Loyola University Chicago
Math 161, Section 002, Fall 2009

Quiz 5 Solutions

Name (print): ______________________ **Signature:** ______________________

You have 30 minutes. Show your work. Notes not allowed! Problems are on both sides of this sheet.

Problem 1. (9 pts) Find the following limits:

$\lim\limits_{x\to 0} \frac{2x}{a^x - 1}$, where $a > 0$ is a constant

Solution:

$$\lim_{x\to 0} \frac{2x}{a^x-1} = \lim_{x\to 0} \frac{2}{\ln a\, a^x} = \frac{2}{\ln a}$$

$\lim\limits_{x\to\infty} \left(1 + \frac{1}{2x}\right)^{3x}$

Solution:

$$\begin{aligned}
\lim_{x\to\infty} \left(1+\frac{1}{2x}\right)^{3x} &= \lim_{x\to\infty} e^{\ln\left(\left(1+\frac{1}{2x}\right)^{3x}\right)} = \lim_{x\to\infty} e^{3x\ln(\left(1+\frac{1}{2x}\right)} = e^{\lim_{x\to\infty} 3x\ln\left(1+\frac{1}{2x}\right)} \\
&= e^{3\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{2x}\right)}{\frac{1}{x}}} = e^{3\lim_{y\to 0} \frac{\ln(1+\frac{y}{2})}{y}} = e^{3\lim_{y\to 0} \frac{\frac{1}{1+\frac{y}{2}}\frac{1}{2}}{1}} \\
&= e^{\frac{3}{2}}
\end{aligned}$$

$\lim\limits_{t\to 0} \frac{t\sin t}{1-\cos t}$

Solution:

$$\lim_{t\to 0} \frac{t\sin t}{1-\cos t} = \lim_{t\to 0} \frac{\sin t + t\cos t}{\sin t} = \lim_{t\to 0} \frac{\cos t + \cos t - t\sin t}{\cos t} = 2$$

Problem 2. (5 pts) You are designing a rectangular poster to contain 50 square inches of printing with a 4 inch margin at the top and bottom and a 2 inch margin at each side. What overall dimensions will minimize the amount of paper used?

Solution: Let x be the height of the whole poster, let y be the width of the whole poster. We need to minimize $A = xy$. Printed area being 50 means that $(x-8)(y-4) = 50$, so $y = 4 + \frac{50}{x-8}$. Then

$$A(x) = x\left(4 + \frac{50}{x-8}\right)$$

Need to minimize this function over $x > 8$.

$$A'(x) = 4 - \frac{400}{(x-8)^2}, \qquad \textit{so } A'(x) = 0 \textit{ gives } x = 18.$$

Is this really a minimum?

$$A''(x) = \frac{800}{(x-8)^3} > 0$$

so the function is concave up, so $x = 18$ is the absolute minimum. When $x = 18$, $y = 9$.

Problem 3. (6 pts) Let a and b be constants and consider the function

$$f(x) = \frac{x+a}{x^2+b^2}.$$

Find and identify critical points of f and say, using interval notation, where the function is increasing and where the function is decreasing.

Solution:

$$f'(x) = \frac{-x^2 - 2ax + b^2}{(x^2+b^2)^2}$$

$f'(x) = 0$ has two solutions:

$$x_1 = -a - \sqrt{a^2+b^2}, \quad x_2 = -a + \sqrt{a^2+b^2}.$$

The sign of $f'(x)$ is determined by the denominator, which is a quadratic function with a negative coefficient at x^2 (so upside-down parabola). Hence, function is decreasing on

$$(-\infty, x_1]$$

and on

$$[x_2, \infty)$$

and it is increasing on

$$[x_1, x_2].$$

Also, x_1 is a local minimum, x_2 is a local maximum.