow to the corresponding arrow with reversed orientation).

Definition 2.1. The *symplectic form* on $E(V)$ is

$$\langle \cdot, \cdot \rangle : E(V) \times E(V) \to \mathbb{C}, (x, y) \to tr \sum_a \epsilon(a) y_{\overline{a}} x_a.$$

The *moment map* is

$$\mu : E(V) \to \mathfrak{gl}(V), x := (x_a)_{a \text{ arrows}} \quad \to \left(\sum_{a:i\to} \epsilon(a) x_{\overline{a}} x_a \right)_{i \in I}$$

where $\mathfrak{gl}(V) := \bigoplus_i \mathfrak{gl}(V_i)$.

Remark 2.1. $\langle \cdot, \cdot \rangle$ is in fact symplectic. The group $\mathrm{GL}(V) := \prod_{I \in i} \mathrm{GL}(V_i)$ acts on $E(V)$, and μ is the moment map for this action. So, the terminology is justified.

Definition 2.2. Set

$$\Lambda(V) := \{x : \mu(x) = 0, x \text{ nilpotent}^2\}.$$

Nilpotent here means that, for some $N > 0$, all paths in $\mathbb{C}Q$ longer then N steps are sent to 0 in the representation x.

Theorem 2.2. [L, Theorem 12.9] *$\Lambda(V)$ is a Lagrangian subvariety of $E(V)$. In particular, it is pure half dimensional.*

Remark 2.3. (i) $\Lambda(V)$ is not irreducible,

(ii) People often write $\Lambda(v)$, only recording the graded dimension of V. One can get away with this because most constructions people consider are $\mathrm{GL}(V)$ invariant.

(iii) $\Lambda(V)$ is sometimes defined as representations of the completed preprojective algebra $\overline{\mathcal{P}} := \overline{\mathbb{C}Q / \sum_{a:i\to} (-1)^{c(a)} \overline{a} a}$ on V, where the completion is with respect to path length. This is easily seen to be equivalent.

3. Realization of $\mathcal{U}^-(\mathfrak{g})$

Denote by $\mathcal{M}(\Lambda(v)/GL(v))$ the space of $\mathrm{GL}(v) = \prod \mathrm{GL}(v_i)$ invariant constructible functions on $\Lambda(V)$. Note that, by $\mathrm{GL}(V)$ invariance, this space is determined up to canonical isomorphism by the graded dimension $v := \dim_I(V)$, so it is justified to forget the underlying vector space of $\Lambda(V)$ in the notation.

[2]The nilpotent condition is not needed in finite type

To define the product of $\bigoplus_v \mathcal{M}(V(v))$, we need to introduce a new variety: Fix V and a dimension vector $v' < \dim V$.

$$\Lambda(V; v') := \{(x, u) : x \in \Lambda(v), ux - \text{invariant subrepresentation} \dim U = v'\}.$$

Let $v'' = \dim V - v'$, and consider the following maps:

$$\begin{tikzcd} & \Lambda(V, v')/\operatorname{GL}(V) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \Lambda(V')/\operatorname{GL}(V') \times \Lambda(V'')/\operatorname{GL}(V'') \arrow[rr, dotted] & & \Lambda(V)/\operatorname{GL}(V) \end{tikzcd}$$

Define $* : \mathcal{M}(\Lambda(v')) \times \mathcal{M}(\Lambda(v'')) \to \mathcal{M}(\Lambda(v))$ by $f * g \mapsto (\pi_2)_!(\pi_1^*)(f \cdot g)$ where $\pi_!$ means *push forward* in the "six functors" sense. For us, it is enough to know the following explicit formulation: for a function $F \in \mathcal{M}(\Lambda(V, v')/GL(V))$,

$$(\pi_2)_!(F) = \sum_{z \in \mathbb{C}} z\, \chi(\{[x, u] : \pi_2[x, u] = x, F[x, u] = z\}),$$

where χ is Euler characteristic.

Remark 3.1. The astute reader will point out that, in writing $\Lambda(V)/GL(V)$, I am really defining a stack, not a variety or scheme. However, in the present context, all operations we need can be defined without this language: π_1^* of a $GL(V') \times GL(V'')$ invariant constructable function on $\Lambda(V') \times \Lambda(V'')$ is clearly a well defined $GL(V)$ invariant function on $\Lambda(V, v')$. To make sense of $(\pi_2)_!$, one needs only notice that the fibers of this map are of the form $\{V' \subset V : V' \text{ is } x \text{ invariant }\}$. This is an ordinary variety (which looks much like a grassmannian), so its Euler characteristic makes sense, and our explicit definition of $(\pi_2)_!$ is well defined.

One can check that $*$ defines an associative product by showing that $f * (g * h)$ and $(f * g) * h)$ can both be defined as $(\pi_2)_! \pi_1^*$, but with a new space is place of $\Lambda(V, v')/\operatorname{GL}(V)$. This new space is roughly 2 step flags of submodules of V.

Let $v_i = 1_i$, the dimension vector which in 1 in degree i, and 0 elsewhere.

Theorem 3.2. *Define a map* $\mathcal{U}^{-}(\mathfrak{g}) \hookrightarrow \bigoplus_v \mathcal{M}(\Lambda(v)/\operatorname{GL}(v))$ *by* $F_i \mapsto$ *function* 1 *on* $\Lambda(1_i)$. *This is an embedding of associative algebras.*

Idea of proof. First check Serre's relations, which only involves rank 2 calculations. This is done by hand, and is a calculation involving Euler characteristics of grassmannians. This show this map is a map of algebras. It remains to show that the map is injective. This can be done in various ways; one argument can be found in [L, Chapter 12]. □

4. Exercises

Exercise 1. For the quiver considered in the previous examples one can show that.

$$F_2^2 F_3 + F_3 F_2^2 = 2 F_2 F_3 F_2.$$

For a slightly more difficult example, try

$$F_1^2 F_2 + F_2 F_1^2 = 2 F_1 F_2 F_1.$$

In general, show that Serres relations always hold.

Exercise 2. One might guess that you could recover $U_q^-(\mathfrak{g})$ by keeping track of the degree in homology contributing to each terms in the Euler characteristic. Show that this does not work. The case of the sl_3 quiver should suffice.

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References

[N] Hiraku Nakajima. Instantons on ALE spaces, quiver varieties and Kac-Moody algebras.

[L] G. Lusztig. Quivers, Perverse sheaves and quantized enveloping algebras. *Journal of the american mathematical society* **4** No 2, April 1991.