A MINUS SIGN THAT USED TO ANNOY ME BUT NOW I KNOW WHY IT IS THERE (TWO CONSTRUCTIONS OF THE JONES POLYNOMIAL)

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Abstract. We consider two well known constructions of link invariants. One uses skein theory: you resolve each crossing of the link as a linear combination of things that don't cross, until you eventually get a linear combination of links with no crossings, which you turn into a polynomial. The other uses quantum groups: you construct a functor from a topological category to some category of representations in such a way that (directed framed) links get sent to endomorphisms of the trivial representation, which are just rational functions. Certain instances of these two constructions give rise to essentially the same invariants, but when one carefully matches them there is a minus sign that seems out of place. We discuss exactly how the constructions match up in the case of the Jones polynomial, and where the minus sign comes from. On the quantum group side, we are led to use a non-standard ribbon element.

Contents

1. Introduction

These expository notes begin by briefly explaining two constructions of the Jones polynomial (neither of which is the original construction due to Jones [J]). The first is via the skien-theoretic construction of the Kauffman bracket [K]. The second is as a $U_q(\mathfrak{sl}_2)$ quantum group link invariant. This second construction uses a circle of idea developed by a number of authors starting in the late 1980s (see [T] and references therein). We attempt to give some explanation of how quantum group knot invariants work in general, but only fully develop the simplest case. We then discuss how the two constructions are related. Much of the content of these notes can be found in, for instance, [O, Appendix H]. The main difference here is that we use the non-standard ribbon element from [ST].

The Kauffman bracket is an isotopy invariant of framed links. The functor used in the quantum group construction involves a category where morphisms are tangles of *directed* framed ribbons. In particular, endomorphisms of the trivial representations are *directed* framed links, and the image of such a link is a Laurent polynomial, which is the invariant. In the case we consider, this invariant does not depend on the directing, and, up to an annoying sign, agrees with the Kauffman bracket. Part of the purpose of these notes is to explain the annoying sign, but the real purpose is to describe how the skein relations used in the Kauffman bracket arise naturally in the quantum group construction. To this end we modify the quantum group construction to obtain a functor from a category whose morphisms are tangles of *undirected* framed ribbons. We find it is necessary to use the non-standard ribbon element from [ST]. With this change, the annoying minus sign disappears, and the two constructions agree exactly!

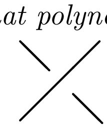
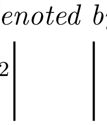
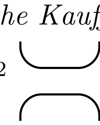
The Jones polynomial is an invariant of directed but unframed links, which can be constructed via a simple modification of the Kauffman bracket (explained in Theorem 2.7 below). We actually compare constructions of invariants of framed but undirected links, so a more accurate subtitle for these notes might be "two constructions of the Kauffman bracket."

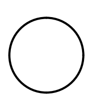
1A. **Acknowledgements.** These notes are based on a talk I gave in 2008 at the university of Queensland in Brisbane Australia. I would like to thank Murray Elder and Ole Warnaar for organizing that visit. I would also like to thank Noah Snyder for many interesting discussions about knot theory.

2. The Kauffman bracket construction of the Jones polynomial

Up to a change in the variable q, the following is the well known construction of the Kauffman bracket [K].

Definition 2.1. *Let L be a link diagram (i.e. A link drawn as a curve is the plane with crossings). Simplify L using the following relations until the result is a polynomial in $q^{1/2}$ and $q^{-1/2}$. That polynomial, denoted by $K(L)$, is the Kauffman bracket of the link diagram.*

(i)  $=$ $q^{1/2}$  $+$ $q^{-1/2}$ 

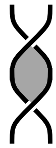
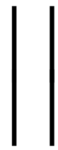
(ii)  $=$ $-q - q^{-1}$

(iii) *If two tangle diagrams are disjoint, their Kauffman brackets multiply.*

Note that (i) depends on which strand is on top.

Comment 2.2. In order for Definition 2.1 to make sense, one needs to assume that all crossings are simple crossings. We will always make this assumption about link diagrams. Once we introduce twists, we will also assume that these occur away from crossings. It is clear that, up to isotopy, any link can be drawn with such a diagram (although not in a unique way).

The Kauffman bracket is not an isotopy invariant of links, but is instead an isotopy invariant of framed links (that is, links tied out of orientable "ribbons"), where the framing is "flat on the page." One can allow twists of the framing to occur in the diagram if one introduces the following extra relation (here both sides represent a single framed string):

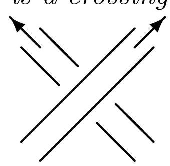
(1)  $= -q^{3/2}$ 

regardless of directing of the ribbon (note that the direction of the twist, i.e. clockwise versus counter clockwise, does matter).

Theorem 2.3. *(see [O, Theorem 1.10]) The Kauffman bracket as calculated using the above relations is an isotopy invariant of framed links.* □

We now describe a modification that leads to an invariant of directed but unframed links. The invariant does still depend on more then just the underlying link (i.e. the choice of directing), but now the amount of choice is much smaller. In fact, for knots (i.e. links with a single component), the invariant does not depend on the directing either (see Comment 2.8).

Definition 2.4. (i) *A positive crossing is a crossing of the form*



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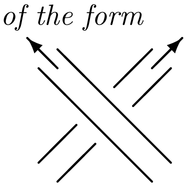
That is, a crossing such that, if you approach the crossing along the upper ribbon in the chosen direction and leave along the lower ribbon, you have made a left turn.

(ii) *A negative crossing is a crossing of the form*



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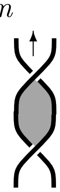
That is, a crossing such that, if you approach the crossing along the upper ribbon in the chosen direction, then leave along the lower ribbon, you have made a right turn.

(iii) *A positive full twist is a twist of the form*



(iv) *A negative full twist is a twist in the opposite direction to a positive full twist.*

(v) *The writhe of a link diagram L, denoted by $w(L)$, is the number of positive crossings minus the number of negative crossings plus the number of positive full twists minus the number of negative full twists.*

Comment 2.5. Here we have drawn every component with its framing. Sometime we will just draw lines, and use the convention that these stand for ribbons lying flat on the page. This is often referred to as the "blackboard framing".

Lemma 2.6. *(see* [K]*) The writhe $w(L)$ is an invariant of directed framed links.* □

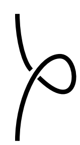
The following is one of the more fundamental theorems in knot theory.

Theorem 2.7. *(see* [O, Theorem 1.5]*) Let L be any link. Then the Jones polynomial,*

$$J(L) := (-q^{3/2})^{-w(L)} K(L), \tag{2}$$

is independent of the framing. Hence $J(L)$ is an isotopy invariant of directed (but not framed) links. □

Theorem 2.7 is sometimes stated in terms of link diagrams, not framed links. The result for framed links follows by noticing that the positive full twist from Definition 2.4 is isotopic to



with the blackboard framing.

Comment 2.8. It is straightforward to see that positive full twists are sent to positive full twists if the direction of the ribbon is reversed, and positive crossings are sent to positive crossings if the directions of both ribbons involved are reversed. It follows that the choice of directing only affects the Jones polynomial for links with at least two components.

3. The quantum group construction of the Jones polynomial

3A. **The quantum group $U_q(\mathfrak{sl}_2)$ and its representations.** $U_q(\mathfrak{sl}_2)$ is an infinite dimensional algebra related to the Lie-algebra $\mathfrak{sl}_2$ of 2×2 matrices with trace zero. It is the algebra over the field of rational functions $\mathbb{C}(q)$ generated by E, F, K and K^{-1}, subject to the relations

$$\begin{aligned} KK^{-1} &= 1, \\ KEK^{-1} &= q^2 E, \\ KFK^{-1} &= q^{-2} F, \\ EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned} \tag{3}$$

For our purposes it is convenient to adjoin a fixed square root $q^{1/2}$ to $\mathbb{C}(q)$.

$U_q(\mathfrak{sl}_2)$ has a finite dimensional representation V_n for each integer n which we now describe. Introduce the "quantum integers"

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+1}. \tag{4}$$

The representation V_n has $\mathbb{C}(q)$-basis $\{v_n, v_{n-2}, \cdots, v_{-n+2}, v_{-n}\}$, and the actions of E, F and K are given by

$$\begin{aligned} E(v_{-n+2j}) &= \begin{cases} [j+1]v_{-n+2j+2} & \text{if } 0 \leq j < n \\ 0 \quad \text{if } j = n, \end{cases} \\ F(v_{n-2j}) &= \begin{cases} [j+1]v_{n-2j-2} & \text{if } 0 \leq j < n \\ 0 \quad \text{if } j = n, \end{cases} \\ K(v_k) &= q^k v_k. \end{aligned} \tag{5}$$

This can be expressed by the following diagram:

$$\begin{tikzcd}[column sep=small]
F: \quad E: & \bullet \arrow[r, shift left, "1"] & \bullet \arrow[l, shift left, "{[n]}"] \arrow[r, shift left, "{[2]}"] & \bullet \arrow[l, shift left, "{[n-1]}"] \arrow[r, shift left, "{[3]}"] & \cdots \arrow[l, shift left, "{[n-2]}"] \arrow[r, shift left, "{[n-2]}"] & \bullet \arrow[l, shift left, "{[3]}"] \arrow[r, shift left, "{[n-1]}"] & \bullet \arrow[l, shift left, "{[2]}"] \arrow[r, shift left, "{[n]}"] & \bullet \arrow[l, shift left, "1"] \\
K: & q^n & q^{n-2} & q^{n-4} & & q^{-n+4} & q^{-n+2} & q^{-n}
\end{tikzcd} \tag{6}$$

There is a tensor product on representations of $U_q(\mathfrak{sl}_2)$, where the action on $a \otimes b \in A \otimes B$ is given by

$$\begin{aligned} E(a \otimes b) &= Ea \otimes Kb + a \otimes Eb, \\ F(a \otimes b) &= Fa \otimes b + K^{-1}a \otimes Fb, \\ K(a \otimes b) &= Ka \otimes Kb. \end{aligned} \tag{7}$$

It turns out that $A \otimes B$ is always isomorphic to $B \otimes A$, and furthermore there is a well known natural system of isomorphisms

$$\sigma^{br}_{A,B} : A \otimes B \to B \otimes A \tag{8}$$

for each pair A, B, called the braiding. The braiding has a standard definition, which can be found in, for example [CP] (or Theorem 5.3 below can also be used as the definition). Here we only ever apply the braiding to representations isomorphic to the standard 2-dimensional representation of $U_q(\mathfrak{sl}_2)$, so we can use the following:

Definition 3.1. *Let V be the 2 dimensional representation of $U_q(\mathfrak{sl}_2)$. Use the ordered basis $\{v_1 \otimes v_1, v_{-1} \otimes v_1, v_1 \otimes v_{-1}, v_{-1} \otimes v_{-1}\}$ for $V \otimes V$. Then $\sigma^{br}_{V,V} : V \otimes V \to V \otimes V$ is given by the matrix*

$$\sigma^{br} = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

To simplify notation, we usually denote $\sigma^{br}_{V,V}$ simply by σ^{br}.

There is a standard action of $U_q(\mathfrak{sl}_2)$ on the dual vector space to V_n. This is defined using the "antipode" S, which is the algebra anti-automorphism defined on generators by:

$$\begin{aligned} S(E) &= -EK^{-1}, \\ S(F) &= -KF, \\ S(K) &= K^{-1}. \end{aligned} \tag{9}$$

For $\hat{v} \in V_n^*$ and $X \in U_q(\mathfrak{sl}_2)$, set $X \cdot \hat{v}$ to be the element of V_n^* defined by

$$(X \cdot \hat{v})(w) := \hat{v}(S(X)w) \tag{10}$$

for all $w \in V_n$. It is straightforward to check that this is in fact an action of $U_q(\mathfrak{sl}_2)$ on V_n^*. It turns out that V_n is always isomorphic to V_n^*, which will be important later on.

Example 3.2. An isomorphism between the standard representation of $U_q(\mathfrak{sl}_2)$ and its dual. Let v_1, v_{-1} be the basis for V. For $i = \pm 1$, let $\hat{v}_i$ be the element of V^* defined by

$$\hat{v}_i(v_j) = \delta_{i,j}. \tag{11}$$

Calculating using the above definition, the action of $U_q(\mathfrak{sl}_2)$ on V^* is given by

$$\begin{array}{ll} F: & \\ & \hat{v}_{-1} \underset{-q}{\overset{-q^{-1}}{\rightleftarrows}} \hat{v}_1, \\ E: & \end{array} \tag{12}$$

Consider the map of vector spaces $f : V \to V^*$ defined by

$$\begin{cases} f(v_1) = \hat{v}_{-1} \\ f(v_{-1}) = -q^{-1}\hat{v}_1 \end{cases} \tag{13}$$

One can easily check that f is in fact an isomorphism of $U_q(\mathfrak{sl}_2)$ representations.

Comment 3.3. If one sets $q = 1$, the representations V_n described above are exactly the irreducible finite dimensional representations of the usual Lie algebra $\mathfrak{sl}_2$, where one identifies

$$E \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \frac{K - K^{-1}}{q - q^{-1}} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14}$$

Of course, one needs to be a bit careful about interpreting the third identification here, since it looks like you divide by 0. This issue is addressed in [CP, Chapters 9 and 11]. For us, this observation will be sufficient justification for thinking of $U_q(\mathfrak{sl}_2)$ as related to ordinary $\mathfrak{sl}_2$.

Comment 3.4. Notice that K acts as the identity on all V_n at $q = 1$. $U_q(\mathfrak{sl}_2)$ actually has some other finite dimensional representations where K does not act as the identity at $q = 1$. So we have not described the full category of finite dimensional representation of $U_q(\mathfrak{sl}_2)$, but only the so called "type **1**" representations. The other representations rarely appear in the theory.

3B. **Ribbon elements and quantum traces.** Much of the following construction can be found in, for example, [CP, Chapter 4] or [O]. The main difference here is that we work with two ribbon elements throughout. Each satisfies the definition of a ribbon element as in [CP]. Consequently we also have two different quantum traces, and two different co-quantum traces. The non-standard ribbon element Q_t is discussed extensively in [ST].

Definition 3.5. *The ribbon elements Q_s and Q_t are elements in some completion of $U_q(\mathfrak{sl}_n)$ defined by*

- *The standard ribbon element Q_s acts on V_n as multiplication by the scalar $q^{-n^2/2-n}$.*
- *The "non-standard" or "half-twist" ribbon element Q_t acts on V_n as multiplication by the scalar $(-1)^n q^{-n^2/2-n}$.*

Comment 3.6. One can also think of Q_s or Q_t as a natural system of automorphisms of each finite dimensional type **1** representations of $U_q(\mathfrak{sl}_2)$. Specifically, Q_s is the system which acts on V_n as multiplication by $q^{-n^2/2-n}$. Similarly, Q_t is the system which acts on V_n as multiplication by $(-1)^n q^{-n^2/2-n}$.

Definition 3.7. *The "grouplike elements" associated to Q_s and Q_t are elements in some completion of $U_q(\mathfrak{sl}_n)$ defined by*

- *g_s acts on $v_{n-2j} \in V_n$ as multiplication by q^{n-2j}.*
- *g_t acts on $v_{n-2j} \in V_n$ as multiplication by $(-1)^n q^{n-2j}$.*

Comment 3.8. The group like elements in Definition 3.7 are related to the ribbon elements in Definition 3.5 as described in [CP, Chapter 4.2C].

Definition 3.9. *(see* [O, Section 4.2]*) Define the following maps:*

(i) *ev is the evaluation map $V^* \otimes V \to F$.*
(ii) *qtr_{Q_s} is the standard quantum trace map $V \otimes V^* \to F$ defined by, for $\phi \in End(V) = V \otimes V^*$, $qtr_{Q_s}(\phi) = trace(\phi \circ g_s)$.*
(iii) *qtr_{Q_t} is the "half-twist" quantum trace map $V \otimes V^* \to F$ defined by, for $\phi \in End(V) = V \otimes V^*$, $qtr_{Q_t}(\phi) = trace(\phi \circ g_t)$.*
(iv) *$coev$ is the coevaluation map $F \to V \otimes V^*$ defined by $coev(1) = Id$, where Id is the identity map in $End(V) = V \otimes V^*$.*
(v) *$coqtr_{Q_s}$ is the standard co-quantum trace map $F \to V^* \otimes V$ defined by*

$$coqtr_{Q_s}(1) = (1 \otimes g_s^{-1}) \circ Flip \circ coev(1),$$

where $Flip$ means interchange the two tensor factors.

(vi) *$coqtr_{Q_t}$ is the "half-twist" co-quantum trace map $F \to V^* \otimes V$ defined by*

$$coqtr_{Q_t}(1) = (1 \otimes g_t^{-1}) \circ Flip \circ coev(1).$$

Comment 3.10. Although this may not be obvious, the maps in Definition 3.9 are all morphisms of $U_q(\mathfrak{sl}_2)$ representations.

Comment 3.11. It is often useful to express the maps from Definition 3.9 in coordinates. So, fix $f \in V^*$, $v \in V$, and $\{e^i\}$, $\{e_i\}$ be dual basis for V^* and V. Then:

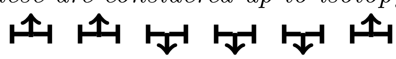
$$\begin{aligned} ev(f \otimes v) &= f(v), \\ qtr_Q(v \otimes f) &= f(gv), \\ coev(1) &= \sum_i e_i \otimes e^i, \\ coqtr_Q(1) &= \sum_i e^i \otimes g^{-1} e_i. \end{aligned} \tag{15}$$

One can choose Q to be either Q_s or Q_t, and then one must use the grouplike element g_s or g_t accordingly.

3C. **Two topological categories.** Quantum group knot invariants work by constructing a functor from a certain topological category to the category of representations of the quantum group. We now define the relevant topological category. In fact, we will need two slightly different topological categories.

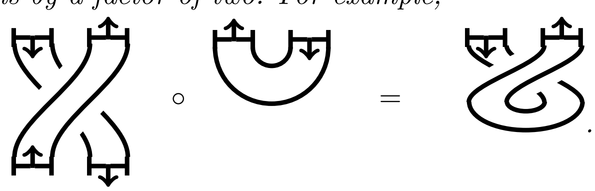
Definition 3.12. *$\mathcal{DRIBBON}$ (directed orientable topological ribbons) is the category where:*

• *An object consists of a finite number of disjoint closed intervals on the real line each directed either up or down. These are considered up to isotopy of the real line. For example:*

 .

‘ • *A morphism between two objects A and B consists of a "tangle of orientable, directed ribbons" in $\mathbb{R}^2 \times I$, whose "loose ends" are exactly $(A, 0, 0) \cup (B, 0, 1) \subset \mathbb{R} \times \mathbb{R} \times I$, such that the direction (up or down) of each interval in $A \cup B$ agrees with the direction of the ribbon whose end lies at that interval. These are considered up to isotopy. For technical details of the definition of "a ribbon", see [CP].*

• *Composition of two morphisms is given by stacking them on top of each other, and then shrinking the vertical axis by a factor of two. For example,*

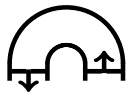
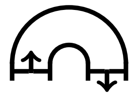


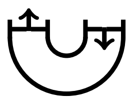
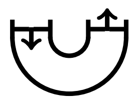
Definition 3.13. *$\mathcal{RIBBON}$ (undirected orientable topological ribbons) is the category obtained from $\mathcal{DRIBBON}$ by forgetting the directings. So an object consists of a finite number of disjoint closed intervals on the real line, a morphism consists of a tangle of undirected ribbons, and composition is still stacking of tangles.*

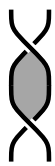
3D. **The functor.** The following is the main ingredient in the quantum group construction of knot invariants. It holds in much greater generality than stated here, which allows for the construction of a great many invariants.

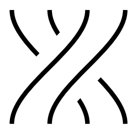
Theorem 3.14. *(see* [CP, Theorem 5.3.2]*) Let V be the standard 2 dimensional representation of $U_q(\mathfrak{sl}_2)$. For each ribbon element Q (i.e. Q_s or Q_t), there is a unique monoidal functor $\mathcal{F}_Q$ from $\mathcal{RIBBON}$ to $U_q(\mathfrak{sl}_2)$-rep such that*

(i) $\mathcal{F}_Q($  $) = V$ *and* $\mathcal{F}_Q($  $) = V^*$,

(ii) $\mathcal{F}_Q\Big($  $\Big) = ev$, $\mathcal{F}_Q\Big($  $\Big) = qtr_Q$,

$\mathcal{F}_Q\Big($  $\Big) = coev$, $\mathcal{F}_Q\Big($  $\Big) = coqtr_Q$,

(iii) $\mathcal{F}_Q\Big($  $\Big) = Q$, *thought of as an automorphism of either* V *or* V^*.

(iv) $\mathcal{F}_Q\Big($  $\Big) = \sigma^{br}$

as a morphism from the tensor product of the bottom two objects to the tensor product of the top two objects, regardless of the directions of the ribbons. □

Comment 3.15. If one or both of the ribbons is directed down, one must be cautious using Definition 3.1 to calculate σ^{br}; one must first choose an explicit isomorphism from V^* to V. By naturality, the resulting morphism σ^{br} will not depend on this choice. This technicality comes up again in Example 4.6, where we deal with it in detail.

Comment 3.16. Notice that $\mathcal{F}_Q$ sends the *negative* full twist to Q. This seems like a strange way to set things up, but it is done to match other fairly standard conventions. In some ways it works well; Q_s acts as multiplication by q to a negative exponent, so positive twists correspond to positive exponents.

Let L be a directed framed link. Then one can draw L as a composition of the elementary features in Theorem 3.14, and hence find the morphism associated to L. This is a morphism from the identity object to itself in the category of $U_q(\mathfrak{sl}_2)$ representations, which is just multiplication by a rational function in $q^{1/2}$ (and this turns out to be a polynomial in $q^{1/2}$ and $q^{-1/2}$). By Theorem 3.14, $\mathcal{F}_Q$ is well defined on tangles up to isotopy. In particular, $\mathcal{F}_Q(L)$, is an isotopy invariant. It is related to the Kauffman bracket as follows:

Theorem 3.17. *(see* [O, Theorem 4.19]*) Fix a framed link* L*. Then* $\mathcal{F}_{Q_s}(L)$ *is independent of the choice of directing of* L*. Furthermore,* $\mathcal{F}_{Q_s}(L) = (-1)^{w(L)+\#L}K(L)$*, where* $w(L)$ *is the writhe of* L *and* $\#L$ *is the number of components of* L*.* □

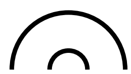
Comment 3.18. Theorem 3.17 is not hard to prove using Corollary 4.4 below and the observation that $\mathcal{F}_{Q_s}(L)/\mathcal{F}_{Q_t}(L)$ is an isotopy invariant that cannot tell the difference between overcrossings and undercrossings, and hence can only depend on the number of components of L and the writhe of each component mod 2 (see also [ST, Proposition 5.22]). Note that $w(L)$ mod 2 does not depend on the directing of L.

The $(-1)^{w(L)+\#L}$ in Theorem 3.17 is the sign referred to in the title of these notes. It is certainly explicitly defined, so in some sense it is not a problem; just an annoyance. Section 4 develops one way to get rid of this sign by using Q_t in place of Q_s, although in some sense this just moves the annoyance into the definition of the ribbon element. The real justification for using Q_t is not so much that it explains the sign, but that it makes Theorem 4.1 functorial.

4. Matching the two constructions using the non-standard ribbon element

We now show how the skein relations used in defining the Kauffman bracket can be explained using the quantum group formulation. This section is similar to [O, Appendix H], although the presentation is simplified by use the non-standard ribbon element Q_t throughout. The idea is to modify the functor from Theorem 3.14 to obtain a function from $\mathcal{RIBBON}$ to $U_q(\mathfrak{sl}_2)-$rep, as opposed to from $\mathcal{DRIBBON}$. One argument for wanting this is that the Kauffman bracket is defined for framed but undirected links, which are morphisms in $\mathcal{RIBBON}$, but not in $\mathcal{DRIBBON}$.

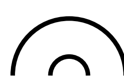
There is only one "elementary" object in $\mathcal{RIBBON}$ (the single interval), as opposed to two in $\mathcal{DRIBBON}$ (the single interval, but with two possible directions). Our morphism will send this single interval to the two dimensional representation V. We must then send each feature in the knot diagram to a morphism between the appropriate tensor powers of V. For instance,

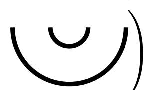


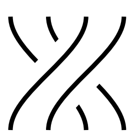
should be sent to a morphism from $V \otimes V$ to the trivial object. This is as opposed to the directed case, where such "caps" are sent to morphisms from $V^* \otimes V$ or $V \otimes V^*$ to the trivial object. To do this, we will use the fact that, in this particular situation, V is isomorphic to V^* (for instance, via the isomorphism from Example 3.2). We obtain:

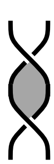
Theorem 4.1. *Choose an isomorphism $f : V \to V^*$. There is a unique functor $\mathcal{F}_f$: $\mathcal{RIBBON} \to U_q(\mathfrak{sl}_2)$-rep such that*

(i) $\mathcal{F}_f$ *takes the object consisting of a single interval to V,*

(ii) $\mathcal{F}_f$ () $= ev \circ (f \otimes Id) = qtr_{Q_t} \circ (Id \otimes f) : V \otimes V \to \mathbb{C}(q)$,

(iii) $\mathcal{F}_f$ () $= (Id \otimes f^{-1}) \circ coev = (f^{-1} \otimes Id) \circ coqtr_{Q_t} : \mathbb{C}(q) \to V \otimes V$,

(iv) $\mathcal{F}_f$ () $= \sigma^{br}$,

(v) $\mathcal{F}_f$ () $= Q_t$ *(or, equivalently, multiplication by $-q^{-3/2}$).*

Furthermore, for any link L, any choice of directing of L, and any choice of f, $\mathcal{F}_f(L) = \mathcal{F}_{Q_t}(L)$.

Comment 4.2. Theorem 4.1 implies that, for any link L, $\mathcal{F}_f(L)$ is independent of the chosen isomorphism f. However, the functor $\mathcal{F}_f$ does depend on this choice. For instance, $\mathcal{F}_f$ applied to a cap clearly depends on f.

Comment 4.3. Quantum trace and co-quantum trace depend on the ribbon element, and the subscript indicates that we are using the ribbon element Q_t. If one tries to use Q_s instead of Q_t, then the two expressions on the right sides in Theorem 4.1 parts (ii) and (iii) are off by a minus sign, and the construction does not work. That the two sides of (ii) and (iii) agree follows from the fact that $U_q(\mathfrak{sl}_2) - \text{rep}$, along with "pivotal structure" related to the ribbon element Q_t, is unimodal, as defined in [T]. For an explanation of this pivotal structure and a proof that it is unimodal see [ST, Section 5B]. It is also not hard to directly verify that the expressions agree.

Proof of Theorem 4.1. This proof is a bit informal. You should draw a non-trivial element of $\mathcal{RIBBON}$, then follow what is being said.

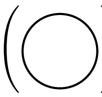
First, verify by a direct calculation that the two expressions on the right for parts (ii) and (iii) agree. Thus $\mathcal{F}_f$ is well defined on framed link diagrams. We will now show that it agrees with $\mathcal{F}_{Q_t}$ calculated with respect to any directing. Since $\mathcal{F}_{Q_t}$ is a functor, this implies that $\mathcal{F}_f$ is as well.

Fix a directing of L. Insert $f \circ f^{-1}$ into $\mathcal{F}_{Q_t}(L)$ somewhere along every segment of L that is directed down. This clearly doesn't change the morphism. By the naturality of σ^{br},

$$(1 \otimes f) \circ \sigma^{br} = \sigma^{br} \circ (f \otimes 1). \tag{16}$$

Use this to pull all the f and f^{-1} through crossings until they are right next to cups and caps. But now you are exactly calculating $\mathcal{F}_f(L)$. Hence $\mathcal{F}_f = \mathcal{F}_{Q_t}$. $\square$

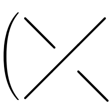
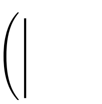
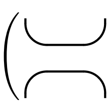
We are now ready to see how skein relations appear. For the rest of this section we use the blackboard framing (i.e. ribbons lie flat on the page, and are drawn simply as lines). A simple calculation shows that

(17) $\mathcal{F}_f\Bigg($  $\Bigg) =$ multiplication by $-q-q^{-1}$.

Another direct calculation shows that

$$\sigma^{br} = q^{1/2} Id + q^{-1/2}(\mathrm{Id} \otimes f^{-1}) \circ coev \circ qtr_{Q_t} \circ (\mathrm{Id} \otimes f) : V \otimes V \to V \otimes V. \tag{18}$$

Equivalently,

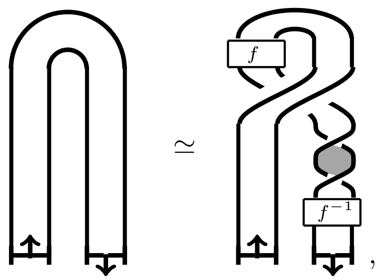
(19) $\mathcal{F}_f\Bigg($  $\Bigg) = q^{1/2}\mathcal{F}_f\Bigg($  $\Bigg) + q^{-1/2}\mathcal{F}_f\Bigg($  $\Bigg)$.

But these are exactly the relations used in Definition 2.1 to define the Kauffman bracket! Noticing that Equation (1) and Theorem 4.1(v) are also compatible, this implies that $\mathcal{F}_f$ of a framed but undirected link gives the Kauffman bracket. Applying Theorem 4.1 we see:

Corollary 4.4. *Let L be a framed link. Then $\mathcal{F}_{Q_t}(L)$ is independent of the chosen directing, and is equal to the Kauffman bracket $K(L)$.* $\square$

Comment 4.5. The non-standard ribbon we use exists in many cases beyond $U_q(\mathfrak{sl}_2)$, and can also help explain the correspondence between various constructions of knot polynomials in those cases.

Example 4.6. A way to verify the definition of quantum trace. Recall that $\mathcal{F}_Q$ is supposed to be defined on $\mathcal{DRIBBON}$, and morphisms there are ribbon tangles *up to isotopy*. One can use an isotopy to change a right going cap to the composition of a twist, a crossing, and a left going cap. Since we have only explicitly defined σ^{br} acting on $V \otimes V$, not acting on $V \otimes V^*$, we also put in copies of f and f^{-1}, where f is the isomorphism from Example 3.2. By the naturality of the braiding, this does not affect the morphism. Diagrammatically,



where the boxes in the diagram mean "put in a copy of the isomorphism f when you apply $\mathcal{F}_Q$." Such "tangles with coupons" are defined precisely in e.g. [CP]. Algebraically, this says

$$\text{qtr}_Q = \text{ev} \circ (f \otimes \text{Id}) \circ \sigma^{br} \circ (\text{Id} \otimes Q^{-1}) \circ (\text{Id} \otimes f^{-1}). \tag{20}$$

Since the action of each element on the right side has been explicitly defined, one can now check that the two sides agree on all basis vectors. Note that both sides depend on the choice of ribbon element Q_s or Q_t.

Comment 4.7. For our purposes, we could simply use the calculation in Example 4.6 to define qtr. However, if we were to more fully develop the theory, the fact that qtr can be defined as $\text{qtr}_Q(\phi) = trace(\phi \circ g)$ for a *grouplike* element g (see Definition 3.9) is important. The reason is that this implies quantum trace is multiplicative on tensor products. See [CP, Remark 1 after Definition 4.2.9].

5. Another advantage: the half twist

We now discuss an invertible element X in a certain completion of $U_q(\mathfrak{sl}_2)$, which is related to the non-standard ribbon element by $Q_t = X^{-2}$. As discussed in [ST], this element has an interesting topological interpretation.

Definition 5.1. *X is the operator that acts on V_n by $Xv_{n-2j} = (-1)^{n-j} q^{n^2/4+n/2} v_{-n+2j}$.*

Comment 5.2. There is actually some choice in how we define X: the operator X' that acts on V_n by $X'v_{n-2j} = i^n q^{n^2/4+n/2} v_{-n+2j}$, where i is the complex number i, also has all the properties discussed below. This type of modification is discussed in [ST, Section 5C] and [KT, Section 8].

One can easily check that $X^{-2} = Q_t$. Comparing with Theorem 3.14(iii), one may hope that X could be interpreted as an isomorphism, and that the functor $\mathcal{F}_{Q_t}$ could be extended

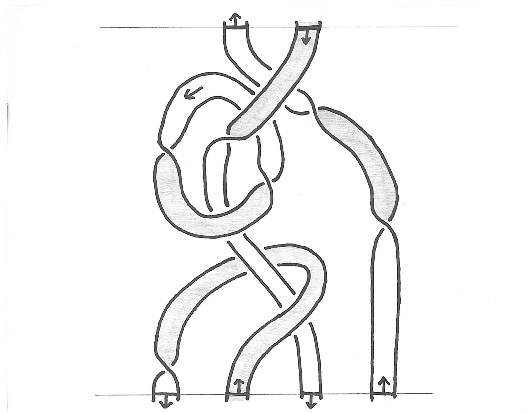


FIGURE 1. A morphism in the topological category of ribbons with half twists

in such a way that

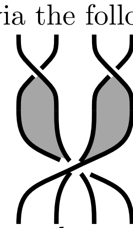
(21) $$\mathcal{F}_{Q_t}\left(\right) = X^{-1}.$$

Another indication that such an extended functor should exist comes from the following result of Kirillov-Reshetikhin [KR, Theorem 3] and Levendorskii-Soibelman, [LS, Theorem 1] (see [KT, Comment 7.3] for this exact statement).

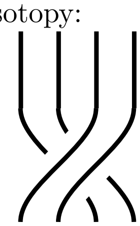
Theorem 5.3. $\sigma^{br} = (X^{-1} \otimes X^{-1}) \circ \mathit{Flip} \circ \Delta(X)$. □

Theorem 5.3 can be interpreted via the following isotopy:

(22)



$\simeq$



.

Here $\mathrm{Flip} \circ \Delta(X)$ should be interpreted as a morphism corresponding to twisting both ribbons at once by 180 degree, as on the bottom of the left side.

Such an extended functor has been defined precisely in [ST], resulting in a functor from a larger category where ribbons are allowed to twist by 180 degrees, not just by 360 degrees (although Mobius bands are still not allowed). Figure 1 shows an example of a morphism in the resulting category. Notice that elementary objects come in both shaded and unshaded versions.

The construction in [ST] can only extend $\mathcal{F}_{Q_t}$, not $\mathcal{F}_{Q_s}$. We feel this gives more evidence that Q_t is natural. One advantage of having such an extended functor is that, since both σ^{br} and Q_t are constructed in term of the "half-twist" X, there is in some sense one less elementary feature.

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