Loyola University Chicago
Math 201, Fall 2011

Quiz 2 Solutions

Name (print): ____________________ **Signature:** ____________________

You have 30 minutes. Show your work. Notes not allowed! Problems are on both sides of this sheet.

Problem 1. (4 pts) Five men and a monkey ... well, OK, not. Convert 291 to base 4.

Solution:

$$291 = 4 \cdot 72 + 3,\ \ 72 = 4 \cdot 18 + 0,\ \ 18 = 4 \cdot 4 + 2,\ \ 4 = 4 \cdot 1 + 0$$

so

$$291 = 4 \cdot 72 + 3 = 4(4 \cdot 18) + 3 = 4(4(4 \cdot 4 + 2)) + 3 = 4^4 + 2 \cdot 4^2 + 3 \cdot 4^0 = (10203)_4$$

Problem 2. (6 pts) Find at least three different nonnegative integer pairs (x, y) that solve the equation

$26x - 18y = 20$

Solution: Extended Euclidean Algorithm shows that $26(-2) + 18(3) = 2$*, so* $26(-2) - 18(-3) = 2$*, so* $26(-20) - 18(-30) = 20$*. EEA also shows that* $26(9) + 18(-13) = 0$*, so* $26(9) - 18(13) = 0$*, so* $26(9k) - 18(13k) = 0$ *for any integer* k*. Then*

$$26(-20 + 9k) - 18(-30 + 13k) = 20,$$

so all solution pairs are given by $x = -20 + 9k$*,* $y = -30 + 13k$*. Both* x *and* y *are positive when* $k \geq 3$*. Some solution pairs, corresponding to* $k = 3, 4, 5$*, are then:*

$$x = 7, y = 9, \quad x = 16, y = 22, \quad x = 25, y = 35.$$

Problem 3. (4 pts) Can 2431 be written as a sum of two positive integers, one of which is divisible by 42 and the other is divisible by 91? *(Say yes or no, and justify your answer.)*

Solution: the question is really this: are there positive integer solutions x and y to

$$42x + 91y = 2431.$$

There are integer solutions to this equation if and only if gcd$(42, 91)$ *divides* 2431*. But gcd*$(42, 91) = 7$ *and* $2431/7 = 347.285\ldots$*, so* 7 *does not divide* 2431*, so the answer is NO.*

Problem 4. (6 pts) Prove that $\gcd(a, c) = \gcd(b, c) = 1$ if and only if $\gcd(ab, c) = 1$.

Solution:

First, we prove that gcd(a, c) *=gcd*$(b, c) = 1$ *implies that gcd*$(ab, c) = 1$*. Note that* $1|ab$ *and* $1|c$*. It is then enough to show that there exist integer* x *and* y *such that* $abx + cy = 1$*. Because gcd*(a, c) *=gcd*$(b, c) = 1$*, there exist integers* k, l, m, n *such that*

$$ak + cl = 1, \quad bm + cn = 1.$$

Multiply these two equations by one another, get

$$abkm + ackn + cblm + ccln = 1,$$

which can be factored like this:

$$ab(km) + c(akn + blm + cln) = 1.$$

Hence, $x = km$*,* $y = akn + blm + cln$*.*

Second, to prove that gcd$(ab, c) = 1$ *implies gcd*(a, c) *=gcd*$(b, c) = 1$*, note that the assumption implies that there exist integer* x*,* y *such that*

$$abx + cy = 1.$$

This can be written as

$$a(bx) + cy = 1, \quad b(ax) + cy = 1,$$

and hence there exist integer solutions k, l, m, n *to equations* $ak + cl = 1$*,* $bm + cn = 1$*. This implies gcd*(a, c) *=gcd*$(b, c) = 1$*.*

Alternative justification: gcd$(a, c) = 1$ *implies that* a *and* c *have no common prime factors. gcd*$(b, c) = 1$ *implies that* b *and* c *have no common prime factors. Prime factors of* ab *consist of prime factors of* a *and of prime factors of* b*, hence there are no common prime factors of* ab *and* c*.*