1. Gelfand–Tsetlin bases

1.1. General construction. Recall: the finite-dimensional irreducible polynomial representations of $\text{GL}_n(\mathbb{C})$ are in bijection with partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, which we represent as Young diagrams with at most $n$ rows. Call this latter set $Y_n$. Let $V_\lambda$ denote the representation corresponding to $\lambda$. Our goal is to construct nice bases for $V_\lambda$ that are well-behaved with respect to restrictions and tensor products.

The idea is work by induction. Suppose that we have a Gelfand–Tsetlin (G-T) basis for all irreducible representations of $\text{GL}_{n-1}$. We consider the restriction of $V_\lambda$ to $\text{GL}_{n-1}$, decompose it as a direct sum of irreducible representations, and take the Gelfand–Tsetlin basis of each of these. This gives a basis for $V_\lambda$ itself. Of course, we have to make a choice when we decompose $V_\lambda$ into irreducible representations of $\text{GL}_{n-1}$, so the notion of Gelfand-Tsetlin basis can only be well defined up to such choices. However, in this case the decomposition is multiplicity free, so in the end we get a basis for $V(\lambda)$ which is well defined up to rescaling each basis vector.

1.2. Combinatorics.

**Definition 1.1.** For $\lambda \in Y_n$ and $\lambda' \in Y_{n-1}$ with $\lambda' \subset \lambda$, say that $\lambda/\lambda'$ is a *horizontal strip* if each column in $\lambda/\lambda'$ has at most 1 element.

Then we have

$$V_\lambda \cong \bigoplus_{\lambda'/\lambda' \text{ horizontal strip}} V_{\lambda'}$$

as $\text{GL}_{n-1}$-representations.

**Definition 1.2.** A *GT-pattern* is a triangular array of numbers $(\lambda_{ij})_{n \geq i \geq j \geq 1}$ such that $\lambda_{ij} \geq \lambda_{i-1,j} \geq \lambda_{i,j+1}$. These are in bijection with semistandard Young tableaux by considering the successive shapes

$$\lambda_1, \bullet \subseteq \lambda_2, \bullet \subseteq \cdots \subseteq \lambda_n, \bullet$$

and labeling the boxes in $\lambda_i, \bullet \setminus \lambda_{i-1}, \bullet$ with the number $i$. Call this bijection $\tau$.

The G-T basis of $V_\lambda$ is parametrized by GT-patterns with $\lambda_{n, \bullet} = \lambda$. To describe the restriction $V_\lambda \downarrow_{\text{GL}_{n-1}}$, take the union of GT-bases for $\text{GL}_{n-1}$ representations, where you forget about the top row.

1.3. Orthogonal Lie algebras. The representations of $\mathfrak{so}_{2n+1}$ are parametrized by sequences $\lambda_1 \geq \cdots \lambda_n \geq 0$ which are either all integers or all half-integers and for $\mathfrak{so}_{2n}$, they are parametrized by $\lambda_1 \geq \cdots \lambda_{n-1} \geq |\lambda_n|$ which are either all integers or all half-integers.

The branching rules are

$$V_\lambda \downarrow_{\mathfrak{so}_{2n+1}} \cong \bigoplus V_\mu$$

where $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_n \geq |\mu_n|$ and

$$V_\lambda \downarrow_{\mathfrak{so}_{2n-1}} \cong \bigoplus V_\mu$$

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where \( \lambda_1 \geq \mu_1 \geq \cdots \geq \mu_{n-1} \geq |\lambda_n| \).

Once again the branchings are multiplicity free, so one can define analogues of G-T bases
and G-T patterns in this case.

2. G-T bases compatible with crystal structure

Now consider \( U_q(\mathfrak{gl}_n(\mathbb{C})) \). One can construct GT bases for the irreducible representations \( V_\lambda \), just
as for \( \text{GL}_n \), which are well defined up to individual rescaling of the basis vectors. Also, consider
the vector representation \( V \). It is well known that \( V^{\otimes N} \cong \bigoplus_T V_{\text{shape}(T)} \), where \( T \) ranges over standard
Young tableaux with \( N \) nodes and at most \( n \) rows.

The main result states the following: there is an appropriate decomposition of choice of \( V^{\otimes N} \)
into irreducible representations \( V(T_R) \) corresponding to each standard Young tableau \( T \), and an
appropriate normalization of the G-T bases for each of these representations, such that when \( q \to 0 \),
the G-T basis vector approach the standard basis vectors of \( V^{\otimes N} \). For \( q^{-1} \to 0 \), the same is true, but
we use different decompositions \( V(T_L) \), and a different normalization of the G-T basis elements. In
modern language, this occurs because both the G-T bases (correctly normalized) and the standard
basis of \( V^{\otimes N} \) are crystal bases. In fact, the results discussed in this section, due to Date, Jimbo
and Miwa [3], were an important precursor to the notion of a crystal basis.

The remainder of this section is occupied with making these statements precise and providing a
proof.

2.1. Action of \( U_q(\mathfrak{gl}_n(\mathbb{C})) \) on G-T basis. \( U_q(\mathfrak{gl}_n(\mathbb{C})) \) is generated by \( q^{\lambda/2}, q^{-\lambda/2}, X_j^+, X_j^- \). Then
denoting the Gelfand-Tsetlin basis elements of the \( U_q(\mathfrak{gl}_n(\mathbb{C})) \)-module \( V_\lambda \) by \( |m\rangle \), the action of
the above generators is given as follows:

\[
q^{\lambda/2}|m\rangle = q^{\sum_{i=1}^j m_{ij} - \sum_{i=1}^{j-1} m_{i,j-1} - \sum_{i=1}^{j-1} m_{ij} - \sum_{i=1}^j m_{i,j-1}}|m\rangle
\]

\[
X_j^+|m\rangle = \sum_{j} c_j(m, m')|m'\rangle
\]

\[
X_j^-|m\rangle = \sum_{j} c_j(m, m')|m'\rangle,
\]

where \( c_j(m, m') \neq 0 \) only if there exists \( i \) such that \( m'_{ij} = m_{ij} - 1 \), \( m'_{ab} = m_{ab} \forall (a, b) \neq (i, j) \), in
which case the coefficients are rather complicated to write down. The highest weight vector is given
by the Gelfand-Tsetlin pattern with first row \((\lambda_1, \cdots, \lambda_n)\), second row \((\lambda_1, \cdots, \lambda_{n-1})\) and so on.

2.2. The embedding \( V_W \subset V_Y \otimes V \). Say \( Y \to W \) if \( W \) is obtained from \( Y \) by adding a box
in the \( \mu \)th row. We will now describe explicitly the decomposition \( V_Y \otimes V \cong \bigoplus_Y \bigoplus_W V_W \). Given
\( |m\rangle \in GT(W) \), define \( |m'\rangle = |m; i_1, n, \cdots, i_j \rangle \in GT(Y) \) (note the slight abuse of notation: \( |m'\rangle \) is
not a single element), where for \( j \leq k \leq n, 1 \leq i_k \leq k \), \( m'_{ik} = m_{ik} - 1 \) if \( j \leq k \leq n, i = i_k \) and
\( m'_{ik} = m_{ik} \) otherwise. Then the above branching rule is determined explicitly by the following,
where the coefficients \( w_q(m; i_1, \cdots, i_j) \) are known as Wigner coefficients.

\[
|m\rangle = \sum_{j=1}^n \sum_{i_n = \mu, i_{n-1}, \cdots, i_1} w_q(m; i_1, \cdots, i_j)|m; i_1, i_2, \cdots, i_j \rangle \otimes v_j
\]

2.3. RSK. We’ll define two bijections \( \alpha \) and \( \beta \) between \( \{1, \ldots, n\}^N \) and \( \prod_Y S(Y) \times T(Y) \)
ranging over all Young diagrams \( Y \) with \( N \) nodes and at most \( n \) rows, where \( S(Y) \) is the set of semistandard
Young tableaux of shape \( Y \), and \( T(Y) \) is the set of standard Young tableaux of shape \( Y \).

First, given a SSYT \( S \) and a number \( x \), define the \( \alpha \)-insertion \( S \leftarrow x \) to be the jdt rectification
of the shape obtained by adjoining \( x \) to the lower left corner of the tableau \( S \). Given a word
\( w = w_1 \cdots w_N \), define \( \alpha_S(w) = (((w_1 \leftarrow w_2) \leftarrow w_3) \cdots) \leftarrow w_N \), and let \( \alpha_T(w) \) record the growth
of the subsequent shapes. The bijection \( \alpha \) is then \( w \to (\alpha_S(w), \alpha_T(w)) \).
The second bijection $\beta$ is defined in the same way, but where $\beta$-insertion, given a SSYT $S$ and a number $x$, $S \downarrow x$ denotes the jdt rectification of the shape obtained by adjoining $x$ to the upper right corner of the tableau $S$. Then $\beta_S(w)$ and $\beta_T(w)$ are defined as above. The importance of $\alpha$-insertion and $\beta$-insertion to study the embedding $V_W \subset V_Y \otimes V$ is detailed in the below proposition:

**Proposition 2.1.** Given $Y \overset{\mu}{\to} W$, node added in the $v$th column. Fix $R \in S(W)$, and let $|m\rangle \in GT(W)$ be the corresponding Gelfand-Tsetlin pattern. Set

$$|m\rangle' = \begin{cases} \tau^{-1}(R \to \nu) & q \to 0 \\ \tau^{-1}(R \uparrow \mu) & q^{-1} \to 0 \end{cases}.$$  

Here $R \to \nu$ (resp $R \uparrow \mu$) are the deletion procedures inverse to $S \leftarrow \mu$ and $S \downarrow \mu$. If the deletion throws away the letter $j$, then under the embedding $V_W \subset V_Y \otimes V$, we have:

$$\lim_{q^{-1} \to 0} |m\rangle = \lim_{q \to 0} (\pm 1)^{\mu-1}|m\rangle' \otimes v_j.$$

### 2.4. Proof of Main Theorem

First we explicitly describe the decomposition $V^\otimes N = \bigoplus_T V_T$ in two different ways: recall that $T$ ranges over all standard tableau with $N$ nodes and $\leq n$ rows. Given a fixed tableau $T$, suppose the entry $k$ entry occurs in the position $(\mu_k, v_k)$. To describe the first decomposition $V^\otimes N \cong \bigoplus_T V(T_R)$, the embedding $V(T_R) \to V^\otimes N$ is defined inductively: let $T_1$ be the subtableau of $T$ consisting of entries $\leq i$, then $V_{T_1} \cong V$, embed $V_{T_2} \to V_{T_1} \otimes V$, and so on until we get $V(T) = V(T_N) \leftarrow V_{T_{N-1}} \otimes V$; composing we get an embedding $V_T \to V^\otimes N$, and the direct sum decomposition $V^\otimes N \cong \bigoplus V(T_R)$ follows inductively from the decomposition $V_Y \otimes V \cong \bigoplus_Y W \otimes V_W$. The second decomposition $V^\otimes N \cong \bigoplus_T V(T_L)$ is defined similarly, but using a slightly different embedding $V(T_2) \to V \otimes V(T_1)$ and so on, using a modification of the Wigner coefficients.

With the notation developed above, for emphasis we now state in full detail the Main Theorem that was quickly mentioned above; and we will note that its proof follows directly from the Proposition above by induction on $N$.

**Theorem 2.2.** If $w = i_1i_2 \cdots i_N$, then $v_{i_1} \otimes \cdots \otimes v_{i_N} \in V^\otimes N \cong \bigoplus_T V(T_R)$, at the limit $q \to 0$, lies in the copy of $V_{TR}$ where $T = \sigma_T(w)$, and is the Gelfand-Tsetl basis element corresponding to the SSYT $S = \alpha_S(w)$. A similar statement holds in the limit $q^{-1} \to 0$, where we instead use the decomposition $V^\otimes N \cong \bigoplus_T V(T_L)$. Thus in both cases, the union of the Gelfand-Tsetl basis coincides with the “obvious” bases of $V^\otimes N$.

This is clear using the proposition and using induction on $N$. Indeed, assume the statement for $N - 1$, consider what the vector $v_{i_1} \otimes \cdots \otimes v_{i_{N-1}}$ corresponds to under the decomposition, and then use the Proposition above to deduce the required statement.

**References**