Dissipativity for dual linear differential inclusions through conjugate storage functions

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***Abstract*— Tools from convex analysis are used to show how dissipativity properties, expressed in terms of convex storage functions, translate when passing from a linear differential inclusion (LDI) to its dual. As special cases, it is shown that a convex, positive definite function is a Lyapunov function for an LDI if and only if its convex conjugate is a Lyapunov function for the LDI's dual, and that passivity and finite $\mathcal{L}_2$-gain are preserved when passing from an LDI with input and output to its dual. Also established is the duality between stabilizability and detectability, including stabilizable and detectable dissipativity, for dual LDIs. Finally, with examples we show how duality effectively doubles the number of tools available for assessing stability of LDIs.**

I. Introduction

Duality is a firmly established concept in linear systems theory. For example, a matrix is Hurwitz if and only if its transpose is Hurwitz. The pair (A, B) is stabilizable if and only if (B^T, A^T) is detectable. For the transfer function $C(sI - A)^{-1}B + D$ and its dual $B^T(sI - A)^{-1}C^T + D^T$, the $\mathcal{H}_\infty$ norms are equivalent. Similarly, one is positive real (respectively, strictly positive real) if and only if the other is positive real (respectively, strictly positive real).

There are various ways to establish this duality. One way is to find an appropriate positive definite matrix P so that $\frac{1}{2}P$ establishes the first property and to show that $\frac{1}{2}P^{-1}$ establishes the second property. For exponential stability, $\frac{1}{2}(A^TP + PA) < 0$ if and only if $\frac{1}{2}(AP^{-1} + P^{-1}A^T) < 0$. Also, $\frac{1}{2}(A^TP + PA - 2PBB^TP) < 0$, which is equivalent to stabilizability of (A, B), if and only if $\frac{1}{2}(AP^{-1} + P^{-1}A^T) < BB^T$, which is equivalent to detectability of (B^T, A^T). Furthermore, it can be shown that $\frac{1}{2}P$ satisfies the bounded real lemma (respectively, the positive real lemma) for (A, B, C, D) if and only if $\frac{1}{2}P^{-1}$ satisfies the bounded real lemma (respectively, the positive real lemma) for (A^T, C^T, B^T, D^T).

Not coincidentally, the function $\xi \mapsto \frac{1}{2}\xi \cdot P^{-1}\xi$ is the convex conjugate (in the sense of convex analysis) of the function of the function $x \mapsto \frac{1}{2}x \cdot Px$ when $P = P^T > 0$. In this paper, we will explore this conjugate relationship and duality for LDIs, at the same time allowing for the possibility of working with nonquadratic functions and their conjugates.

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Since the codification of the absolute stability problem in the middle of the twentieth century (see, for example, [1]), researchers have looked for Lyapunov functions to guarantee exponential stability and/or input-output properties for systems that can be modeled as switching among a (possibly infinite) family of linear systems. The classical circle criterion gives necessary and sufficient condition for the existence of a quadratic Lyapunov function that certifies exponential stability in the absolute stability problem. However, it is well known that a system can be absolutely stable without the existence of a quadratic Lyapunov function. For example, see [9]. In [8] it is noted that convex, positively homogeneous of degree two Lyapunov functions always exist for exponentially stable switching linear systems. In [4] it is shown that, moreover, these functions can always be taken to be everywhere continuously differentiable and smooth except at the origin.

In order to give computationally tractable methods to search for Lyapunov functions, researchers have focused on classes of functions over which to look. In the papers [13], [7], and [3], the authors consider homogeneous polynomial Lyapunov functions and provide linear matrix inequality (LMI) conditions for exponential stability. In [2], a matrix condition for a Lyapunov function that is the maximum of positive semidefinite quadratics is outlined.

One of the contributions of this paper is showing that the origin of an LDI is exponentially stable if and only if the origin of the dual LDI is exponentially stable and that Lyapunov functions for the dual systems are related through the convex conjugacy. Thus, the tools for computation of Lyapunov functions mentioned above can be applied to the dual system to check stability of the original system. In a sense, this observation doubles the number of tools available for assessing stability of LDIs. In some cases, there is a vast difference in the stability range that a certain class of Lyapunov functions can certify for a system compared to what it can certify for the system's dual. We illustrate this with an example at the end of the paper. Further benefits are shown in our companion paper [6], where stability regions of saturated systems are estimated.

In addition to results on exponential stability, we present duality results for dissipativity when using convex, positively homogeneous storage functions and convex/concave supply rates. These results are used to show, for example, that passivity and finite $\mathcal{L}_2$ gain are preserved when passing from an LDI with input and output to its dual. Duality of stabilizability and detectability, and stabilizable and detectable dissipativity, is established as well.

II. LDIs and Dissipation inequalities

We are interested in dissipation properties, established via convex storage functions, for a linear system

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C & D_1 & D_2 \end{bmatrix} \begin{bmatrix} x \\ u \\ d \end{bmatrix} \tag{1}$$

and its dual

$$\begin{bmatrix} \dot{\xi} \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} A^T & C^T \\ B_1^T & D_1^T \\ B_2^T & D_2^T \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix}, \tag{2}$$

as well as a linear differential inclusion

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} \in \text{co} \left\{ \begin{bmatrix} A & B_1 & B_2 \\ C & D_1 & D_2 \end{bmatrix}_i \right\}_{i=1}^m \begin{bmatrix} x \\ u \\ d \end{bmatrix} \tag{3}$$

and its dual

$$\begin{bmatrix} \dot{\xi} \\ z_1 \\ z_2 \end{bmatrix} \in \text{co} \left\{ \begin{bmatrix} A^T & C^T \\ B_1^T & D_1^T \\ B_2^T & D_2^T \end{bmatrix}_i \right\}_{i=1}^m \begin{bmatrix} \xi \\ w \end{bmatrix}. \tag{4}$$

Our most general Lyapunov inequalities will be of the form

$$\begin{aligned} \partial V(x) \cdot (Ax + B_1 u + B_2 d) &\leq -\gamma V(x) \\ &\quad -k(Cx + D_1 u + D_2 d, u, d), \end{aligned} \tag{5}$$

where the function k is convex in its first two arguments, and concave in the third. For such functions, if (5) holds for each vertex of the convex hull, we have

$$\begin{aligned} &\partial V(x) \cdot \sum_{i=1}^m \lambda_i (A_i x + B_{1i} u + B_{2i} d) + \gamma V(x) \\ &\quad \leq -\sum_{i=1}^m \lambda_i k\left((C_i x + D_{1i} u + D_{2i} d), u, d\right) \\ &\quad \leq -k\left(\sum_{i=1}^m \lambda_i (C_i x + D_{1i} u + D_{2i} d), u, d\right) \end{aligned}$$

for any $\lambda_i \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$. Thus, checking the inequalities for the vertices of the LDI is sufficient for verify the Lyapunov condition for all possible matrices in the convex hull. For ease of notation, in what follows we will usually suppress indices when considering LDIs. Our assumptions applied to these systems should be understood to hold for all matrix vertices.

Our main results show that inequalities of the form (5) can be equivalently restated in terms of inequalities like

$$\begin{aligned} \partial V^*(\xi) \cdot (A^T \xi + C^T w) &\leq \gamma V^*(\xi) \\ &\quad + k^*(w, -B_1^T \xi - D_1^T w, -B_2^T \xi - D_2^T w) \end{aligned} \tag{6}$$

which involve a function V^* conjugate (in the sense of convex analysis) to V and a k^*, a convex-concave conjugate of k. In many cases of practical interest, which we discuss here as corollaries and examples, verifying this inequality for vertices of the LDI is also sufficient.

III. Convex Analysis Preliminaries

The standard reference for the objects and concepts of convex analysis we summarize here is [10]. Given any function $f : \mathbb{R}^n \mapsto \mathbb{R}$, its conjugate function is defined, for $\xi \in \mathbb{R}^n$ by

$$f^*(\xi) = \sup_{x \in \mathbb{R}^n} \{\xi \cdot x - f(x)\}.$$

Basic examples are:

- ⋄ For a positive definite matrix P,

$$f(x) = \frac{1}{2} x \cdot Px \quad \Longleftrightarrow \quad f^*(\xi) = \frac{1}{2} \xi \cdot P^{-1} \xi. \tag{7}$$

- ⋄ For any $p > 1$, $q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$,

$$f(x) = \frac{1}{p} \|x\|^p \quad \Longleftrightarrow \quad f^*(\xi) = \frac{1}{q} \|\xi\|^q. \tag{8}$$

More elaborate examples are presented in Section IV.

As in this paper we are mostly interested in functions f that are convex, positive definite, and positively homogeneous of degree $p > 1$, from now on assume that f has these properties. Then:

(i) $f^*(\xi)$ is finite for every $\xi \in \mathbb{R}^n$.

(ii) f^* is a convex, positive definite, and positively homogeneous of degree $q > 1$ where $1/p + 1/q = 1$.

(iii) If

$$\frac{\alpha}{p} \|x\|^p \leq f(x) \leq \frac{\beta}{p} \|x\|^p$$

for some $\alpha > 0$, $\beta > 0$ (such constants exist for any continuous, positively homogeneous of degree p and positive definite function), then

$$\frac{\beta^{1-q}}{q} \|\xi\|^q \leq f^*(\xi) \leq \frac{\alpha^{1-q}}{q} \|\xi\|^q.$$

For example, positive homogeneity of f^* can be verified directly from the definition:

$$\begin{array}{rcl} f^*(\lambda\xi) & = & \sup_x \{(\lambda\xi) \cdot x - f(x)\} \\ & = & \lambda^q \sup_x \left\{\xi \cdot (x/\lambda^{q-1}) - f(x)/\lambda^q\right\} \\ & = & \lambda^q \sup_x \left\{\xi \cdot (x/\lambda^{q-1}) - f(x/\lambda^{q/p})\right\} \\ & = & \lambda^q \sup_x \left\{\xi \cdot x - f(x)\right\} = \lambda^q f^*(\xi). \end{array}$$

since $q - 1 = q/p$. Bounds on $f^*(\xi)$ follow from (8) and the fact that conjugacy reverses inequalities.

A fundamental property of convex functions, key to many results involving duality, is that the conjugate of f^* is the function f. That is,

$$(f^*)^*(x) = \sup_{\xi} \{x \cdot \xi - f^*(y)\} = f(x).$$

So, for example, the property of f^* described in (ii) above is equivalent to the same property of f. Similarly, bounds on f in (iii) are equivalent to those on f^*.

A *subgradient* of a convex function f at x is a vector $v \in \mathbb{R}^n$ such that

$$f(x') \geq f(x) + v \cdot (x' - x) \quad \forall x' \in \mathbb{R}^n.$$

If f is differentiable, then $\nabla f(x)$ is the unique subgradient at x. At points of nondifferentiability, the *subdifferential* $\partial f(x)$, being the set of all subgradients, has more than 1 element.

A fundamental relationship between ∂f and ∂f^* is:

$$\xi \in \partial f(x) \iff x \in \partial f^*(\xi).$$

This immediately leads to the following observation (the inequality $\partial f(x) \cdot Ax < 0$ should be understood as $\xi \cdot Ax < 0$ for all $\xi \in \partial f(x)$). When the convex function f and its conjugate f^* are positive definite, we have

$$\partial f(x) \cdot Ax < 0 \ \forall x \neq 0 \iff \partial f^*(\xi) \cdot A^T \xi < 0 \ \forall \xi \neq 0.$$

Indeed, suppose the condition on the left holds. Pick any $\xi \neq 0$, and any $x \in \partial f^*(x)$. Then $x \neq 0$, since $0 \in \partial f^*(x)$ would imply x minimizes f^*. Thus $x \cdot A^T \xi = \xi \cdot Ax < 0$, since $x \in \partial f^*(\xi)$ is equivalent to $\xi \in \partial f(x)$.

A more precise relationship exists for positively homogeneous functions.

Lemma 3.1: $f(x) = 1/p$ and $\xi \in \partial f(x)$ if and only if $f^*(\xi) = 1/q$ and $x \in \partial f^*(\xi)$.

Proof: The subdifferential inclusions are equivalent, thus we only need to show that $f(x) = 1/p$ and $y \in \partial f(x)$ implies $f^*(y) = 1/q$. As $y \in \partial f(x)$, then x maximizes $y \cdot x - \partial f(x)$, and so $f^*(y) = y \cdot x - f(x) = y \cdot x - 1/p$. Furthermore, $\lambda = 1$ maximizes the function $y \cdot \lambda x - f(\lambda x) = y \cdot \lambda x - \lambda^p/p$ over $\lambda \geq 0$. The derivative being 0 at $\lambda = 1$ yields $y \cdot x = 1$. Thus $f^*(y) = 1 - 1/p$. ■

Given any function $g : \mathbb{R}^n \mapsto \mathbb{R}$, its convex hull $\operatorname{co} g$ is the greatest convex function bounded above by g. Under mild assumptions, for example when g^* is finite everywhere (this always holds if g is positively homogeneous of degree $p > 1$ and positive definite) we have

$$\operatorname{co} g(x) = \min \left\{ \sum_{i=1}^{n+1} \lambda_i g(x_i) \mid \sum_{i=1}^{n+1} \lambda_i x_i = x \right\} \quad (9)$$

where the minimum is taken over x_i's and λ_i's such that $\sum_{i=1}^{n+1} \lambda_i = 1$, $\lambda_i \geq 0$ (we denote such set as Δ_n). If $\operatorname{co} g(x) = \sum_{i=1}^{n+1} \lambda_i g(x_i)$ then $\operatorname{co} g(x_i) = g(x_i)$ at each x_i with nonzero λ_i. Furthermore, if g is differentiable at each such x_i, then $\nabla \operatorname{co} g(x) = \nabla g(x_i)$ for each such i (in particular, $\operatorname{co} g$ is differentiable at x).

Now consider convex functions $f_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i = 1, 2, ..l$, and define

$$f(x) = \max_{i=1,2,...l} f_i(x). \quad (10)$$

The conjugate function f^* is the convex hull of the function $g(\xi) = \min_{i=1,2,...l} f_i^*(\xi)$. A similar relationship holds for level sets – level sets of f are intersections of level sets of all f_i's while level sets of f^* are convex hulls of (smallest convex sets containing) level sets of all f_i^*'s.

Lemma 3.2: Consider a positive definite function f given by (10) and any function h. The following are equivalent:

(a) for all $i = 1, 2, ...k$ and x such that $f(x) = f_i(x)$,

$$\partial f_i(x) \cdot (Ax + Bu) \leq -\gamma f_i(x) + h(x, u),$$

(b) for all x, $\partial f(x) \cdot (Ax + Bu) \leq -\gamma f(x) + h(x, u)$.

Proof: Fix $\bar{x}$ and let $i_1, i_2, ...i_s$ be the set of all indices for which $f_{i_k}(\bar{x}) = \bar{x}$. Then (a) implies that

$$\partial f_{i_k}(\bar{x}) \cdot (A\bar{x} + Bu) \leq -\gamma f(\bar{x}) + h(\bar{x}, u)$$

for $k = 1, 2, ...s$. The subdifferential $\partial f(\bar{x})$ is the convex hull of the union of $\partial f_{i_k}(\bar{x})$. More precisely, for $r = \min\{n+1, s\}$, given any $\bar{\xi} \in \partial f(\bar{x})$, there exist $\xi_1, \xi_2, ...\xi_r$ with $\xi_k \in \partial f_{i_k}(\bar{x})$ and $(\lambda_1, \lambda_2, ...\lambda_r) \in \Delta_r$ such that $\sum_{k=1}^r \lambda_k \xi_{i_k} = \bar{y}$. As $\xi_k \cdot (A\bar{x} + Bu) \leq -\gamma f(\bar{x}) + h(\bar{x}, u)$ for each k, multiplying these inequalities by λ_k and summing them yields $\bar{\xi} \cdot (A\bar{x} + Bu) \leq -\gamma f(\bar{x}) + h(\bar{x}, u)$. Thus (b) holds. The reverse implication is simple. ■

IV. Composite Quadratic Function

For positive definite symmetric Q_i, $i = 1, 2, ..l$, consider

$$q(x) = \max_{j=1,2,...l} \frac{1}{2} x \cdot Q_i x. \quad (11)$$

It turns out that the conjugate of q, which is the convex hull of functions $\xi \mapsto \frac{1}{2} \xi \cdot Q_i^{-1} \xi$, is the same as the composite quadratic function used in [5] for stability analysis. Indeed,

$$\max_{\lambda \in \Delta_l} \sum_{i=1}^{l} \lambda_i \frac{1}{2} x \cdot Q_i x = \max_{\lambda \in \Delta_l} \frac{1}{2} x \cdot \left(\sum_{i=1}^{l} \lambda_i Q_i \right) x$$

since the maximum of a linear function of λ over a simplex is attained on one of the vertices. Consequently,

$$\begin{aligned} q^*(\xi) &= \sup_{x \in \mathbb{R}^n} \left\{ \xi \cdot x - \max_{\lambda \in \Delta_l} \frac{1}{2} x \cdot \left(\sum_{i=1}^{l} \lambda_i Q_i \right) x \right\} \\ &= \sup_{x \in \mathbb{R}^n} \min_{\lambda \in \Delta_l} \left\{ \xi \cdot x - \frac{1}{2} x \cdot \left(\sum_{i=1}^{l} \lambda_i Q_i \right) x \right\} \\ &= \min_{\lambda \in \Delta_l} \sup_{x \in \mathbb{R}^n} \left\{ \xi \cdot x - \frac{1}{2} x \cdot \left(\sum_{i=1}^{l} \lambda_i Q_l \right) x \right\}. \end{aligned}$$

Switching $\sup$ and $\min$ is possible, as the function in the brackets above is concave in x, convex in γ, and the minimum is taken over a compact set, see for example Corollary 37.3.2 in [10]. Now, calculating the conjugate of a quadratic function yields

$$q^*(\xi) = \min_{\lambda \in \Delta_k} \frac{1}{2} \xi \cdot \left(\sum_{i=1}^{l} \lambda_i Q_i \right)^{-1} \xi. \quad (12)$$

The function (12) is exactly the composite quadratic function of [5].

The dual description of (12) leads to an alternate way to analyze its properties. For example, the function q is strongly convex with constant ρ, where $\rho > 0$ is smaller than every eigenvalue of Q_i, $i = 1, 2, ..l$. (Strong convexity means that $q(x) - \frac{1}{2} \rho \|x\|^2$ is convex.) This is equivalent to q^* being differentiable and ∇q^* being Lipschitz continuous with constant $1/\rho$.

Numerical examples in Section VIII illustrate the use of both q and q^* in stability and $\mathcal{L}_2$-gain analysis.

V. Lyapunov Inequalities

The subdifferential mappings of a pair of conjugate convex functions are inverses of one another. Lemma 3.1 stated this more exactly for positively homogeneous functions. A key consequence of such symmetry is that a conjugate of a Lyapunov function for a linear system is a Lyapunov function for the dual system.

Theorem 5.1: Let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be a convex, positive definite, positively homogeneous of degree $p > 1$ function; let A be any matrix. Then, the condition

$$\partial V(x) \cdot Ax \leq -\gamma p V(x) \quad \text{for all } x \in \mathbb{R}^n \tag{13}$$

is equivalent to

$$\partial V^*(\xi) \cdot A^T \xi \leq -\gamma q V^*(\xi) \quad \text{for all } \xi \in \mathbb{R}^n. \tag{14}$$

Proof: By positive homogeneity of V, inequality (13) is equivalent to

$$\partial V(\bar{x}) \cdot A\bar{x} \leq -\gamma \quad \text{for all } \bar{x} \text{ s.t. } V(\bar{x}) = 1/p. \tag{15}$$

Indeed, given any $x \neq 0$ (so that $V(x) \neq 0$), consider $\bar{x} = x/s$, where $s = (pV(x))^{1/p}$. Then $V(\bar{x}) = 1/p$, while

$$\partial V(\bar{x}) = \frac{1}{s^{p-1}} \partial V(x).$$

Thus (15) becomes $\dfrac{1}{s^{p-1}} \partial V(x) \cdot A \dfrac{x}{s} \leq \gamma$ which is exactly (19). Similarly, (14) is equivalent to

$$\partial V^*(\bar{\xi}) \cdot A\bar{\xi} \leq -\gamma \quad \text{for all } \bar{\xi} \text{ s.t. } V(\bar{\xi}) = 1/q. \tag{16}$$

Now, (15) means that $\bar{\xi} \cdot A\bar{x} \leq -\gamma$ for any element $\bar{\xi}$ of $\partial V(\bar{x})$ with $V(\bar{x}) = 1/p$. Such $\bar{x}$ and $\bar{\xi}$ can be equivalently characterized by $\bar{x} \in \partial V^*(\bar{\xi})$, $V^*(\bar{\xi}) = 1/q$, see Lemma 3.1. Thus (15) is equivalent to (16). ■

For V (and automatically V^*) positively homogeneous of degree 2, the decay rates in (13) and (14) are the same. Such functions naturally appear in linear systems and LDIs

$$\dot{x} \in \text{co}\,\{A_i\}_{i=1}^m\, x \tag{17}$$

and their duals

$$\dot{\xi} \in \text{co}\left\{A_i^T\right\}_{i=1}^m \xi\ . \tag{18}$$

Below, we write $\mathcal{L}$ for the set of all convex, positive definite, and positively homogeneous of degree 2 functions.

Theorem 5.2: The origin of (17) is asymptotically stable if and only if there exist $\gamma > 0$ and a function $V \in \mathcal{L}$ such that

$$\partial V(x) \cdot Ax \leq -\gamma V(x) \quad \text{for all } x \in \mathbb{R}^n \tag{19}$$

for all $A \in \{A_i\}_{i=1}^m$.

This result can be found in [8], here in Example 5.4 we write down one possible Lyapunov function.

Corollary 5.3: The origin of (17) is exponentially stable (with decay rate γ) if and only if (18) is exponentially stable (with decay rate γ).

An immediate practical consequence of this is that to verify exponential stability of (17) with a particular computation, one can also carry out that computation with transpose matrices. This can dramatically improve the results, as we illustrate in Example 8.2.

Example 5.4: Suppose that (17) is exponentially stable with decay rate γ. One way to construct a Lyapunov function verifying this is to consider

$$V(x) = \frac{1}{2} \sup \|\phi(\tau, x)\|^2 e^{2\gamma\tau}, \tag{20}$$

where the supremum is taken over all solutions $\phi(\cdot, x)$ and all $\tau \geq 0$. We have $V \in \mathcal{L}$ and in particular, it is a convex function. The conjugate function V^* turns out to be

$$V^*(\xi) = \frac{1}{2} \operatorname{co} \inf e^{-2\gamma\tau} \|\psi(\tau, \xi)\|^2$$

with the infimum taken over all τ and all solutions $\psi(\cdot, \xi)$ to $\dot{\xi}(t) \in \text{co}\left\{-A_i^T\right\}_{i=1}^m \xi$. Theorem 5.1 states that this is a Lyapunov function for the system (18).

Lemma 3.2 and its dual interpretation lead to practical conditions for stability of LDIs, with Lyapunov functions given by (11) or (12).

Corollary 5.5: Suppose that there exist positive definite and symmetric matrices $Q_1, Q_2, ...Q_l$ and numbers $\lambda_{ijk} \geq 0$ for $i, k = 1, 2, ...l$, $j = 1, 2, ...m$ such that

$$A_j^T Q_k + Q_k A_j \leq \sum_{i=1}^{l} \lambda_{ijk}(Q_i - Q_k) - \gamma Q_k \tag{21}$$

for all $j = 1, 2, ...m$, $k = 1, 2, ..l$. Then

$$\partial V(x) \cdot Ax \leq -\gamma V(x) \quad \forall x \in \mathbb{R}^n, A \in \operatorname{co}\Omega, \tag{22}$$

where V is the maximum of quadratic functions $x \mapsto \frac{1}{2}x \cdot Q_i x$ (recall (11)).

Proof: Since $\lambda_{ijk} \geq 0$, the inequality (21) implies that for any x with $x \cdot Q_i x \leq x \cdot Q_k x$ for all $i = 1, 2, ...l$, it holds that

$$x \cdot (A_j^T Q_k + Q_k A_j)x \leq -\gamma x \cdot Q_k x.$$

Invoking Lemma 3.2 with $g_i = \frac{1}{2}x \cdot Q_i x$ finishes the proof. ■

Corollary 5.6: Suppose that there exist positive definite and symmetric matrices $R_1, R_2, ...R_l$ and numbers $\lambda_{ijk} \geq 0$ for $i, k = 1, 2, ...m$, $j = 1, 2, ...l$ such that

$$R_k^{-1} A_i^T + A_i R_k^{-1} \leq \sum_{j=1}^{l} \lambda_{ijk}(R_j^{-1} - R_k^{-1}) - \gamma R_k^{-1} \tag{23}$$

for all $i = 1, 2, ...m$, $k = 1, 2, ..l$. Then

$$\partial V(x) \cdot Ay \leq -\gamma V(x) \quad \forall x \in \mathbb{R}^n, A \in \operatorname{co}\Omega, \tag{24}$$

where V is the convex hull of quadratic functions $x \mapsto \frac{1}{2}x \cdot R_j x$.

Proof: Corollary 5.5 implies that

$$\partial V^*(\xi) \cdot A^T \xi \leq -\gamma V^*(\xi) \quad \forall \xi \in \mathbb{R}^n, A \in \operatorname{co}\Omega,$$

with V^* being the maximum of quadratic functions $\xi \mapsto \frac{1}{2}\xi \cdot R_i^{-1}\xi$. This is equivalent to the desired conclusion. ∎

When a maximum (respectively, a convex hull) of two quadratic functions is considered in corollaries above, conditions (21) (respectively, (23)) are also necessary, see [2], page 73.

Now consider a control system

$$\dot{x} \in \text{co}\left\{\left[\begin{array}{cc} A & B \end{array}\right]_i\right\}_{i=1}^m \left[\begin{array}{c} x \\ u \end{array}\right] \tag{25}$$

and its dual system with output

$$\left[\begin{array}{c} \dot{\xi} \\ z \end{array}\right] \in \text{co}\left\{\left[\begin{array}{c} A^T \\ B^T \end{array}\right]_i\right\}_{i=1}^m \xi \ . \tag{26}$$

We say the system (25) is stabilizable by linear feedback (switched linear feedback) if there exists K (m matrices K_i) such that the origin of the system

$$\dot{x} \in \text{co}\left\{A_i + B_i K\right\}_{i=1}^m x$$

(respectively the origin of the system

$$\dot{x} \in \text{co}\left\{A_i + B_i K_i\right\}_{i=1}^m x \ )$$

is exponentially stable. The system (26) is stabilizable by linear output injection (switched linear output injection) if there exists L (m matrices L_i) such that the origin for

$$\dot{\xi} \in \text{co}\left\{A_i^T + L B_i^T\right\}_{i=1}^m \xi$$

(the origin for

$$\dot{\xi} \in \text{co}\left\{A_i^T + L_i B_i^T\right\}_{i=1}^m \xi \ )$$

is exponentially stable.

Corollary 5.7: The system (25) is stabilizable by linear feedback (respectively, switched linear feedback) if and only if the system (26) is stabilizable by linear (respectively, switched linear) output injection.

Additional relationships between stabilizability and "detectability", which is strongly related to stabilization by output injection, will be given in a later section.

Finally, note that restricting our attention to convex Lyapunov functions does not limit the breadth of LDIs for which stability can be guaranteed: any Lyapunov function for such system can be "convexified".

Lemma 5.8: Let W be a function satisfying (19) and define $V = \text{co}\, W$. Suppose that W is differentiable at every point x with $W(x) = V(x)$, and that V^* is finite everywhere. Then V satisfies (19).

Proof: Finiteness of V^* guarantees that V can be described through (9). Thus, given any x and any representation $V(x) = \sum_{i=1}^{n+1} \lambda_i W(x_i)$, we have $\nabla V(x) = \nabla W(x_i)$ for any i with nonzero λ_i. As W satisfies (19), we have (with the sum taken over i's with nonzero λ_i):

$$\begin{aligned} \nabla V(x) \cdot Ax &= \nabla V(x) \cdot A\left(\Sigma \lambda_i x_i\right) = \Sigma \lambda_i \nabla V(x) \cdot A x_i \\ &= \Sigma \lambda_i \nabla W(x_i) \cdot A x_i \leq \Sigma \lambda_i \left(-\gamma W(x_i)\right) \\ &= -\gamma V(x) \end{aligned}$$

Thus V satisfies (19). ∎

VI. DISSIPATIVITY

In this section we consider LDIs with external disturbance

$$\left[\begin{array}{c} \dot{x} \\ y \end{array}\right] = \text{co}\left\{\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]_i\right\}_{i=1}^m \left[\begin{array}{c} x \\ d \end{array}\right] \tag{27}$$

and its dual

$$\left[\begin{array}{c} \dot{\xi} \\ z \end{array}\right] = \text{co}\left\{\left[\begin{array}{cc} A^T & C^T \\ B^T & D^T \end{array}\right]_i\right\}_{i=1}^m \left[\begin{array}{c} \xi \\ w \end{array}\right] \ . \tag{28}$$

We consider infinitesimal dissipation inequalities of the form

$$\begin{aligned} &\partial V(x) \cdot (Ax + Bd) \\ &\qquad \leq -\gamma p V(x) - h(Cx + Dd, d) \ . \end{aligned} \tag{29}$$

The function V is called a storage function and the function $-h$ is called a supply rate [11]. We will relate dissipativity with storage function $V(x)$ and supply rate $-h(y,d)$ for the system (27) to dissipativity with storage function $V^*(\xi)$ and supply rate $h^*(w,-z)$ for the system (28). While a more general result covering storage functions positively homogeneous of degree $p > 1$ is valid, we restrict our attention to $V \in \mathcal{L}$. Supply rates are as follows:

Assumption 6.1: Supply rate h and dual supply rate h^* are positively homogeneous of degree 2 and such that

$$h(c,d) = \sup_w \left\{c \cdot w - f(w,d)\right\}, \tag{30}$$

$$h^*(w,z) = \inf_d \left\{-z \cdot d + f(w,d)\right\}, \tag{31}$$

for some function f. No conditions on f are needed, in particular it can take on values of $\pm\infty$.

This assumption is quite mild. Consider h such that $h(c,d)$ is convex in c for a fixed d, concave in d for a fixed c, and finite everywhere. Let h^* be given by

$$h^*(w,z) = \inf_d \sup_c \left\{w \cdot c + z \cdot d - h(c,d)\right\}. \tag{32}$$

Such a pair satisfies Assumption 6.1, with

$$f(w,d) = \sup_c \left\{w \cdot c - h(c,d)\right\}.$$

The same conclusion holds if h is convex/concave as before, and for some closed convex cones K_1, K_2, we have $h(c,d)$ is finite if $c \in K_1$, $d \in K_2$, $h(c,d) = +\infty$ if $c \notin K_1$, $d \in K_2$, and $h(c,d) = -\infty$ if $d \notin K_2$. In both cases, a formula symmetric to (32) holds:

$$h(c,d) = \sup_w \inf_z \left\{c \cdot w + d \cdot z - h^*(w,z)\right\}. \tag{33}$$

Example 6.2: (Quadratic h and h^***).** Suppose

$$h(c,d) = \frac{1}{2} c \cdot Qc - \frac{1}{2} d \cdot Rd + c \cdot Sd.$$

Then h is convex-concave if and only if Q and R are positive semidefinite. If (and only if) the matrix $M = \left[\begin{array}{cc} Q & S \\ S^T & -R \end{array}\right]$ describing the gradient of h, that is

$\nabla h(c,d) = M\begin{bmatrix} c \\ d\end{bmatrix}$, is invertible, the dual supply rate h^* defined by (32) is finite everywhere. Then, h^* is also quadratic and $\nabla h^*(w,z) = M^{-1}\begin{bmatrix} w \\ z\end{bmatrix}$.

Theorem 6.3: Let h and h^* be as in Assumption 6.1. The following conditions are equivalent:

(a) for all x, d,

$$\begin{aligned}&\partial V(x)\cdot(Ax+Bd) \\ &\qquad \le -\gamma V(x) - h(Cx+Dd,d)\end{aligned} \tag{34}$$

(b) for all ξ, w,

$$\begin{aligned}&\partial V^*(\xi)\cdot(A^T\xi + C^T w) \\ &\qquad \le -\gamma V^*(\xi) + h^*(w, -B^T\xi - D^T w).\end{aligned} \tag{35}$$

Proof: (Outline.) Using homogeneity of V and h, similarly as in the proof of Theorem 5.1, one obtains that (a) is equivalent to: for all x with $V(x)=1/2$, for all d,

$$\partial V(x)\cdot(Ax+Bd) \le -\gamma - h(Cx+Dd,d).$$

Thus, (a) is equivalent to: for all x with $V(x)=1/2$ and any $\xi\in\partial V(x)$, we have

$$\xi\cdot Ax + \gamma \le \inf_d \left\{-\xi\cdot Bd - h(Cx+Dd,d)\right\}.$$

Similarly, (b) is equivalent to: for all ξ with $V^*(\xi)=1/2$ and any $x\in\partial V^*(\xi)$, we have

$$x\cdot A^T\xi + \gamma \le \inf_w \left\{-w\cdot Cx + h^*(w,-B^T\xi - D^T w)\right\}.$$

Using (30) and (31) in the two formulas displayed above shows that they are the same. Lemma 3.1 finishes the proof. ∎

Example 6.4: Consider a quadratic convex-concave h with a quadratic conjugate h^*, that is, for an invertible matrix $M = \begin{bmatrix} Q & S \\ S^T & -R\end{bmatrix}$ with symmetric and positive semidefinite Q, R, let

$$h(c,d) = \frac{1}{2}\begin{bmatrix} c \\ d\end{bmatrix}\cdot M\begin{bmatrix} c \\ d\end{bmatrix},\ h^*(w,z) = \frac{1}{2}\begin{bmatrix} w \\ z\end{bmatrix}\cdot M^{-1}\begin{bmatrix} w \\ z\end{bmatrix}.$$

Throughout this example, we will say that a system is M-dissipative if it has a storage function in $\mathcal{L}$ supporting some $\gamma\ge 0$ and the supply rate $-h(y,d)$ with h defined above. According to Theorem 6.3, the system (27) is M-dissipative if and only if its dual (28) is $\widehat{M}$-dissipative, with

$$\widehat{M} = \begin{bmatrix} 0 & I \\ -I & 0\end{bmatrix} M^{-1}\begin{bmatrix} 0 & I \\ -I & 0\end{bmatrix}.$$

Special cases include

- ◇ *Passivity.* By passivity, we mean M-dissipativity with $M = \begin{bmatrix} 0 & -I \\ -I & 0\end{bmatrix}$. See, e.g., [12]. In this case $\widehat{M} = M$. Thus, passivity for an LDI is equivalent to passivity for its dual.
- ◇ *Passivity with extra feedforward.* By passivity with extra feedforward, we mean M-dissipativity with $M = \begin{bmatrix} 0 & -I \\ -I & -R\end{bmatrix}$ for some $R = R^T \ge 0$. In this case, we have $M^{-1} = \begin{bmatrix} R & -I \\ -I & 0\end{bmatrix}$ and then $\widehat{M} = M$. Thus, passivity with extra feedforward for an LDI is equivalent to this property for its dual.
- ◇ *Finite $\mathcal{L}_2$-gain.* By $\mathcal{L}_2$-stable with gain $\gamma>0$ we mean M-dissipative with $M = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I\end{bmatrix}$. In this case we get $\widehat{M} = \gamma^{-2}M$. Since M-dissipativity is not affected by positive scaling, $\mathcal{L}_2$-stability with gain γ for an LDI is equivalent to this property for its dual.
- ◇ *Weighted $\mathcal{L}_2$-gain.* For $M = \begin{bmatrix} Q & 0 \\ 0 & -R\end{bmatrix}$ one obtains $\widehat{M} = \begin{bmatrix} R^{-1} & 0 \\ 0 & -Q^{-1}\end{bmatrix}$. So, in passing from a system to its dual, the weights on the inputs in the supply rate become inverted weights on the outputs and vice-versa.

The next example shows that the results on Lyapunov inequalities can be derived from the dissipation results by judicious use of positive infinity.

Example 6.5: Consider two convex, proper and lower semicontinuous functions f and g, such that one of f, g and one of f^*, g^* is finite everywhere (as usual, f and g are positively homogeneous of degree p). Then consider $h(c,d) = f(c) - g(d)$, which is equivalent to $h^*(w,z) = f^*(w) - g^*(z)$ (relations (32), (33) are true). Below, we will use the indicator function of 0:

$$\delta_0(\xi) = 0 \text{ if } \xi = 0,\quad \delta_0(\xi) = +\infty \text{ if } \xi\ne 0. \tag{36}$$

It is a convex, proper, and lower semicontinuous function, positively homogeneous of any degree. Its conjugate equals 0 everywhere.

Through careful use of the indicator function, Theorem 5.1 can be be derived from Theorem 6.3. In (34), consider $C=0$, $D=0$, and $h(c,d) = \delta_0(c) - g(d)$. Then $h(Cx+Dd,d) = \delta_0(0) - g(d)$, and (34) becomes

$$\partial V(x)\cdot(Ax+Bd) \le -\gamma p V(x) + g(d).$$

We also get $h^*(w,z) = -g^*(z)$, and the equivalent inequality (35) turns to

$$\partial V^*(\xi)\cdot A^T\xi \le -\gamma q V^*(\xi) - g^*(-B^T y).$$

Setting $B=0$, $g(d)=0$ for all d (and so $g^*(-B^T\xi) = \delta_0(0) = 0$) yields Theorem 5.1.

VII. Stabilizable and Detectable Dissipativity

Theorem 7.1: Suppose that positive definite of degree 2 functions k, k^* satisfy

$$k(z,u,d) = \sup_w \left\{z\cdot w - f(w,u,d)\right\},$$
$$k^*(w,a,b) = \inf_d\sup_u \left\{a\cdot u - b\cdot d + f(w,u,d)\right\},$$

for some function f such that, for all α, β, and d, the $\sup_u\inf_v$ and $\inf_v\sup_u$ of $\alpha\cdot u + \beta\cdot v - f(w,u,d)$ are equal. Then, the following are equivalent:

(a) for all x, d, there exists u such that

$$\begin{aligned}\partial V(x)\cdot(Ax+B_1u+B_2d) &\le -\gamma V(x)\\ &\quad - k(Cx+D_1u+D_2d,u,d),\end{aligned} \tag{37}$$

(b) for all ξ, w

$$\begin{aligned}\partial V^*(\xi)\cdot(A^T\xi+C^Tw) &\le -\gamma V^*(\xi)\\ &\quad + k^*(w,-B_1^T\xi-D_1^Tw,-B_2^T\xi-D_2^Tw)\end{aligned} \tag{38}$$

The proof is similar to that of Theorem 6.3. The technical assumption on k and k^* holds for example when $k(z,u,d) = g_1(z)+g_2(u)-g_3(d)$ with g_i being proper, lower semicontinuous, and convex functions, and $k^*(w,a,b) = g_1^*(w)+g_2^*(a)-g_3^*(b)$. This is the case in the examples of this section. We add that in several cases, (a) can be equivalently restated as

(a') for all x, there exists u such that, for all d the inequality (37) holds.

This amounts to switching $\sup_d \inf_u$ to $\inf_u \sup_d$ of

$$\xi\cdot B_1u+\xi\cdot B_2d+k(Cx+D_1u+D_2d,u,d)$$

for all $\xi\in\partial V(x)$ and all x. For a quadratic k, this is possible when $k(Cx+D_1u+D_2d,u,d)$ is positive definite in u for a fixed d and negative definite in d for a fixed u.

Corollary 7.2: Let g be a convex function and g^* its conjugate in the convex sense. The following conditions are equivalent:

(a) for all x, there exists u such that

$$\begin{aligned}&\partial V(x)\cdot(Ax+Bu)\\ &\qquad\le -\gamma V(x)-g(Cx+Du,u),\end{aligned} \tag{39}$$

(b) for all ξ, for all w,

$$\begin{aligned}&\partial V^*(\xi)\cdot(A^T\xi+C^Tw)\\ &\qquad\le -\gamma V^*(\xi)+g^*(w,-B^T\xi-D^Tw).\end{aligned} \tag{40}$$

Proof: In Theorem 7.1, consider $k(z,u,d)=g(z,u)$, $B_1=B$, $B_2=0$, $D_1=D$, and $D_2=0$. Then (37) reduces to (39). We also have $k^*(w,a,b) = g^*(w,a)+\delta_0(b)$. (Recall (36).) Then $h^*(w,-B_1^T\xi-D_1^Tw,-B_2^T\xi-D_2^Tw) = g^*(w,-B^T\xi-D^Tw)+\delta_0(0)$, and thus (38) reduces to (40). ∎

Example 7.3: In (37), consider $D=0$, $\gamma=0$, and for a closed convex cone (say, the nonnegative orthant), let $g(z,u) = \frac{1}{2}\|z\|^2+\frac{1}{2}\|u\|^2$ if $u\in K$, $g(z,u)=+\infty$ if $u\notin K$. Then (a) states that for all x,

$$H(x,-\partial V(x))\ge 0 \tag{41}$$

where H is the (maximized) Hamiltonian for an optimal control problem with dynamics $\dot x(t)=Ax(t)+Bu(t)$ and cost $g(Cx,u)$ (this problem is a linear-quadratic regulator with input constraint). Precisely, we have

$$H(x,p)=p\cdot Ax-\frac{1}{2}\|Cx\|^2+\sup_{u\in K}\left\{(B^Tp)\cdot u-\frac{1}{2}\|u\|^2\right\}.$$

Condition (b) translates to: for all ξ,

$$H(-\partial V^*(\xi),\xi)\ge 0. \tag{42}$$

Of course, equivalence of (41), (42) can be deduced from the fact that ∂V^* is the inverse of ∂V.

Example 7.4: Stabilizability: for all x there exists u such that

$$\partial V(x)\cdot(Ax+Bu)\le -\gamma V(x) \tag{43}$$

can be written as (39). It suffices to take $C=0$, $D=0$, and $g(z,u)=\delta_0(z)$. Then also $g^*(w,a)=\delta_0(a)$ and (40) is equivalent to detectability: for all ξ,

$$B^T\xi=0 \quad\Rightarrow\quad \partial V^*(\xi)\cdot A^T\xi\le -\gamma V^*(\xi) \tag{44}$$

Indeed, (40) trivially holds when $B^T\xi\neq 0$, while when $B^T\xi=0$, it reduces to the inequality above.

VIII. Numerical Examples

Example 8.1: In [4], an LDI given by $\mathrm{co}\{A_1,A_2\}$ with

$$A_1=\begin{bmatrix}-1 & -1\\ 1 & -1\end{bmatrix},\quad A_2=\begin{bmatrix}-1 & -a\\ -1/a & -1\end{bmatrix}$$

and $a>1$, was used to show that existence of a common quadratic Lyapunov function is not necessary for exponential stability. The maximal a ensuring existence of such function was found to be $a_q = 3+\sqrt{8} = 5.8284$, while the LDI was shown, via analytical methods not leading to a Lyapunov function, to be stable for all $a\in[1,10]$.

Here we show using Corollaries 5.5 and 5.6 that the composite quadratic function q^* (12) and the maximum of quadratics q (11) can lead to good estimates of largest a guaranteeing stability.

With q^* formed by two quadratics ($l=2$), the maximal a is 8.14. With $l=3$, the maximal a is 8.95. The three matrices Q_i determined under $a=8.95$ are as follows:

$$Q_1=\begin{bmatrix}26.1802 & -0.0273\\ -0.0273 & 2.9146\end{bmatrix},\ Q_2=\begin{bmatrix}16.5961 & 3.0303\\ 3.0303 & 3.6388\end{bmatrix}$$

$$Q_3=\begin{bmatrix}32.5579 & -3.0335\\ -3.0335 & 1.8518\end{bmatrix}.$$

Corresponding ellipsoids (points x with $x\cdot Q_i^{-1}x=1$) and their convex hull (points x with $q^*(x)=1$) are in the upper two plots of Fig. 1. Also plotted there are directions of $\dot x = A_1x$ (left) and that of $\dot x = A_2x$ (right) along the boundary of the convex hull. Lower plots of Fig. 1 are the ellipsoids $x\cdot Q_ix=1$ and their intersection, along with the direction of $\dot y = A_1^Ty$ and $\dot y = A_2^Ty$ along the boundary of the intersection.

Example 8.2: The following third-order LDI that was discussed in [3]. For matrices

$$A_1=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -4\end{bmatrix},\quad M=\begin{bmatrix}-2 & 0 & -1\\ 1 & -10 & 3\\ 3 & -4 & 2\end{bmatrix},$$

let $A_2=A_1+aM$ with $a>0$, and consider the LDI with the state matrix belonging to the set $\mathrm{co}\{A_1,A_2(a)\}$. The

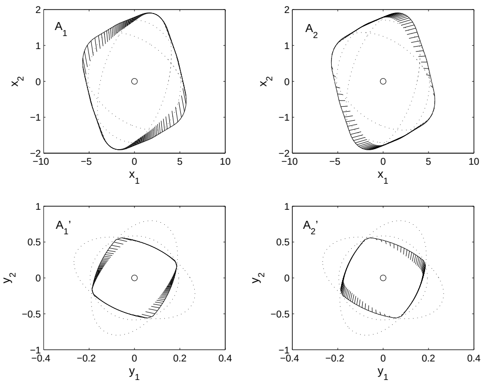

Fig. 1. Vector fields and invariant sets

maximal a that ensures the existence of a common quadratic function is $a_q = 1.9042$. The maximal a that ensures the existence of a common forth-order homogeneous Lyapunov function was found to be $a_h = 75.1071$ by [3].

By Corollary 5.3, exponential stability of the LDI is equivalent to that of the dual LDI described by $\mathrm{co}\{A_1^T, A_2(a)^T\}$. For this dual system, we used the method from [3] to determine a parameter range of a over which a common fourth-order homogeneous Lyapunov function exists. It turns out that there is no upper bound for a. Let A_{m1} be the augmented matrix for A_1 and $A_{m2}(a)$ be the augmented matrix for $A_2(a)$. Let $L(\alpha)$ be the matrix containing auxiliary parameters (see, page 1032 of [3]). Then for each $a > 0$, there exist a symmetric positive definite matrix $Q = \mathbb{R}^{6\times 6}$ and parameters $\alpha, \beta \in \mathbb{R}^6$ such that

$$QA_{m1} + A_{m1}^T Q + L(\alpha) \leq -0.0606Q,$$

$$QA_{m2}(a) + A_{m2}^T(a)Q + L(\beta) \leq -0.0606Q.$$

No numerical problem arises even for $a = 10^{20}$.

We also performed stability analysis on these dual systems using the composite quadratic Lyapunov function (12) with $l = 2$. Stability can be verified with such function for a up to 430. For $a = 430$, it can be verified that there exist $Q_1 > 0$ and $Q_2 > 0$ satisfying

$$\begin{array}{l} Q_1A_1^T + A_1Q_1 < 5.1(Q_2 - Q_1) \\ Q_2A_1^T + A_1Q_2 < 0 \\ Q_1A_2^T + A_2Q_1 < 0 \\ Q_2A_2^T + A_2Q_2 < 2654.7(Q_1 - Q_2) \end{array}$$

The same algorithm used for the dual LDI shows that there is no upper bound for a. Actually, for each $a > 0$, there exist $Q_1, Q_2 > 0$ and $k_{ij} \geq 0$ such that

$$Q_kA_i^T + A_iQ_k < k_{jk}(Q_j - Q_k) - 0.0530Q_k$$

for $i, j, k = 1, 2,\ j \neq k$. We tested a up to 10^{20} and no numerical issues occur. For $a = 10^8$, $k_1 = 3.5473$, $k_2 = 0$, $k_3 = 87614$, $k_4 = 7.4699 * 10^8$.

Example 8.3: We next use the max function q (11) and the composite quadratic function q^* (12) to bound the $\mathcal{L}_2$-gain of an LDI with input and output. With A_1, A_2 as in Example 8.2, for $S = [1 \quad 1 \quad 1]$ let $B_1 = B_2 = S^T$, $C_1 = C_2 = S$, $D_1 = D_2 = 0$.

Lemma 3.2 and Corollary 5.5 suggest that γ is a upper bound for the $\mathcal{L}_2$-gain, verified through q (with two quadratics), if there exist $Q_1, Q_2 > 0$ with $k_{ijk} > 0$, $i, j, k = 1, 2$, $j \neq k$ such that

$$\begin{bmatrix} A_i^TQ_k + Q_kA_i + C_i^TC_i + k_{ijk}(Q_k - Q_j) & Q_kB_i \\ B_i^TQ_k & -\gamma^2 I \end{bmatrix} < 0$$

Invoking Theorem 6.3 and Example 6.4 shows that γ is a upper bound for the $\mathcal{L}_2$-gain, verified through q^* (with two quadratics), if the above inequality holds with B_i's switched to C_i^T's and vice versa, and A_i's to A_i^T's.

Case 1: $a = 1$. The system is quadratically stable. With the quadratic Lyapunov function, the gain is estimated as $\gamma_2 = 14.7475$. With q, the gain is estimated as $\gamma = 8.4213$, the parameters are $k_1 = 3.9800$, $k_2 = k_3 = 0$, $k_4 = 12.4726$. With q^*, the gain is estimated as $\gamma^* = 9.3988$. $k_1 = k_4 = 0$, $k_2 = 9.7841$, $k_3 = 18.6882$.

Case 2: $a = 100$. The system is not quadratically stable. With q, the gain is estimated as $\gamma = 34.8763$, the parameters are $k_1 = 3.3615$, $k_2 = k_3 = 0$, $k_4 = 756.3307$. With q^*, the gain is estimated as $\gamma^* = 85.1115$. The parameters are $k_1 = 5.1000$, $k_2 = k_3 = 0$, $k_4 = 652.3610$.

Case 3: $a = 10^5$. The stability is not confirmed with q^*. With q, the gain is estimated as $\gamma = 57.6956$, the parameters are $k_1 = 3.5587$, $k_2 = k_3 = 0$, $k_4 = 7.4682 \times 10^5$.

Case 4: $a = 10^8$, $\gamma = 57.7388$.

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