Loyola University Chicago
Math 161-005, Spring 2013

Midterm 2 Sample C solutions

Name (print): ____________________ **Signature:** ____________________

Please do not start working until instructed to do so.

You have 75 minutes.

You must show your work to receive full credit.

No calculators.

You may use one one-sided 8.5 by 11 sheet of handwritten (by you) notes.

Problem 1. ________

Problem 2. ________

Problem 3. ________

Problem 4. ________

Problem 5. ________

Problem 6. ________

Problem 7. ________

Total. ________

Problem 1. *(30 points)* Find the following derivatives. Put a $\boxed{\text{box around your final answer}}$.

a. *(5 points)* $\left(4x^5 - 7\sqrt{x} + 3^x\right)'$

Solution:

$$\left(4x^5 - 7\sqrt{x} + 3^x\right)' = 20x^4 - \frac{7}{2\sqrt{x}} + \ln 3\, 3^x$$

b. *(5 points)* $\frac{d}{dx}\left(x^3\cos(2x) - \frac{\sin(x)}{x-1}\right)$

Solution:

$$\frac{d}{dx}\left(x^3\cos(2x) - \frac{\sin(x)}{x-1}\right) = 3x^2\cos(2x) + x^3(-2\sin(2x)) - \frac{\cos(x)(x-1) - \sin(x)}{(x-1)^2}$$

c. *(5 points)* $\frac{dy}{dx}$, in terms of x and y, if $xy^2 + y^3 = 6x - 8$

Solution: implicitly differentiate and solve for y':

$$y^2 + x2yy' + 3y^2y' = 6, \quad y'(2xy + 3y^2) = 6 - y^2, \quad y' = \frac{6 - y^2}{2xy + 3y^2}$$

d. *(5 points)* $\frac{d}{dx}\left(\arccos(4 + x^5)\right)$

Solution:

$$\frac{d}{dx}\left(\arccos(4 + x^5)\right) = -\frac{5x^4}{\sqrt{1 - (4 + x^5)^2}}$$

e. *(5 points)* $\frac{d}{dy}\left(\arctan\left(\frac{\arcsin(78)}{e^\pi}\right)\right)$

Solution: the whole mess in the brackets is a constant, so

$$\frac{d}{dy}\left(\arctan\left(\frac{\arcsin(78)}{e^\pi}\right)\right) = 0$$

f. *(5 points)* $\frac{d}{dz}\left(x^3\cos(z) - \ln(z) + \frac{z^2}{y}\right)$

Solution:

$$\frac{d}{dz}\left(x^3\cos(z) - \ln(z) + \frac{z^2}{y}\right) = -x^3\sin(z) - \frac{1}{z} + \frac{2z}{y}$$

Problem 2. *(10 points total)* Use linear approximation to approximate $\sqrt{23}$.

Solution: Notice that $\sqrt{25} = 5$ *and so we can rely on the tangent line approximation to* $f(x) = \sqrt{x}$ *at* $x = 25$*. Find this tangent line:* $f'(x) = \frac{1}{2\sqrt{x}}$, $f'(5) = \frac{1}{10}$*, and so the equation of the tangent line is*

$$y - 5 = \frac{1}{10}(x - 25), \quad y = 5 + \frac{1}{10}(x - 25).$$

Now plug $x = 23$ *into this equation, get*

$$\sqrt{23} \approx 5 + \frac{1}{10}(23 - 25) = 5 - \frac{2}{10} = 4.8.$$

Note that this approximation is pretty good: the exact answer is $\sqrt{23} = 4.79583152331\ldots$*, so the approximation is off by* $0.00416847668\ldots$.

Problem 3. *(10 points total)* Simplify $\cot(\arccos(3x))$ to an expression not involving trig functions.

Solution: Let $\theta = \arccos(3x)$*, so that* $\cos(\theta) = 3x$*. Draw a right triangle representing this information, with one angle being* θ*, adjacent side* $3x$ *and hypotenuse* 1*. Then the opposite side is* $\sqrt{1 - (3x)^2}$*. Consequently,*

$$\cot(\arccos(3x)) = \cot(\theta) = \frac{3x}{\sqrt{1 - (3x)^2}}$$

Problem 4. *(10 points)* Find the equations of the tangent line and of the normal line to the ellipse $3(x + y)^2 + x^2 = 28$ at the point $(1, 2)$.

Solution: First, find the slope of the ellipse at $(1, 2)$ *through implicit differentiation:*

$$3\,2(x + y)(1 + y') + 2x = 0, \quad 6(1 + 2)(1 + y') + 2 = 0, \quad 1 + y' = -\frac{2}{18}, \quad y' = -1 - \frac{1}{9} = -\frac{10}{9}.$$

The slope of the tangent line is then $-\frac{10}{9}$ *and the slope of the normal line is the negative reciprocal, so* $\frac{9}{10}$*. The equation of the tangent line and of the normal line are then*

$$y - 2 = -\frac{10}{9}(x - 1), \quad y - 2 = \frac{9}{10}(x - 1),$$

which can be simplified further.

Problem 5. *(10 points total)* Write the definition of the derivative of a function f at a point x and then use this definition to find the derivative of $f(x) = \dfrac{2}{3x+4}$.

Solution: The derivative $f'(x)$ is the limit below, if it exists:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{2}{3(x+h)+4} - \frac{2}{3x+4}}{h} \\ &= \lim_{h\to 0} \frac{2(3x+4) - 2(3(x+h)+4)}{h(3(x+h)+4)(3x+4)} = \lim_{h\to 0} \frac{-6h}{h(3(x+h)+4)(3x+4)} \\ &= \lim_{h\to 0} \frac{-6}{(3(x+h)+4)(3x+4)} \\ &= \frac{-6}{(3x+4)^2} \end{aligned}$$

Problem 6. *(10 points)* For what value of c is the curve $y = \dfrac{c}{x+1}$ tangent to the line through the points $(0,3)$ and $(5,-2)$.

Solution: The line through $(0,3)$ and $(5,-2)$ has slope $\frac{0-5}{3-(-2)} = -1$ and the equation

$$y - 3 = -1(x-0), \quad y = 3 - x.$$

We need to have the curve $y = \dfrac{c}{x+1}$ be tangent to this line, so in the first place, the curve needs to have slope -1, and so

$$y' = -\frac{c}{(x+1)^2} = -1, \quad c = (x+1)^2.$$

Hmmm... c depends on x? Odd. The curve also needs to touch the line at the point of tangency, so the value of $y = \dfrac{c}{x+1}$ and of $y = 3-x$ should be the same. Use $c = (x+1)^2$ in the equation of the curve to get

$$\frac{c}{x+1} = \frac{(x+1)^2}{x+1} = x+1 = 3-x$$

and solve to get $x = 1$. Then $c = (1+1)^2 = 4$.

A more rigorous way to do this: the line through $(0,3)$ and $(5,-2)$ is $y = 3-x$. Suppose that this line is tangent to the curve $y = \dfrac{c}{x+1}$ at the point where $x = a$. Then the values are the same and the slopes are the same, so

$$\frac{c}{a+1} = 3 - a \quad \text{and} \quad -\frac{c}{(a+1)^2} = -1.$$

Solve to get $a = 1$ (which we were not asked about) and $c = 4$.

Problem 7. *(10 points)* Consider the hyperbola $(x+2y)(2x+y)=-1$. Find all points on it at which the tangent line to the hyperbola is horizontal.

Solution: Horizontal tangent lines are where $\frac{dy}{dx}=0$. *Use implicit differentiation and product rule to get*

$$(1+2y')(2x+y)+(x+2y)(2+y')=0.$$

Set $y'=0$ *and simplify to get*

$$2x+y+2x+4y=0, \quad 4x+5y=0, \quad y=-\frac{4}{5}x.$$

Plug this y *into the equation of the hyperbola* $(x+2y)(2x+y)=-1$*, get*

$$(x-\frac{8}{5}x)(2x-\frac{4}{5}x)=-1, \quad -\frac{3}{5}x\frac{6}{5}x=-1, \quad \frac{18}{25}x^2=1, \quad x^2=\frac{25}{18},$$

and so $x=\dfrac{5}{2\sqrt{3}}$ *or* $x=-\dfrac{5}{2\sqrt{3}}$. *Now recall that* $y=-\frac{4}{5}x$ *and so the points where the tangent line is horizontal are*

$$\left(\frac{5}{2\sqrt{3}},-\frac{4}{2\sqrt{3}}\right), \quad \left(-\frac{5}{2\sqrt{3}},\frac{4}{2\sqrt{3}}\right).$$

Extra credit: Given your solution above, find a <u>*quick*</u> *way to find all points in this hyperbola at which the tangent line to the hyperbola is vertical.*

Solution: Use symmetry! Note that switching x *to* y *and* y *to* x *in the equation of the hyperbola* $(x+2y)(2x+y)=-1$ *gives exactly the same equation. Geometrically, this switch corresponds to reflecting the hyperbola about the line* $y=x$. *The fact that the formula is the same means the hyperbola is symmetric about the line* $y=x$. *This reflection also turns vertical lines to horizontal lines (and vice-versa). We now the horizontal tangents are at the points*

$$\left(\frac{5}{2\sqrt{3}},-\frac{4}{2\sqrt{3}}\right), \quad \left(-\frac{5}{2\sqrt{3}},\frac{4}{2\sqrt{3}}\right).$$

Reflecting back gives us vertical tangent lines at

$$\left(-\frac{4}{2\sqrt{3}},\frac{5}{2\sqrt{3}}\right), \quad \left(\frac{4}{2\sqrt{3}},-\frac{5}{2\sqrt{3}}\right).$$