A Factorization Approach to C^1 Stabilization of Nonlinear Triangular Systems

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***Abstract*— We expand the class of systems to which backstepping is applicable, by allowing states that are used as virtual controls to appear in non-invertible maps. Representing these maps as products of C^1 bijective maps and sign definite C^0 gains, we develop a recursive design which is robust to multiplicative uncertainties. When the linearization of the system is controllable, our design achieves global asymptotic stability, otherwise it guarantees global practical stability. The designed feedback system possesses desirable inverse optimality properties.**

I. Introduction

Much of the recent research for general triangular systems

$$\begin{aligned} \dot{x}_1 &= F_1(x_1, x_2), \\ \dot{x}_2 &= F_2(x_1, x_2, x_3), \\ &\vdots \\ \dot{x}_n &= F_n(x_1, ..., x_n, u), \end{aligned} \tag{1}$$

focused on the existence of stabilizing controls under the least restrictive assumptions about nonlinearities F_i. Coron and Praly [1] initiated this line of work with a sufficient condition for existence of continuous (not necessarily smooth) locally asymptotically stabilizing (LAS) control laws for (1). They proved that such laws exist if all $k_i = \min\{j \in Z^+ \mid \frac{\partial^j F_i}{\partial x_{i+1}^j}(0) \neq 0\}$, $i = 1, .., n$ are finite and odd. Under this condition, Čelikovski and Aranda-Bricaire [10] used a homogeneous approximation of (1) to construct continuous feedback laws to achieve LAS. Tsinias [8] proved the existence of continuous dynamic GAS control laws for systems (1) with $F_i(x_1, ..., x_{i+1}) = \sum_{j=0}^{p_i-1} x_{i+1}^j a_{ij}(x_1, ..., x_i) + x_{i+1}^{p_i}$ where p_i is an odd integer and $a_{ij}(0, ..., 0) = 0$. Applying this existence result, Tzamzi and Tsinias [9] constructed continuous GAS control laws for $F_i(x_1, ..., x_{i+1}) = \sum_{j=1}^{i} c_{ij} x_j + x_{i+1}^{p_i}$. When $F_i(x_1, ..., x_{i+1}) = \overline{F}_i(x_1, ..., x_i) + x_{i+1}^{p_i}$, $|\overline{F}_i(x_1, ..., x_i)| \leq D_i \in I\!R$, and $\overline{F}_i(0, ..., 0) = 0$, they constructed bounded continuous GAS control laws. Further advance was made by Lin and Qian [6] with their 'adding a power integrator' procedure. In [3] they developed an adaptive version of their design. The designs in [8], [9], [6], [3] are for 'power integrators' $x_{i+1}^{p_i}$, where p_i is odd, resulting in non-Lipschitz control laws. Qian and Lin [4], [5] have also designed C^1 GAS control laws for special forms of F_i's, such as, $F_1(x_1, x_2) = x_1^q + x_2^3$ with $q \geq 3$.

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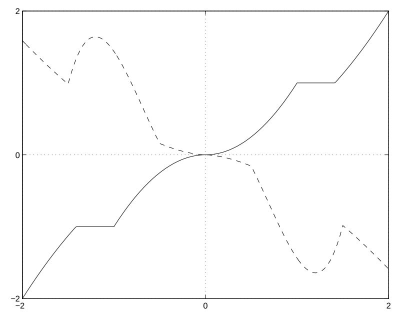

Fig. 1. Representative examples of allowed $\phi_i(x)$

In this paper we expand the class of triangular systems in a direction that has not been explored before. We consider $F_i(x_1, ..., x_{i+1}) = f_i(x_1, ..., x_i) + \phi_i(x_{i+1})$, where ϕ_i is not required to be bijective away from zero. A pair of such nonlinearities ϕ_i is shown in Figure (1). To circumvent the obstacle of non-invertibility of ϕ_i we introduce factorization $\phi_i(s) = \widetilde{\mu}_i(s)\widetilde{\phi}_i(s)$, where $\widetilde{\phi}_i$ is a 'nice' nonlinearity and $\widetilde{\mu}_i$ is a gain specified only by its bounds. When ϕ_i is 'bad' locally, $\phi_i'(0) = 0$, our C^1 feedback laws achieve global practical stability (GPS) of $x = 0$. Our design retains the simplicity of standard backstepping with quadratic Lyapunov functions and expands its applicability to nonlinearities that do not satisfy the Coron-Praly condition at $x = 0$.

Section II describes the factorization of ϕ_i. Section III states the main result and displays three distinct cases with the help of examples. Design procedure and the proof of the main result are in Section IV. Section V extends the class of systems to which our result applies. Optimality properties of the designed control laws are discussed in Section VI, while Section VII contains conclusions.

In what follows, s always denotes a scalar variable, x a vector in $I\!R^n$, x_i its i^{th} component, and $\overline{x}_i \in I\!R^i$ the first i coordinates of x.

II. Factorization

The factorization relies on the following fact.

Lemma 2.1: *Let $\phi : I\!R \mapsto I\!R$ be continuous, $\phi(s) \neq 0$ for $s \neq 0$, and $\lim_{s\to-\infty}\phi(s) = -\lim_{s\to+\infty}\phi(s)$ equals to* *either ∞ or $+\infty$ (automatically, $\phi(0)=0$).*

(a) *Suppose that ϕ is differentiable at $s=0$ with $\phi^{'}(s)\neq 0$. Then there exist C^1 functions $\widetilde{\phi}$, M and a C^0 function $\widetilde{\mu}$ such that, for all $s\in I\!R$,*

$$\phi(s)=\widetilde{\mu}(s)\,\widetilde{\phi}(s), \tag{2}$$

$$\widetilde{\phi}(0)=0 \text{ and } \widetilde{\phi}^{'}(s)>0, \tag{3}$$

$$1\leq\widetilde{\mu}(s)\leq M(s). \tag{4}$$

(b) *Suppose that for some $d>0$, ϕ is differentiable on $(-d,d)$ with $\phi'(0)=0$ but either $\phi'(s)>0$ for all $s\neq 0$ in $(-d,d)$, or $\phi'(s)<0$ for all $s\neq 0$ in $(-d,d)$. Assume moreover that ϕ^{-1} is a well-defined function between $\phi(-d)$ and $\phi(d)$. Then for any δ in $(-d,d)$, there exist C^1 functions $\widetilde{\phi}$, M and a C^0 function $\widetilde{\mu}$ such that*

$$\phi(s+\delta)-\phi(\delta)=\widetilde{\mu}(s)\,\widetilde{\phi}(s)$$

and (3), (4) hold for all $s\in I\!R$.

An explicit formula for $\widetilde{\phi}$ can be given. Consider ϕ as in (a), with $\phi^{'}(0)>0$. For $s>0$ define

$$\widetilde{\phi}(s)=\frac{1}{s}\int_0^s \min_{x\geq t} f(x)\,dt.$$

The integrand is the greatest nondecreasing function bounding ϕ from below. Defining $\widetilde{\phi}$ symmetrically for $s<0$ and setting $\widetilde{\phi}(0)=0$ yields a function with desired properties (note that $\widetilde{\phi}^{'}(0)=\frac{1}{2}\phi^{'}(0)$).

Of course, other constructions are possible. We illustrate this by factoring

$$\phi(s)=\begin{cases}|s|s, & |s|\leq 1,\\ 1, & 1\leq|s|\leq\sqrt{2},\\ \frac{1}{2}|s|s, & |s|\geq\sqrt{2},\end{cases}$$

shown in Figure 1, solid. Note $\phi^{'}(0)=0$ but $\phi^{'}(s)>0$ for $s\in(-1,1)$, $s\neq 0$. We obtain (with $\delta=0.5$)

$$\widetilde{\phi}(s)=\begin{cases}1.572s^3-2.26s^2+1.605s-0.03,\\ \qquad\text{for } \frac{1}{10}\leq s\leq\sqrt{2}-\frac{1}{2},\\ 1.482s^3+2.111s^2+1.178s-0.008,\\ \qquad\text{for } -\frac{7}{10}\leq s\leq -\frac{1}{10},\\ 0.925s^3+3.819s^2+5.861s+2.468,\\ \qquad\text{for } -\sqrt{2}-\frac{1}{2}\leq s\leq -1,\\ \phi(s+\delta)-\phi(\delta), \quad \text{otherwise.}\end{cases}$$

Figure 2 shows $\widetilde{\phi}$ along with $\phi(s+\delta)-\phi(\delta)$ and the corresponding $\widetilde{\mu}(s)\geq 1$ and $M(s)$. A constant bound is obtained, $\max_a\widetilde{\mu}(a)=1.86$. However, this bound is too conservative and may lead to unnecessarily large control signal. Instead, $M(s)=\frac{1.9+0.1s^4}{1+0.1s^4}$ is much less conservative.

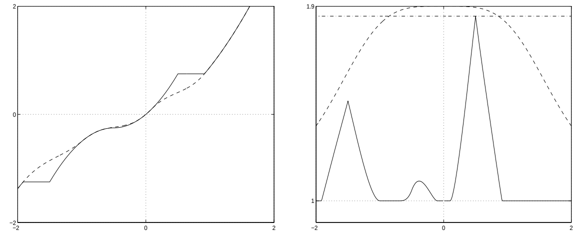

Fig. 2. Left graph: dashed line - $\widetilde{\phi}(x)$; solid line - $\phi(x+\delta)-\phi(\delta)$. Right graph: solid line - $\widetilde{\mu}(x)$; dashed line - $M(x)$; dash-dotted line - $\max_x M(x)$.

III. Main Result and Discussion

With $\phi_i: I\!R\to I\!R$ that satisfy the assumptions of Lemma 2.1, and C^1 maps $f_i: I\!R^i\to I\!R$, $f_i(0)=0$, we consider

$$\begin{aligned}\dot{x}_1&=f_1(x_1)+\phi_1(x_2),\\ \dot{x}_2&=f_2(x_1,x_2)+\phi_2(x_3),\\ &\vdots\\ \dot{x}_n&=f_n(x_1,...,x_n)+\phi_n(u).\end{aligned} \tag{5}$$

Our task is to construct a C^1 feedback control law $u: I\!R^n\to I\!R$ which renders $x=0$ globally asymptotically stable (GAS) if the linearization of (5) at $x=0$ is controllable. If not, then u is to render $x=0$ globally practically stable (GPS).

Standard backstepping does not apply to (5) because it requires that ϕ_i be bijective. Even if bijective, ϕ_i may have points where $\phi_i^{'}=0$, thus disallowing division by $\phi_i^{'}$. Moreover, when $\phi_i^{'}(0)=0$, the linearization of (5) at $x=0$ is not controllable.

Definition 3.1: *An equilibrium of (5) obtained with $u=u^*=const$ is called linearly controllable, and denoted by x^*, if the linearization of (5) at (x^*,u^*) is controllable.*

When $x=0$ of (5) is not linearly controllable, in order to design C^1 GAS control laws, further restrictions must be imposed on f_i, see [4], [5]. Instead, we design C^1 control laws which achieve GPS of $x=0$.

We now state our main result. Its proof is the design procedure in Section IV.

Theorem 3.2: *If $x=0$ in (5) is linearly controllable then a C^1 feedback control law which renders it globally asymptotically stable can be constructed by a recursive procedure. If $x=0$ is not linearly controllable but for any small $\epsilon>0$ there exists a linearly controllable x^*, $\|x^*\|\leq\epsilon$, then a C^1 feedback control law can be constructed to render $x=0$ globally practically stable.*

A key assumption in Theorem 3.2 is the existence of x^*, which we compute starting with a choice of x_1^* and recursively solving the equations

$$f_i(\overline{x}_i^*)+\phi_i(x_{i+1}^*)=0, \quad i=1,..n \tag{6}$$

with $x_{n+1}^*=u^*$. If $x_{i+1}^*\neq 0$, $\forall i=1,n$ this is an acceptable x^*. If $x_{i+1}^*=0$ for some i, the computation can continue provided $\phi_i^{'}(0) > 0$. If $\phi_i^{'}(0) = 0$, a different choice of x_1^* is needed. It may happen that $x_{i+1}^* = 0$ for any small x_1^*, as when $f_i(\overline{x}_i) \equiv 0$ or $f_i(\overline{x}_i) = f_{i-1}(\overline{x}_{i-1}) + \phi_{i-1}(x_i)$. Then, there are no linearly controllable equilibria near $x = 0$.

The three cases which can occur in the design of C^1 control laws for system (5) are now described with the help of examples.

Case of $x^* = 0$. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 + \phi_1(x_2) = x_1 + \widetilde{\mu}_1(x_2)\widetilde{\phi}_1(x_2), \\ \dot{x}_2 &= x_1 x_2 + \phi_2(u) = x_1 x_2 + \widetilde{\mu}_2(u)\widetilde{\phi}_2(u), \end{aligned} \tag{7}$$

where ϕ_1 and ϕ_2 satisfy Lemma 2.1.1 with $1 \le |\widetilde{\mu}_1(x_2)| \le M_1$ and $1 \le |\widetilde{\mu}_2(u)| \le M_2$. Differentiating $V_1 = \frac{1}{2}x_1^2$ and adding and subtracting $x_1\gamma_1(x_1) \triangleq -3x_1^2$ we get

$$\begin{aligned} \dot{V}_1 &= x_1(x_1 + \gamma_1(x_1)) + \widetilde{\mu}_1(x_2)x_1(\widetilde{\phi}_1(x_2) \\ &\quad -\gamma_1(x_1)) + (\widetilde{\mu}_1(x_2) - 1)x_1\gamma_1(x_1) \\ &\le -\tfrac{3}{2}x_1^2 + \tfrac{1}{2}M_1^2(\widetilde{\phi}_1(x_2) + 3x_1)^2. \end{aligned}$$

For $V_2 \triangleq V_1 + \frac{1}{2}(\widetilde{\phi}_1(x_2) + 3x_1)^2$ we obtain

$$\begin{aligned} \dot{V}_2 &\le -\tfrac{1}{2}x_1^2 + (\widetilde{\phi}_1(x_2) + 3x_1)^2\sigma(x_1, x_2) \\ &\quad +\widetilde{\phi}_1^{'}(x_2)(\widetilde{\phi}_1(x_2) + 3x_1)\widetilde{\mu}_2(u)\widetilde{\phi}_2(u), \end{aligned}$$

where $\sigma(x_1, x_2) = 50 + \frac{7}{2}M_1^2 + (\widetilde{\phi}_1^{'}(x_2)x_2)^2 > 0$. To render $\dot{V}_2 < 0 \ \forall x \ne 0$ we require

$$\widetilde{\phi}_2(u) < -\frac{1}{\widetilde{\phi}_1^{'}(x_2)}(\widetilde{\phi}_1(x_2) + 3x_1)\sigma(x_1, x_2).$$

Because $\widetilde{\phi}_1^{'}$ is only C^0, and can not be used for a C^1 control law, we select C^1 maps $\kappa_1, \pi_1 : I\!R \to I\!R$, $0 < \kappa_1(x_2) \le \widetilde{\phi}_1^{'}(x_2) \le \pi_1(x_2)$ and get

$$\widetilde{\phi}_2(u) = -\frac{100 + 7M_1^2 + 2\pi_1^2(x_2)x_2^2}{2\kappa_1(x_2)}(\widetilde{\phi}_1(x_2) + 3x_1)$$

which can be solved for u because $\widetilde{\phi}_2^{-1}$ exists by construction. This control law yields $\dot{V}_2 \le -V_2$, and, hence, $x = 0$ is GES.

Case of $x^* \ne 0$. Let ϕ_1 and ϕ_2 in (7) satisfy conditions of Lemma 2.1 (b) and $\phi_1^{'}(0) = \phi_2^{'}(0) = 0$. Because now a C^1 control law that achieves asymptotic stabilization of $x = 0$ does not exist, we select arbitrarily small x_1^*, compute x_2^*, u^* such that $x_1^* + \phi_1(x_2^*) = 0$, $x_1^* x_2^* + \phi_2(u^*) = 0$ and rewrite (7) using $\hat{x} = x - x^*$, $\hat{u} = u - u^*$

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_1 + \phi_1(\hat{x}_2 + x_2^*) - \phi_1(x_2^*), \\ \dot{\hat{x}}_2 &= \hat{x}_1(\hat{x}_2 + x_2^*) + x_1^*\hat{x}_2 + \phi_2(\hat{u} + u^*) - \phi_2(u^*). \end{aligned} \tag{8}$$

The linearization of (8) at $\hat{x} = 0$ is controllable. Using $\phi_1(\hat{x}_2 + x_2^*) - \phi_1(x_2^*) = \widetilde{\mu}_1(\hat{x}_2)\widetilde{\phi}_1(\hat{x}_2)$ and $\phi_2(\hat{u} + u^*) - \phi_2(u^*) = \widetilde{\mu}_2(\hat{u})\widetilde{\phi}_2(\hat{u})$, we bring (8) to the form of (7). A C^1 control achieving GAS of $\hat{x} = 0$ guarantees GPS of $x = 0$.

Case of x^* not existing. Local behavior around $x = 0$ is more complex for the system

$$\dot{x}_1 = x_1 + \phi_1(x_2), \qquad \dot{x}_2 = \phi_2(u). \tag{9}$$

Although ϕ_1 and ϕ_2 satisfy conditions of Lemma 2.1 (b), there are no solutions of $x_1^* + \phi_1(x_2^*) = 0$, $\phi_2(u^*) = 0$ for which the linearization of (9) is stabilizable. Thus, there does not exist a C^1 AS control law for any (x^*, u^*). We show that it is still possible to achieve GPS of $x = 0$. With arbitrarily small $x_1^\bullet, u^\bullet$ we compute $x_2^\bullet$ such that $x_1^\bullet + \phi_1(x_2^\bullet) = 0$ and, using $\hat{x} = x - x^\bullet$, $\hat{u} = u - u^\bullet$, we get

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_1 + \phi_1(\hat{x}_2 + x_2^\bullet) - \phi_1(x_2^\bullet) = \hat{x}_1 + \widetilde{\mu}_1(\hat{x}_2)\widetilde{\phi}_1(\hat{x}_2), \\ \dot{\hat{x}}_2 &= \phi_2(\hat{u} + u^\bullet) - \phi_2(u^\bullet) + \phi_2(u^\bullet) \\ &= \widetilde{\mu}_2(\hat{u})\widetilde{\phi}_2(\hat{u}) + \phi_2(u^\bullet). \end{aligned}$$

For $V_2 = \frac{1}{2}\hat{x}_1^2 + \frac{1}{2}(\widetilde{\phi}_1(\hat{x}_2) + 3x_1)^2$ we obtain

$$\begin{aligned} \dot{V}_2 &\le -\tfrac{1}{2}\hat{x}_1^2 + (\widetilde{\phi}_1(\hat{x}_2) + 3\hat{x}_1)^2\sigma(\hat{x}_1, \hat{x}_2) \\ &\quad +\widetilde{\mu}_2(\hat{u})\widetilde{\phi}_1^{'}(\hat{x}_2)(\widetilde{\phi}_1(\hat{x}_2) + 3\hat{x}_1)\widetilde{\phi}_2(\hat{u}) + \phi_2^2(u^\bullet) \end{aligned}$$

with $\sigma(\hat{x}_1, \hat{x}_2) = 9 + \frac{1}{4}\pi_1^2(\hat{x}_2) + M_1(3 + \frac{81}{2}M_1)$ and $\kappa_1(\hat{x}_2)$, $\pi_1(\hat{x}_2)$ as in the case of $x^* = 0$. Then

$$\widetilde{\phi}_2(\hat{u}) = -\frac{1}{\kappa_1(\hat{x}_2)}(\widetilde{\phi}_1(\hat{x}_2) + 3\hat{x}_1)\left(\sigma(\hat{x}_1, \hat{x}_2) + \frac{1}{2}\right) \tag{10}$$

guarantees $\dot{V}_2 \le -V_2 + \phi_2^2(u^\bullet)$, forcing $\hat{x}(t)$ to converge to $\{\hat{x} \in I\!R^2 : V_2(\hat{x}) \le \phi_2^2(u^\bullet)\}$. Because $u^\bullet$ is arbitrary, and V_2 does not depend on $u^\bullet$, the control law (10) achieves GPS of $\hat{x} = 0$, that is of $x = x^\bullet$ in the original coordinates. Since $\|x^\bullet\|$ can be arbitrarily small, $x = 0$ is also GPS.

A distinguishing feature of this case is that $u^\bullet$ directly influences $\widetilde{\phi}_2$, and $\widetilde{\phi}_2^{'}(0) \to 0$ as $|u^\bullet| \to 0$. This makes it more difficult to prove GPS for higher-order systems (5) that do not possess x^*.

In the above constructions the use was made of the fact that if $g : I\!R^i \to I\!R$, $g(0) = 0$, is C^1 then there exist C^1 maps $\overline{g}_j : I\!R^i \to I\!R_+$, $j = 1, i$ and $\widetilde{g} : I\!R^i \to I\!R_+$ such that

$$\begin{aligned} |g(\overline{x}_i)| &\le \textstyle\sum_{j=1}^{i} |x_j|\overline{g}_j(\overline{x}_i), \\ |x_i||g(\overline{x}_i)| &\le \textstyle\sum_{j=1}^{i-1} x_j^2 + x_i^2\widetilde{g}(\overline{x}_i). \end{aligned} \tag{11}$$

IV. Design Procedure

Case of $x^* = 0$. By Lemma 2.1, there exist C^1 maps $\widetilde{\phi}_i$ and C^0 maps $\widetilde{\mu}_i$, M_i such that $\phi_i(x_{i+1}) = \widetilde{\mu}_i(x_{i+1})\widetilde{\phi}_i(x_{i+1})$ and $1 \le \widetilde{\mu}_i(x_{i+1}) \le M_i(x_{i+1})$ for $i = 1, ...n$. An equivalent representation of (5) is then

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + \widetilde{\mu}_1(x_2)\widetilde{\phi}_1(x_2), \\ \dot{x}_2 &= f_2(\overline{x}_2) + \widetilde{\mu}_2(x_3)\widetilde{\phi}_2(x_3), \\ &\ \vdots \\ \dot{x}_n &= f_n(x) + \widetilde{\mu}_n(u)\widetilde{\phi}_n(u). \end{aligned} \tag{12}$$

In the first step we let $V_1 = \frac{1}{2}x_1^2$ and get

$$\begin{aligned} \dot{V}_1 &= x_1(f_1(x_1) + \gamma_1(x_1)) + \widetilde{\mu}_1(x_2)x_1(\widetilde{\phi}_1(x_2) \\ &\quad -\gamma_1(x_1)) + (\widetilde{\mu}_1(x_2) - 1)x_1\gamma_1(x_1), \end{aligned} \tag{13}$$

where a C^1 map $\gamma_1 : I\!R \to I\!R$, $\gamma_1(0) = 0$, is chosen to satisfy for, $c_1 > 0$,

$$x_1\gamma_1(x_1) \le 0, \qquad x_1(f_1(x_1) + \gamma_1(x_1)) \le -(n+1)x_1^2.$$

Such a γ_1 exists because f_1 is C^1 and $f_1(0) = 0$. With $M_1(x_2) \geq \widetilde{\mu}_1(x_2) \geq 1$, we obtain

$$\dot{V}_1 \leq -nx_1^2 + \tfrac{1}{4}M_1(x_2)^2(\widetilde{\phi}_1(x_2) - \gamma_1(x_1))^2. \qquad (14)$$

For the i^{th} step we use induction. Suppose that $V_{i-1} : I\!R^{i-1} \to I\!R_+$ and C^1 maps $\gamma_j : I\!R^j \to I\!R$, $\gamma_j(0) = 0$, $j = 0, i-1$, $\gamma_0 = 0$, $\widetilde{\phi}_0(x_1) = x_1$, satisfy

$$\begin{aligned}\dot{V}_{i-1} \leq & -\textstyle\sum_{j=0}^{i-2}(n-i+2)(\widetilde{\phi}_j(x_{j+1}) - \gamma_j(\overline{x}_j))^2 \\ & +\tfrac{1}{4}M_{i-1}^2(x_i)(\widetilde{\phi}_{i-2}'(x_{i-1}))^2(\widetilde{\phi}_{i-1}(x_i) - \gamma_{i-1}(\overline{x}_{i-1}))^2.\end{aligned}$$

Our goal is to construct $V_i : I\!R^i \to I\!R_+$ and a C^1 map $\gamma_i : I\!R^i \to I\!R$, $\gamma_i(0) = 0$ such that the analog of the inequality above holds for V_i. Considering $V_i = V_{i-1} + \frac{1}{2}(\widetilde{\phi}_{i-1}(x_i) - \gamma_{i-1}(\overline{x}_{i-1}))^2$ we get

$$\begin{aligned}\dot{V}_i \leq & -\textstyle\sum_{j=0}^{i-2}(n-i+2)(\widetilde{\phi}_j(x_{j+1}) - \gamma_j(\overline{x}_j))^2 \\ & +(\widetilde{\phi}_{i-1}(x_i) - \gamma_{i-1}(\overline{x}_{i-1}))(\widetilde{\phi}_{i-1}'(x_i)\dot{x}_i - \textstyle\sum_{j=1}^{i-1}\frac{\partial\gamma_{i-1}}{\partial x_j}\dot{x}_j) \\ & +\tfrac{1}{4}M^2(x_i)(\widetilde{\phi}_{i-2}'(x_{i-1}))^2(\widetilde{\phi}_{i-1}(x_i) - \gamma_{i-1}(\overline{x}_{i-1}))^2 \\ \leq & -\textstyle\sum_{j=0}^{i-2}(n-i+2)(\widetilde{\phi}_j(x_{j+1}) - \gamma_j(\overline{x}_j))^2 \\ & +(\widetilde{\phi}_{i-1}(x_i) - \gamma_{i-1}(\overline{x}_{i-1}))\widetilde{\phi}_{i-1}'(x_i)\widetilde{\mu}_i(x_{i+1})\widetilde{\phi}_i(x_{i+1}) \\ & +|\widetilde{\phi}_{i-1}(x_i) - \gamma_{i-1}(\overline{x}_{i-1})||G_i(\overline{x}_i)|\end{aligned} \qquad (15)$$

where C^1 map $G_i : I\!R^i \to I\!R$, $G_i(0) = 0$ satisfies

$$\begin{aligned}|G_i(\overline{x}_i)| \geq & |\tfrac{1}{4}M_{i-1}^2(x_i)(\widetilde{\phi}_{i-2}'(x_{i-1}))^2(\widetilde{\phi}_{i-1}(x_i) \\ & -\gamma_{i-1}(\overline{x}_{i-1})) - \textstyle\sum_{j=1}^{i-1}\frac{\partial\gamma_{i-1}}{\partial x_j}\dot{x}_j + \widetilde{\phi}_{i-1}'(x_i)f_i(\overline{x}_i)|.\end{aligned} \qquad (16)$$

Using (11) we construct a C^1 map $\widetilde{G}_i : I\!R^i \to I\!R_+$ such that

$$\begin{aligned}|\widetilde{\phi}_{i-1}(x_i) - \gamma_{i-1}(\overline{x}_{i-1})||G_i(\overline{x}_i)| \leq & \textstyle\sum_{j=0}^{i-2}(\widetilde{\phi}_j(x_{j+1}) \\ & -\gamma_j(\overline{x}_j))^2 + (\widetilde{\phi}_{i-1}(x_i) - \gamma_{i-1}(\overline{x}_{i-1}))^2\widetilde{G}_i(\overline{x}_i).\end{aligned} \qquad (17)$$

We then select $\gamma_i : I\!R^i \to I\!R$ to satisfy

$$\begin{aligned}& \gamma_i(\overline{x}_i)(\widetilde{\phi}_{i-1}(x_i) - \gamma_{i-1}(\overline{x}_{i-1})) \leq \\ & -\tfrac{\widetilde{G}_i(\overline{x}_i)+n-i+1}{\widetilde{\phi}_{i-1}'(x_i)}(\widetilde{\phi}_{i-1}(x_i) - \gamma_{i-1}(\overline{x}_{i-1}))^2\end{aligned} \qquad (18)$$

Map $\widetilde{\phi}_{i-1}$ is C^1. In order to avoid non-differentiability of γ_i, we replace $\widetilde{\phi}_{i-1}'$ in (18) with C^1 map $\kappa_{i-1} : I\!R \to I\!R$ such that $\forall x_i \in I\!R$, $0 < \kappa_{i-1}(x_i) \leq \widetilde{\phi}_{i-1}'(x_i)$. Throughout the design, whenever the smoothness of γ_i is needed, $\widetilde{\phi}_{i-1}'$ is replaced by a suitable κ_{i-1}. Using (15), (18), and adding and subtracting $\widetilde{\phi}_{i-1}'(x_i)\gamma_i(\overline{x}_i)(\widetilde{\phi}_{i-1}(x_i) - \gamma_{i-1}(\overline{x}_{i-1}))$ we get:

$$\begin{aligned}\dot{V}_i \leq & -\textstyle\sum_{j=0}^{i-1}(n-i+1)(\widetilde{\phi}_j(x_{j+1}) - \gamma_j(\overline{x}_j))^2 \\ & +\tfrac{1}{4}M_i(x_{i+1})^2(\widetilde{\phi}_{i-1}'(x_i))^2(\widetilde{\phi}_i(x_{i+1}) - \gamma_i(\overline{x}_i))^2 \\ & +(\widetilde{\mu}_i(x_{i+1}) - 1)\widetilde{\phi}_{i-1}'(x_i)\gamma_i(\overline{x}_i)(\widetilde{\phi}_{i-1}(x_i) - \gamma_{i-1}(\overline{x}_{i-1})).\end{aligned}$$

By (18) and $\widetilde{\mu}_i(x_{i+1}) \geq 1$ the last term above is non-positive, and it can be omitted. The inequality is thus the desired analog of the bound on $\dot{V}_{i-1}$.

For $i = n$, (18) yields the control law

$$\widetilde{\phi}_n(u) \triangleq \gamma_n(x) = -\tfrac{\widetilde{G}_n(x)+1}{\kappa_{n-1}(x_n)}(\widetilde{\phi}_{n-1}(x_n) - \gamma_{n-1}(\overline{x}_{n-1})) \qquad (19)$$

which guarantees that $\dot{V}_n \leq -2V_n$, hence achieves GAS of $x = 0$ for (5).

Case of $x^* \neq 0$. When the linearization at $x = 0$ of (5) is not controllable, we select a linearly controllable x^*, $\|x^*\| \leq \epsilon$. With $\hat{x} = x - x^*$, $\hat{u} = u - u^*$ and using factorization $\phi_i(\hat{x}_{i+1} + x_{i+1}^*) - \phi_i(x_{i+1}^*) = \widetilde{\mu}_i(\hat{x}_{i+1})\widetilde{\phi}_i(\hat{x}_{i+1})$, we obtain:

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{f}_1(\hat{x}_1) + \widetilde{\mu}_1(\hat{x}_2)\widetilde{\phi}_1(\hat{x}_2), \\ \dot{\hat{x}}_2 &= \hat{f}_2(\hat{\overline{x}}_2) + \widetilde{\mu}_2(\hat{x}_3)\widetilde{\phi}_2(\hat{x}_3), \\ &\vdots \\ \dot{\hat{x}}_n &= \hat{f}_n(\hat{x}) + \widetilde{\mu}_n(\hat{u})\widetilde{\phi}_n(\hat{u}),\end{aligned} \qquad (20)$$

where $\hat{f}_i(\hat{\overline{x}}_i) = f_i(\hat{\overline{x}}_i + \overline{x}_i^*) + \phi_i(x_{i+1}^*)$, $\hat{f}_i(0) = 0$. Since the linearization of (20) at $\hat{x} = 0$ is controllable, we use the design procedure for system (12) to construct a C^1 control law

$$\widetilde{\phi}_n(\hat{u}) = -\frac{\hat{G}_n(\hat{x}) + 1}{\kappa_{n-1}(\hat{x}_n)}(\widetilde{\phi}_{n-1}(\hat{x}_n) - \hat{\gamma}_{n-1}(\hat{\overline{x}}_{n-1})). \qquad (21)$$

Above, $\breve{G}_i(\hat{\overline{x}}_i)$, $\hat{G}_i(\hat{\overline{x}}_i)$ and $\hat{\gamma}_i(\hat{\overline{x}}_i)$, satisfy (16), (17) and (18) respectively, with $\hat{f}_i(\hat{\overline{x}}_i)$ substituted for $f_i(\overline{x}_i)$, $i = 1, \ldots n$. Control law (21) renders $x = x^*$ of system (5) GAS. Since $\|x^*\| \leq \epsilon$, and ϵ can be arbitrarily small, it follows that $x = 0$ of (5) is GPS.

Case of x^* not existing. This case is illustrated on the system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + \phi_1(x_2), \\ &\vdots \\ \dot{x}_{k-1} &= f_{k-1}(\overline{x}_{k-1}) + \phi_{k-1}(x_k), \\ \dot{x}_k &= \phi_k(x_{k+1}), \\ \dot{x}_{k+1} &= f_{k+1}(\overline{x}_{k+1}) + \phi_{k+1}(x_{k+2}), \\ &\vdots \\ \dot{x}_n &= f_n(x) + \phi_n(u),\end{aligned} \qquad (22)$$

where maps ϕ_i, $i = 1, \ldots n$, $i \neq k$, satisfy assumptions of Lemma 2.1 (a), and ϕ_k satisfies those of (b). Because $x_{k+1}^* = 0$, the linearization is not controllable. We construct a C^1 control law to achieve global ultimate boundedness with an arbitrarily small bound on a subset of states.

Let $x^\bullet = [\overline{0}_k, x_{k+1}^\bullet, \overline{0}_{n-k-1}]$ where $x_{k+1}^\bullet$ is sufficiently small. With $\hat{x} = x - x^\bullet$, and applying factorization $\phi_i(\hat{x}_{i+1}) = \widetilde{\mu}_i(\hat{x}_{i+1})\widetilde{\phi}_i(\hat{x}_{i+1})$, $i = 1, n$, $i \neq k$ and $\phi_k(\hat{x}_{k+1} + x_{k+1}^\bullet) - \phi_k(x_{k+1}^\bullet) = \widetilde{\mu}_k(\hat{x}_{k+1})\widetilde{\phi}_k(\hat{x}_{k+1})$, $i = k$, we get:

$$\begin{aligned}\dot{\hat{x}}_1 &= f_1(\hat{x}_1) + \widetilde{\mu}_1(\hat{x}_2)\widetilde{\phi}_1(\hat{x}_2), \\ &\vdots \\ \dot{\hat{x}}_{k-1} &= f_{k-1}(\hat{\overline{x}}_{k-1}) + \widetilde{\mu}_{k-1}(\hat{x}_k)\widetilde{\phi}_{k-1}(\hat{x}_k), \\ \dot{\hat{x}}_k &= \widetilde{\mu}_k(\hat{x}_{k+1})\widetilde{\phi}_k(\hat{x}_{k+1}) + \phi_k(x_{k+1}^\bullet), \\ \dot{\hat{x}}_{k+1} &= f_{k+1}(\hat{\overline{x}}_{k+1}) + \widetilde{\mu}_{k+1}(\hat{x}_{k+2})\widetilde{\phi}_{k+1}(\hat{x}_{k+2}), \\ &\vdots \\ \dot{\hat{x}}_x &= f_n(\hat{x}) + \widetilde{\mu}_n(u)\widetilde{\phi}_n(u).\end{aligned}$$

The linearization at $\hat{x}=0$ is controllable, but $\hat{x}=0$ is not an equilibrium because $\phi_k(x^\bullet_{k+1}) \neq 0$. The design for the system above closely follows the one for (12), except that terms due to $\phi_k(x^\bullet_{k+1})$ appear in $\dot{V}_i$, $i=k,n$:

$$\frac{\partial(\widetilde{\phi}_{i-1}(\hat{x}_i)-\gamma_{i-1}(\hat{\overline{x}}_{i-1}))^2}{\partial \hat{x}_k}\phi_k(x^\bullet_{k+1}) \leq \phi_k^2(x^\bullet_{k+1}) + \frac{1}{4}(\widetilde{\phi}_{i-1}(\hat{x}_i)-\gamma_{i-1}(\hat{\overline{x}}_{i-1}))^2\left(\frac{\partial(\widetilde{\phi}_{i-1}(\hat{x}_i)-\gamma_{i-1}(\hat{\overline{x}}_{i-1}))}{\partial \hat{x}_k}\right)^2$$

We compensate for these terms by constructing a C^1 map G_i, $G_i(0)=0$, $i=k,n$ to satisfy

$$|G_i(\hat{\overline{x}}_i)| \geq |\tfrac{1}{4}(M_{i-1}^2(\hat{x}_i)(\widetilde{\phi}'_{i-2}(\hat{x}_{i-1}))^2 + \left(\frac{\partial(\widetilde{\phi}_{i-1}(\hat{x}_i)-\gamma_{i-1}(\hat{\overline{x}}_{i-1}))}{\partial \hat{x}_k}\right)^2)(\widetilde{\phi}_{i-1}(\hat{x}_i)-\gamma_{i-1}(\hat{\overline{x}}_{i-1})) - \sum_{j=1}^{i-1}\frac{\partial \gamma_{i-1}}{\partial \hat{x}_j}\dot{\hat{x}}_j + \widetilde{\phi}'_{i-1}(\hat{x}_i)f_i(\hat{\overline{x}}_i)|$$

instead of (16). The resulting control law is of the form (19) and ensures that

$$\dot{V}_n \leq -2V_n + p_k\phi_k^2(x^\bullet_{k+1}), \quad p_k = n-k+1. \tag{23}$$

This proves global ultimate boundedness of $\hat{x}(t)$, which converges to the positively invariant set $\{\hat{x}\in I\!\!R^n : 2V_n(\hat{x}) \leq p_k\phi_k^2(x^\bullet_{k+1})\}$ in finite time. Thus, for each summand in V_n, $(\widetilde{\phi}_{i-1}(\hat{x}_i)-\gamma_{i-1}(\hat{\overline{x}}_{i-1}))^2$, $i=1,...n$, as $t\to\infty$, we have the bound,

$$|\widetilde{\phi}_{i-1}(\hat{x}_i(t))-\gamma_{i-1}(\hat{\overline{x}}(t)_{i-1})| \leq \sqrt{p_k}|\phi_k(x^\bullet_{k+1})|.$$

Using this bound and the fact that the first k summands of V_n are independent of $x^\bullet_{k+1}$, we can show that for any $\epsilon_2 > 0$, there exists $x^\bullet_{k+1}$ such that

$$\|\hat{\overline{x}}_{k+1}(t)\| \leq \epsilon_2, \quad as\ t\to\infty. \tag{24}$$

If in addition

$$\lim_{|s|\to 0}\frac{|\phi_k(s)|}{|\phi'_k(s)|} = 0, \tag{25}$$

then (24) holds for $\hat{\overline{x}}_{k+2}$.

Corollary 4.1: *Let k be the largest integer such that the first $k-1$ equations of (5), with x_k as the control, possess a linearly controllable equilibrium $\overline{x}^*_k$, $\|\overline{x}^*_k\| \leq \epsilon_1$ for any $\epsilon_1 > 0$. Then a C^1 feedback control law that achieves global ultimate boundedness and guarantees $\|\overline{x}_{k+1}(t)\| \leq \epsilon_2$ as $t\to\infty$, for any $\epsilon_2 > 0$, can be constructed by a recursive procedure. If ϕ_k satisfies (25), then such a control law guarantees $\|\overline{x}_{k+2}(t)\| \leq \epsilon_2$ as $t\to\infty$.*

The control laws (19) and (21) are defined implicitly, via $\widetilde{\phi}_n(u)$. When $\widetilde{\phi}_n^{-1}$ is not readily available, we enlarge the system (5) or (20) with an integrator, $\dot{u} = \gamma_{n+1}(x,u)$, to obtain a system affine in controls. Then, we construct a C^1 map $\gamma_{n+1} : I\!\!R^{n+1} \to I\!\!R$ such that $(x,u)=(0,0)$ is GAS/GPS.

V. An Extension

In [6], the following triangular systems were considered

$$\begin{aligned} \dot{x}_1 &= \textstyle\sum_{k_1=1}^{j_1}\beta_{k_1}(x_1)\alpha_{k_1}(x_2) + \phi_1(x_2), \\ \dot{x}_2 &= \textstyle\sum_{k_2=1}^{j_2}\beta_{k_2}(\overline{x}_2)\alpha_{k_2}(x_3) + \phi_2(x_3), \\ &\vdots \\ \dot{x}_n &= \textstyle\sum_{k_n=1}^{j_n}\beta_{k_n}(x)\alpha_{k_n}(u) + \phi_n(u), \end{aligned} \tag{26}$$

with $\phi_i(x_{i+1}) = x_{i+1}^{p_i}$, where p_i is an odd integer, $\alpha_{k_i}(x_{i+1}) = x_{i+1}^{k_i}$ and $\beta_{k_i} : I\!\!R^i \to I\!\!R$, $\beta_{k_i}(0)=0$ is C^1, $k_i = 1, p_i - 1$, $i=1,...n$, and a GAS C^0 feedback control law was constructed. We now let maps ϕ_i satisfy Lemma 2.1 and $\alpha_{k_i} : I\!\!R \to I\!\!R$ be C^1, $k_i = 1, j_i$, such that

$$\lim_{|x_{i+1}|\to\infty}\frac{|\alpha_{k_i}(x_{i+1})|}{|\phi_i^-(x_{i+1})|} = 0. \tag{27}$$

Corollary 5.1: *Let the condition (27) hold for $k_i = 1,...j_i$, $i=1,...n$. If $x=0$ of (26) is linearly controllable then a C^1 control law which renders it globally asymptotically stable can be constructed by a recursive procedure. If $x=0$ is not linearly controllable but for any small $\epsilon > 0$ there exists a linearly controllable equilibrium x^*, $\|x^*\| \leq \epsilon$ then such a control law renders $x=0$ globally practically stable.*

We illustrate Corollary 5.1 on the system

$$\begin{aligned} \dot{x}_1 &= x_1\frac{x_2}{\ln(x_2^2+2)} + (1+x_2^2\cos^2 x_2)x_2 \\ &\triangleq \beta_1(x_1)\alpha_1(x_2) + \widetilde{\mu}_1(x_2)\widetilde{\phi}_1(x_2), \\ \dot{x}_2 &= u, \end{aligned} \tag{28}$$

to which no existing designs can be applied. Take $\widetilde{\phi}_1(x_2) = x_2$ and $M_1(x_2) = 1+x_2^2$. Then (27) holds, since

$$\lim_{|x_2|\to\infty}\frac{\alpha_1(x_2)}{x_2} = 0.$$

The C^1 map $\eta_1(x_1) \triangleq x_1(e^{|x_1|}+\sqrt{2})$ satisfies

$$|\frac{x_2}{\ln(x_2^2+2)}||x_1| \leq \frac{1}{2}|x_2| + |\eta_1(x_1)|. \tag{29}$$

Using

$$|\alpha_1(s_1)-\alpha_1(s_2)| \leq \max_{s\in\boldsymbol{R}}|\alpha'_1(s)||s_1-s_2| \leq \frac{3}{2}|s_1-s_2|$$

and (29), it can be shown that $V_2 = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2-\gamma_1(x_1))^2$ is a CLF for (28), where

$$\gamma_1(x_1) = -2x_1\left(2+\sqrt{2}+\frac{9}{4}x_1^2 + e^{\sqrt{x_1^2+1}}\right) \tag{30}$$

is C^∞. The control law

$$u = \frac{\partial\gamma_1}{\partial x_1}\dot{x}_1 - \frac{1}{4}(5+M_1^2(x_2))(x_2-\gamma_1(x_1))$$

yields $\dot{V}_2 \leq -2V_2$ and renders $x=0$ GAS.

VI. Inverse Optimality

It is of interest to investigate inverse optimality properties of the designed feedback control laws, because it guards against excessive control effort and ensures a sector margin $(\frac{1}{2}, \infty)$, as shown in [2], [7].

A stabilizing feedback control law $u = \gamma(x)$ for $\dot{x} = f(x) + g(x)u$ is said to be inversely optimal if it minimizes

$$J = \int_0^\infty (l(x) + r(x)u^2)dt \tag{31}$$

for some $r, l : \mathbb{R}^n \to \mathbb{R}_+$. If a Lyapunov function V and a control law $u = \gamma(x)$ satisfy

$$\gamma(x) = -\frac{1}{2} r^{-1}(x) \frac{\partial V}{\partial x} g(x) \triangleq -\frac{1}{2} r^{-1}(x) L_g V(x)$$

for some $r > 0$, and l defined by

$$l(x) \triangleq -\frac{\partial V}{\partial x} f(x) + \frac{1}{4} r^{-1}(x) L_g V^2(x) \tag{32}$$

satisfies $l > 0$, then $u = \gamma(x)$ is inversely optimal to (31) and V is its optimal value function, Theorem 3.19 in [7].

For system (5) and

$$J = \int_0^\infty (l(x) + r(x)\phi_n^2(u))dt \tag{33}$$

we now examine the optimality of $\widetilde{\phi}_n(u) = \gamma_n(x)$ defined by (19). As a Lyapunov function we use V_n constructed in Section IV. Noting that $L_g V_n(x) = \phi'_{n-1}(x_n)(\widetilde{\phi}_{n-1}(x_n) - \gamma_{n-1}(\overline{x}_{n-1}))$, we write

$$\begin{aligned} &\phi_n(u) = \widetilde{\mu}_n(u)\widetilde{\phi}_n(u) = -\widetilde{\mu}_n(x)\tfrac{\widetilde{G}_n(x)+c_n}{\kappa_{n-1}(x_n)} * \\ &(\widetilde{\phi}_{n-1}(x_n) - \gamma_{n-1}(\overline{x}_{n-1})) \triangleq -\tfrac{1}{2} r^{-1}(x) L_g V_n(x) \end{aligned} \tag{34}$$

where $\widetilde{\mu}_n(x) \triangleq \widetilde{\mu}_n(\widetilde{\phi}_n^{-1}(\gamma_n(x)))$ and $r : \mathbb{R}^n \to \mathbb{R}_+$ is a C^1 map defined by

$$r(x) = \frac{\kappa_{n-1}(x_n)\widetilde{\phi}'_{n-1}(x_n)}{2\widetilde{\mu}_n(x)(\widetilde{G}_n(x) + c_n)}. \tag{35}$$

An upper bound on l defined by (32) for $V = V_n$ could be obtained from (15) with $i = n$. The term c_n is added to ensure $l > 0$.

$$\begin{aligned} l(x) \geq &\textstyle\sum_{i=0}^{n-2} (\widetilde{\phi}_i(x_{i+1}) - \gamma_i(\overline{x}_i))^2 + \\ &\widetilde{\mu}_n(x) \tfrac{\widetilde{\phi}'_{n-1}(x_n)}{2\kappa_{n-1}(x_n)} (\widetilde{\phi}_{n-1}(x_n) - \gamma_{n-1}(\overline{x}_{n-1}))^2 (\widetilde{G}_n(x) + c_n) \\ &-(\widetilde{\phi}_{n-1}(x_n) - \gamma_{n-1}(\overline{x}_{n-1}))^2 \widetilde{G}_n(x). \end{aligned}$$

Applying Theorem 3.19 in [7] and selecting

$$c_n \geq \widetilde{G}_n(x) \tag{36}$$

we can prove the following corollary.

Corollary 6.1: *Let (5) have a linearly controllable equilibrium at $x = 0$. Then C^1 control law (34), (36) achieves GAS of $x = 0$ and minimizes performance index (33) with l and r given by (32) and (35) respectively, and V_n is the optimal value function.*

Similar inverse optimality properties can be guaranteed for control law (21) as well.

VII. Conclusion

We have developed a constructive procedure for stabilization of a wider class of triangular systems with C^1 control laws. We allow that controlling nonlinearities be non-invertible and non-smooth. When $x = 0$ is not linearly controllable, to avoid non-Lipschitz control laws, we stabilize another linearly controllable equilibrium close to $x = 0$. In the absence of linearly controllable equilibria, our control laws guarantee ultimate boundedness and force a subset of states to be arbitrarily small. The designed control laws possess desirable inverse optimality properties.

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