

Set-valued Lyapunov functions for difference inclusions

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Abstract

The paper relates set-valued Lyapunov functions to pointwise asymptotic stability in systems described by a difference inclusion. Pointwise asymptotic stability of a set is a property which requires that each point of the set be Lyapunov stable and that every solution to the inclusion, from a neighborhood of the set, be convergent and have the limit in the set. Weak set-valued Lyapunov functions are shown, via an argument resembling an invariance principle, to imply this property. Strict set-valued Lyapunov functions are shown, in the spirit of converse Lyapunov results, to always exist for closed sets that are pointwise asymptotically stable.

Keywords: set-valued Lyapunov function, converse Lyapunov result, difference inclusion, consensus, asymptotic stability, continuum of equilibria

1. Introduction

Set-valued Lyapunov functions were suggested in Moreau (2005) for the purpose of studying convergence to a consensus in multi-agent systems. In Moreau (2005), Angeli and Bliman (2006), and Lorenz and Lorenz (2010), they were used in sufficient conditions for a property requiring that each equilibrium be Lyapunov stable and that solutions to a system, from a neighborhood of the set of equilibria, be convergent to an equilibrium. This property reduces to asymptotic stability for an isolated equilibrium, but is more restrictive for a continuum, compact or not, of equilibria. Difference equations were considered in Moreau (2005), difference inclusions in Angeli and Bliman (2006), while Lorenz and Lorenz (2010) focused on generalizing the class of set-valued Lyapunov functions.

The same property was studied before, in Bhat and Bernstein (2003), in the setting of differential equations. Extensive discussion of the property is included there, including motivation in kinetics of chemical reactions. Sufficient conditions were also given in Bhat and Bernstein (2003), in terms of classical Lyapunov functions and non-tangential behavior of the vector field. Some of these results were extended to a broader setting in Hui et al. (2009). The usual Lyapunov inequalities, on their own, can not fully characterize the property; this was also pointed out in Moreau (2005) in motivating the set-valued approach. In Hui et al. (2008), also in the setting of differential equations, a converse Lyapunov result was in fact given in terms of a classical Lyapunov function, but only yielded a necessary, not a sufficient condition. Further motivation, from consensus problems in the continuous-time setting, was also included in Hui et al. (2008).

Working with difference equations and inclusions, this

paper develops some basic theory of set-valued Lyapunov functions and relates them to pointwise asymptotic stability, which is the term chosen here for the property discussed above.¹ A stricter decrease condition, than those used in Moreau (2005), Angeli and Bliman (2006), is proposed, to match more closely the usual Lyapunov inequalities. A sufficient condition for asymptotic stability, using a nonincreasing set-valued Lyapunov function, is given. It relies on invariance arguments, in the spirit of results due to Barabasin, Krasovskii, and LaSalle, and extracts and unifies some arguments used by (Moreau, 2005, Theorem 4) and (Angeli and Bliman, 2006, Theorem 5).

This paper also states two converse set-valued Lyapunov results, showing the existence of a strictly decreasing set-valued Lyapunov function for pointwise asymptotically stable systems. One applies to continuous difference inclusions and noncompact sets of equilibria, the other to compact sets of equilibria for difference inclusions under constraints. In short, strict set-valued Lyapunov functions are both necessary and sufficient for pointwise asymptotic stability. As it is for the usual asymptotic stability, converse set-valued Lyapunov results imply appropriately understood robustness of pointwise asymptotic stability. For background on converse Lyapunov results for the usual asymptotic stability and their relation to robustness, consult Kellett and Teel (2005) and the references therein.

While the papers Moreau (2005), Angeli and Bliman (2006) consider communication links in consensus analysis,

¹The term “semistability” was used in Bhat and Bernstein (2003) to describe a property of a single equilibrium point, and in Hui et al. (2008) and Hui et al. (2009) to describe a property of a differential equation, which match what is required here for pointwise asymptotic stability of the set of all equilibria.

set-valued Lyapunov conditions are used there in settings where communication is no longer an issue. For example, they are applied not to the dynamics, which may depend on the communication graph, but to iteration of the dynamics over a sufficiently long time interval, over which the communication graphs have a connectedness property. This motivates the focus of this paper, set-valued Lyapunov functions, and is behind the absence here of any issues related to communication links.

2. Background and definitions

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping, with nonempty values at each $x \in \mathbb{R}^n$, and consider the difference inclusion on \mathbb{R}^n :

$$x^+ \in F(x). \quad (1)$$

Solutions to (1) are functions $\phi : \mathbb{N} \rightarrow \mathbb{R}^n$ such that, for every $j \in \mathbb{N}$, $\phi(j+1) \in F(\phi(j))$. That is, only complete solutions are considered. For a given set $C \subset \mathbb{R}^n$, $\mathcal{S}(C)$ denotes the set of all solutions to (1) with $\phi(0) \in C$.

Classical definitions of asymptotic stability focus on a single equilibrium. Many reasons suggest considering sets of equilibria, one of which comes from analysis of convergence to consensus.

Example 2.1. Let $n = km$ and write $x \in \mathbb{R}^n$ as $x = (x_1, x_2, \dots, x_k)$, where $x_i \in \mathbb{R}^m$, $i = 1, 2, \dots, k$. This can be thought of as x representing the states x_i of k subsystems or agents. Consensus between agents corresponds to all x_i being the same. The set of all consensus states,

$$\mathcal{A} = \{x \in \mathbb{R}^n \mid x_1 = x_2 = \dots = x_k\}, \quad (2)$$

is an m -dimensional subspace of \mathbb{R}^n . \triangle

Extensions of the usual concept of asymptotic stability to an equilibrium set require, usually, that solutions from points near the equilibrium set remain near it (stability) and converge to it (local attractivity). This does not require that solutions from points near the equilibrium set have limits — it is only their distance from the equilibrium set that has to converge to 0.

A more restrictive definition, requiring that each point of an equilibrium set be stable and that every solution have a limit in the equilibrium set, is now stated. Below, \mathbb{B} is the closed unit ball in \mathbb{R}^n , so $a + \delta\mathbb{B}$ is the closed ball of radius δ centered at a . The notation $\text{rge } \phi$ represents the range of ϕ , so $\text{rge } \phi \subset A + \varepsilon\mathbb{B}$ means that $\phi(j)$ is in the closed ε -neighborhood of the set A , for all $j \in \mathbb{N}$.

Definition 2.2 (pointwise asymptotic stability).

The set $A \subset \mathbb{R}^n$ is locally pointwise asymptotically stable for (1) if:

- every point $a \in A$ is stable, in the sense that, for every $a \in A$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that every $\phi \in \mathcal{S}(a + \delta\mathbb{B})$ satisfies $\text{rge } \phi \subset a + \varepsilon\mathbb{B}$;

- there exists a neighborhood U of A such that for every solution $\phi \in \mathcal{S}(U)$, the limit $\lim_{j \rightarrow \infty} \phi(j)$ exists and $\lim_{j \rightarrow \infty} \phi(j) \in A$.

The basin of attraction, denoted $\mathcal{B}(A)$, of a locally pointwise asymptotically stable set $A \subset \mathbb{R}^n$ is the set of all $x \in \mathbb{R}^n$ such that for every solution $\phi \in \mathcal{S}(x)$, the limit $\lim_{j \rightarrow \infty} \phi(j)$ exists and $\lim_{j \rightarrow \infty} \phi(j) \in A$. If $\mathcal{B}(A) = \mathbb{R}^n$, A is globally pointwise asymptotically stable.

Unless A is a single point, the usual Lyapunov functions, which are functions that are positive definite with respect to A and which decrease along solutions to (1), are inadequate to characterize pointwise asymptotic stability of the set A . (Positive definiteness with respect to A means that the function is 0 on A and positive elsewhere.) Indeed, they can not ensure stability of every point in A and can only guarantee that the distance of a solution from A decrease to 0, but not that the solution have a limit. To overcome this, Moreau (2005) proposed a concept of a set-valued Lyapunov function. More precisely:

Definition 2.3. Given sets U and $A \subset U$, a set-valued Lyapunov function candidate for A on U is a set-valued mapping $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that

- $x \in W(x)$ for every $x \in U$;
- $W(x) = \{x\}$ for every $x \in A$;
- W is outer semicontinuous and locally bounded at every point of U .

In this paper, a set-valued mapping $M : U \rightrightarrows \mathbb{R}^n$ is understood to associate, to every point $x \in U$, a set $M(x) \subset \mathbb{R}^n$. The mapping M is locally bounded at $x \in U$ if $M(N) := \bigcup_{z \in N} M(z)$ is bounded, for some neighborhood N of x . The mapping M is outer semicontinuous at x if for any sequences $x_i \rightarrow x$, $y_i \rightarrow y$ with $y_i \in M(x_i)$ one has $y \in M(x)$. It is continuous if for every $y \in M(x)$, every sequence $x_i \rightarrow x$, there exist $y_i \in M(x_i)$ such that $y_i \rightarrow y$. When M is locally bounded and has closed values, outer semicontinuity agrees with what is sometimes called upper semicontinuity. Consult Rockafellar and Wets (1998) for these and other set-valued analysis notions used below.

The result (Moreau, 2005, Theorem 4), extended from difference equations to inclusions in (Angeli and Bliman, 2006, Theorem 5), states that the existence of a set-valued Lyapunov function candidate for A on \mathbb{R}^n , with the decrease condition:

there exists a function $\mu : 2^{\mathbb{R}^n} \rightarrow \mathbb{R}_{\geq 0}$, defined on subsets of \mathbb{R}^n , such that $x \mapsto \mu(W(x))$ is locally bounded at every point of \mathbb{R}^n , and a function $\beta : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, positive definite with respect to A , such that

$$\mu(W(y)) \leq \mu(W(x)) - \beta(x) \quad (3)$$

for all $y \in F(x)$, all $x \in A$

ensures pointwise asymptotic stability of A . In Moreau (2005), μ is the diameter of a set and is applied to convex sets $W(x)$, while Angeli and Bliman (2006) extends the concept to nonconvex sets $W(x)$. Both Moreau (2005) and Angeli and Bliman (2006) include examples beyond what is mentioned below. Remark 4.5 sheds more light on μ and β in the case of W being the reachable set.

A common example of a set-valued Lyapunov function candidate is based on the concept of a convex hull.

Example 2.4. In the setting of Example 2.1, let $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be given by

$$W(x) = (\text{con}\{x_1, x_2, \dots, x_k\})^k$$

for each $x \in \mathbb{R}^n$. That is, for $x = (x_1, x_2, \dots, x_k)$ with $x_i \in \mathbb{R}^m$, $W(x)$ is the Cartesian product of k convex hulls $\text{con}\{x_1, x_2, \dots, x_k\}$ of the points x_1, x_2, \dots, x_k . Then W is a continuous set-valued Lyapunov function candidate for A on \mathbb{R}^n , with A as in (2). \triangle

A general class of set-valued mappings, resembling set-valued Lyapunov function candidates and subsuming the function in Example 2.4, was considered in Lorenz and Lorenz (2010), along with a weaker decrease condition:

Example 2.5. In the setting of Example 2.1, let $g_i : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $i = 1, 2, \dots, l$, be continuous and such that

$$x_i \in w(x) := \text{con}\{g_1(x), g_2(x), \dots, g_l(x)\}$$

for all $x \in \mathbb{R}^n$, $i = 1, 2, \dots, k$. For example, with the use of $l = 2^m$ functions, one may obtain $w(x)$ to be the smallest m -dimensional box, i.e., a Cartesian product of intervals, containing x_1, x_2, \dots, x_k . Let $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be given by

$$W(x) = (w(x))^k.$$

Such W is continuous and meets the first condition for a set-valued Lyapunov function candidate for A on \mathbb{R}^n , with A as in (2). The condition that ensures that $W(x) = \{x\}$ holds for all $x \in A$ is that $g_i((z, z, \dots, z)) = z$ for all $z \in \mathbb{R}^m$, $i = 1, 2, \dots, l$. \triangle

A natural set-valued mapping, the values of which decrease along solutions to (1), is the infinite-horizon reachable set. This is discussed further in the example below.

Example 2.6. Let $R : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be the set-valued mapping given, at each x , as the reachable set from x :

$$R(x) = \{y \in \mathbb{R}^n \mid \exists \phi \in \mathcal{S}(x), j \in \mathbb{N} \text{ such that } y = \phi(j)\}$$

Alternatively, $R(x) = \bigcup_{\phi \in \mathcal{S}(x)} \text{rge } \phi$. It is always the case that $x \in R(x)$ and that $R(y) \subset R(x)$ for any $y \in F(x)$. If a closed set $A \subset \mathbb{R}^n$ is pointwise asymptotically stable, then there exists a neighborhood U of A such that R is a set-valued Lyapunov function candidate for A on U . Furthermore, R is outer semicontinuous at every point

$x \in A$. These properties come almost directly from the definitions. Outer semicontinuity of R at points $x \notin A$ should not be expected, since $R(x)$ may easily fail to be closed. For example, for the system $x^+ = x/2$, and a point $x \neq 0$, $0 \notin R(x)$ but $0 \in \overline{R(x)}$, where the latter set is the closure of $R(x)$. This motivates considering a set-valued mapping $\overline{R} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by

$$\overline{R}(x) = \overline{R(x)}.$$

Still, the existence of a neighborhood U of a closed set A such that \overline{R} is a set-valued Lyapunov function candidate for A on U , even with the additional property that \overline{R} be outer semicontinuous at every $x \in U$, does not imply pointwise asymptotic stability of A . This is visible in the case of $F(x) = x$ for all $x \in \mathbb{R}^n$, any closed A , and U being any neighborhood of A . \triangle

3. Sufficient Lyapunov conditions

Below, a simple and very strict decrease condition on a set-valued Lyapunov function candidate is considered. Later, Theorem 4.4 and Theorem 4.7 will show that functions satisfying this decrease condition always do exist, for pointwise asymptotically stable sets. Below, $W(F(x))$ denotes the set $\bigcup_{y \in F(x)} W(y)$, and similar meaning is attached to $W(C)$, for any set C .

Theorem 3.1. *Let $A \subset \mathbb{R}^n$ be closed, U be a neighborhood of A such that $F(U) \subset U$, and suppose that there exists a set-valued Lyapunov function candidate $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ for A on U such that*

there exists a function $\alpha : U \rightarrow \mathbb{R}_{\geq 0}$, continuous and positive definite with respect to A , such that

$$W(F(x)) + \alpha(x)\mathbb{B} \subset W(x) \quad \text{for all } x \in U. \quad (4)$$

If $U = \mathbb{R}^n$, then A is globally pointwise asymptotically stable. In general, if $W(U) \subset U$ or $\lim_{i \rightarrow \infty} \alpha(x_i) = 0$ implies $\lim_{i \rightarrow \infty} x_i \in A$ for every convergent sequence of points $x_i \in U$, then A is locally pointwise asymptotically stable and $U \subset \mathcal{B}(A)$.

Proof. Pick any $x \in A$. Since W is outer semicontinuous and locally bounded at x , for every $\varepsilon > 0$ there exists $\delta > 0$ such that $W(x + \delta\mathbb{B}) \subset W(x) + \varepsilon\mathbb{B} = x + \varepsilon\mathbb{B}$. For any $\phi \in \mathcal{S}(U)$, (4) implies that $W(\phi(1)) \subset W(\phi(0))$, and Definition 2.3 gives $\phi(1) \in W(\phi(0))$. Repeated use of these two facts yields $\text{rge } \phi \subset W(\phi(0))$. Then, $\text{rge } \phi \subset x + \varepsilon\mathbb{B}$ for any $\phi \in \mathcal{S}(x + \delta\mathbb{B})$. This shows stability of x .

Now consider any solution $\phi \in \mathcal{S}(U)$. Since $\text{rge } \phi \subset W(\phi(0))$, local boundedness of W , hence boundedness of $W(\phi(0))$, implies that every solution is bounded. Furthermore, one has $W(\phi(1)) + \alpha(\phi(0)) \subset W(\phi(0))$, $W(\phi(2)) + \alpha(\phi(1)) + \alpha(\phi(0)) \subset W(\phi(0))$, etc. and hence $\sum_{j=1}^{\infty} \alpha(\phi(j))$ is finite. This yields $\alpha(\phi(j)) \rightarrow 0$ as $j \rightarrow \infty$. If $W(\phi(0)) \subset U$, which always holds when $W(U) \subset U$ and,

in particular, when $U = \mathbb{R}^n$, compactness of $W(\phi(0))$ combined with continuity and positive definiteness of α implies that the set of cluster points of the sequence $\{\phi(j)\}_{j=1}^{\infty}$ is a subset of A . When $W(\phi(0))$ is not a subset of U , the condition that $\lim_{i \rightarrow \infty} \alpha(x_i) = 0$ implies $\lim_{i \rightarrow \infty} x_i \in A$ for every convergent sequence of points $x_i \in U$ results in the same conclusion: cluster points of $\{\phi(j)\}_{j=1}^{\infty}$ are in A . But any point of A is stable, and consequently, if $a \in A$ is a cluster point, it must be that $\lim_{j \rightarrow \infty} \phi(j) = a$. \square

The results (Moreau, 2005, Theorem 4) and (Angeli and Bliman, 2006, Theorem 5), which relied on the decrease condition (3), were stated for time-dependent systems. Theorem 3.1 and its proof applies, essentially without change, to such a case when (4) is replaced by $W(F(j, x)) + \alpha(x)\mathbb{B} \subset W(x)$ for all $x \in U$, $j \in \mathbb{N}$.

The sufficient condition for pointwise asymptotic stability in Theorem 3.1 does not require any regularity from F defining the dynamics. Some conditions about the regularity of W could be weakened as well. Some regularity of F is needed when a decrease condition weaker than (4) is used and one relies on invariance arguments.

Assumption 3.2. *The set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous, locally bounded, and has nonempty values at each $x \in \mathbb{R}^n$.*

Theorem 3.3. *Under Assumption 3.2, let $A \subset \mathbb{R}^n$ be closed, U be a neighborhood of A such that $F(U) \subset U$, and suppose that there exists a set-valued Lyapunov function candidate $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ for A on U which is continuous at every point of \bar{U} , the closure of U , such that*

$$W(F(x)) \subset W(x) \quad \text{for all } x \in U. \quad (5)$$

Furthermore, assume that for any $x \in \bar{U}$, if $W(y) = W(x)$ for some $y \in F(x)$ then $x \in A$. Then A is locally pointwise asymptotically stable and $U \subset \mathcal{B}(A)$.

Proof. Stability of every $x \in A$ is shown as in the proof of Theorem 3.1. Consider any $\phi \in \mathcal{S}(U)$ and, for $i = 1, 2, \dots$, let $W_i = W(\phi(i))$. By (5), the sequence $\{W_i\}_{i=1}^{\infty}$ is nonincreasing, in the sense that $W_{i+1} \subset W_i$, $i = 1, 2, \dots$. Hence, the sequence has a limit, $\lim_{i \rightarrow \infty} W_i = \bigcap_{i=1}^{\infty} W_i =: \widehat{W}$; see (Rockafellar and Wets, 1998, Exercise 4.3). Let $\Omega(\phi)$ be the ω -limit of ϕ , i.e.,

$$\Omega(\phi) = \{x \in \mathbb{R}^n \mid x \text{ is a cluster point of } \{\phi(j)\}_{j=1}^{\infty}\}.$$

Then $\Omega(\phi)$ is a nonempty, compact subset of \bar{U} which is weakly forward invariant for (1): for any $x \in \Omega(\phi)$ there exists a solution ψ to (1) such that $\psi(0) = x$ and $\text{rge } \psi \subset \Omega(\phi)$. For any $x \in \Omega(\phi)$, continuity of W implies that $W(x) = \widehat{W}$. Indeed, if $x = \lim_{k \rightarrow \infty} \phi(j_k)$ for some subsequence $\{j_k\}_{k=1}^{\infty}$ of \mathbb{N} , then $W(x) = \lim_{k \rightarrow \infty} W_{j_k} = \widehat{W}$. Weak forward invariance of (1) is equivalent to the property that, for any $x \in \Omega(\phi)$, there exists $y \in F(x)$ such that $y \in \Omega(\phi)$. Thus, for any $x \in \Omega(\phi)$, there exists $y \in F(x)$

such that $W(y) = W(x) = \widehat{W}$. Hence $\Omega(\phi) \subset A$. Stability of every point in A now implies that ϕ has a limit in A . \square

Remark 3.4. *Without the assumption that $W(y) = W(x)$ for some $y \in F(x)$ implies $x \in A$, Theorem 3.3 would conclude that any solution ϕ converges to a weakly forward invariant set $\Omega(\phi)$, and this set is such that for any $x \in \Omega(\phi)$ there exists $y \in F(x)$ such that $W(y) = W(x)$. This resembles the usual invariance principle which concludes, from a nonincrease condition using a Lyapunov-like function, that solutions converge to an invariant set on which the Lyapunov-like function is constant.*

Time-dependent version of Theorem 3.3 requires some asymptotic properties of $F(j, x)$, as $j \rightarrow \infty$, as is the case for the usual invariance principles.

Example 3.5. The following system is motivated by an example in Lorenz and Lorenz (2010). Consider a continuous function $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by

$$F(x) = \begin{pmatrix} x_1 \\ a(|x_2 - x_3|)x_1 + [1 - a(|x_2 - x_3|)]x_2 \\ \frac{1}{2}(x_2 + x_3) \end{pmatrix},$$

where $a : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a continuous and nonincreasing function with $a(0) = 1/2$ and $a(d) = 0$ for all $d \geq 1$. Then, for solutions to $x^+ = F(x)$, x_1 is constant; x_3 moves towards x_2 ; while x_2 moves towards x_1 only when $|x_2 - x_3| < 1$, otherwise it remains constant. In the terminology of Lorenz and Lorenz (2010), this is a proper averaging map and convergence to consensus can be concluded. Here, it is noted that the ‘‘invariance principle’’, Theorem 3.3, applies. Let $W(x) = (x_1, \text{con}\{x_1, x_2\}, \text{con}\{x_2, x_3\})$, where $\text{con}\{x_1, x_2\}$ is just the interval containing x_1 and x_2 , so either $[x_1, x_2]$ or $[x_2, x_1]$. Such W is a set-valued Lyapunov function candidate, for the set \mathcal{A} as in Example 2.1, but does not satisfy (4) for any α . It does satisfy (5). Furthermore, if $W(F(x)) = W(x)$ then $x_1 = x_2 = x_3$. Theorem 3.3 implies pointwise asymptotic stability of \mathcal{A} . \triangle

4. Converse set-valued Lyapunov results

Converse set-valued Lyapunov results require some preliminary results.

Proposition 4.1. *Under Assumption 3.2, let $A \subset \mathbb{R}^n$ be a nonempty closed set that is locally pointwise asymptotically stable for (1). Then:*

- (a) *The basin of attraction $\mathcal{B}(A)$ of A is open.*
- (b) *The set-valued mapping $L : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by*

$$L(x) = \left\{ y \in \mathbb{R}^n \mid \exists \phi \in \mathcal{S}(x) \text{ such that } \lim_{j \rightarrow \infty} \phi(j) = y \right\}$$

is outer semicontinuous and locally bounded at every point of $\mathcal{B}(A)$. In particular, if $L(x)$ is a singleton at every $x \in \mathcal{B}(A)$, as it always is when F is single-valued, then, restricted to $\mathcal{B}(A)$, L is a continuous function.

(c) The set-valued mapping $\bar{R} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, as defined in Example 2.6, is outer semicontinuous and locally bounded at every point of $\mathcal{B}(A)$. Furthermore, $\bar{R}(x) = R(x) \cup L(x)$ for all $x \in \mathcal{B}(A)$.

Proof. The following argument and notation will be used in the proof. For any $x \in \mathcal{B}(A)$ and any sequence $\{\phi_i\}_{i=1}^\infty$ of solutions to (1) such that $\lim_{i \rightarrow \infty} \phi_i(0) = x$, a locally uniformly convergent subsequence $\{\phi_{i_k}\}_{k=1}^\infty$ can be extracted from the sequence. Denote the limit of the subsequence by $\bar{\phi}$. Then $\bar{\phi}$ is a solution to (1). Furthermore, since $\bar{\phi}(0) = x \in \mathcal{B}(A)$, $\bar{\phi}$ is convergent and $a := \lim_{j \rightarrow \infty} \bar{\phi}(j) \in A$.

To prove (a), suppose that $\mathcal{B}(A)$ is not open, i.e., for some $x \in \mathcal{B}(A)$ there exists a sequence $\{\phi_i\}_{i=1}^\infty$ of solutions to (1), each of which either does not have a limit or does have a limit which does not belong to A , while $\lim_{i \rightarrow \infty} \phi_i(0) = x$. Apply the argument in the paragraph above, and, using the pointwise asymptotic stability of A , pick $\delta > 0$ such that every $\phi \in \mathcal{S}(a + \delta\mathbb{B})$ has a limit and the limit is in A . There exists $J \in \mathbb{N}$ such that $\bar{\phi}(J) \in (\delta/2)\mathbb{B}$. Then $\phi_{i_k}(J) \in a + \delta\mathbb{B}$ for all large enough k , and consequently, solutions ϕ_{i_k} are convergent and have limit in A , for all large enough k . This contradiction proves (a).

To prove local boundedness of R , suppose, on the contrary, that some sequence $\{\phi_i\}_{i=1}^\infty$ of solutions to (1) with $\lim_{i \rightarrow \infty} \phi_i(0) = x \in \mathcal{B}(A)$ is not eventually uniformly bounded, and thus, without loss of generality, that, for all $j = 1, 2, \dots$, $|\phi_i(j_i)| > i$ for some j_i . Pass to a locally uniformly convergent subsequence $\{\phi_{i_k}\}_{k=1}^\infty$. As before, let $\bar{\phi}$ be the limit and $a = \lim_{j \rightarrow \infty} \bar{\phi}(j)$. Using pointwise asymptotic stability of A , pick $\delta > 0$ such that $\phi \in \mathcal{S}(a + \delta\mathbb{B})$ implies $\text{rge } \phi \in a + \mathbb{B}$. With J as in the paragraph above, there exists $K \in \mathbb{N}$ such that $\bar{\phi}_{i_k}(J) \in a + \delta\mathbb{B}$ for all $k \geq K$. Since ϕ_{i_k} converge uniformly to $\bar{\phi}$ on $\{1, 2, \dots, K\}$ and $\phi_{i_k}(j) \in a + \mathbb{B}$ for $j > J$, the subsequence $\{\phi_{i_k}\}_{k=K}^\infty$ is uniformly bounded. This is a contradiction. Hence R is locally bounded at every $x \in \mathcal{B}(A)$, and so is \bar{R} . Furthermore, since $L(x) \subset \bar{R}$ for all $x \in \mathcal{B}(A)$, L is locally bounded at such x as well.

To finish proving (b), take $x \in \mathcal{B}(A)$, a sequence $\{x_i\}_{i=1}^\infty$ convergent to x , a sequence $\{y_i\}_{i=1}^\infty$ with $y_i \in L(x_i)$ that is convergent to some y . It needs to be shown that $y \in L(x)$. Let $\{\phi_i\}_{i=1}^\infty$ be a sequence of solutions to (1) with $\phi_i(0) = x_i$, $\lim_{j \rightarrow \infty} \phi_i(j) = y_i$. As before, pass to a locally uniformly convergent subsequence. With the same notation as above, pick an $\varepsilon > 0$ and let $\delta > 0$ be such that every $\phi \in \mathcal{S}(a + \delta\mathbb{B})$ has a limit in A and $\text{rge } \phi \subset a + \varepsilon\mathbb{B}$. As in the paragraph above, there exists $J \in \mathbb{N}$ such that $\phi_{i_k}(J) \in a + \delta\mathbb{B}$, for all large enough k . Then, for such k , $y_{i_k} = \lim_{j \rightarrow \infty} \phi_{i_k}(j) \in a + \varepsilon\mathbb{B}$. Since ε is arbitrary, $y = \lim_{k \rightarrow \infty} y_{i_k} = a \in L(x)$. Hence (b) is shown.

To complete the proof of (c), take any sequence $\{x_i\}_{i=1}^\infty$ convergent to $x \in \mathcal{B}(A)$, a sequence $\{y_i\}_{i=1}^\infty$ with $y_i \in R(x_i)$ that is convergent to some y . It will be first shown that $y \in \bar{R}(x)$. Let $\{\phi_i\}_{i=1}^\infty$ be a sequence of solutions to (1) with $\phi_i(0) = x_i$, $y_i = \phi_i(j_i)$ for some $j_i \in \mathbb{N}$. If the sequence $\{j_i\}_{i=1}^\infty$ is uniformly bounded, then passing to a locally uniformly convergent subsequence $\{\phi_{i_k}\}_{k=1}^\infty$ can be done so that $j_{i_k} = J$, for some $J \in \mathbb{N}$ and all k . Then $y = \bar{\phi}(J)$ and hence $y \in R(x)$. Now suppose that $\{j_i\}_{i=1}^\infty$ is not uniformly bounded and pass to a locally uniformly convergent subsequence of solutions $\{\phi_{i_k}\}_{k=1}^\infty$ so that $\{j_{i_k}\}_{i=1}^\infty$ is still not uniformly bounded. Arguments very similar to those at the end of the previous paragraph now show that $y = \lim_{k \rightarrow \infty} \phi_{i_k}(j_{i_k}) = a \in L(x)$. It was thus shown that $y \in R(x) \cup L(x)$. Repeating this argument with $x_i = x$ shows that $\bar{R}(x) \subset R(x) \cup L(x)$. The opposite inclusion is clear, hence $\bar{R}(x) = R(x) \cup L(x)$. Finally, since $y_i \in R(x_i)$, $x_i \rightarrow x$, $y_i \rightarrow y$ implies $y \in \bar{R}(x)$, it is straightforward to show that the same conclusion holds with $y_i \in \bar{R}(x_i)$. The proof of (c) is finished. \square

A function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a class \mathcal{K}_∞ function if α is continuous, increasing, unbounded, and $\alpha(0) = 0$. A function $\omega : \mathcal{B}(A) \rightarrow \mathbb{R}_{\geq 0}$ is a proper indicator of a compact set $K \subset \mathcal{B}(A)$ on $\mathcal{B}(A)$ if ω is continuous, positive definite with respect to K , and $\omega(x_i) \rightarrow \infty$ if $\lim_{i \rightarrow \infty} |x_i| = \infty$ or $\lim_{i \rightarrow \infty} x_i \notin \mathcal{B}(A)$, for any sequence of points $x_i \in \mathcal{B}(A)$. The preliminary result below shows the so-called \mathcal{KL} -stability, with respect ω and $\bar{\omega}$, of a pointwise asymptotically stable set. The bound (6) is an equivalent formulation of the \mathcal{KL} -stability; see Sontag (1998) or Kellett and Teel (2005) for background on \mathcal{KL} -stability.

Proposition 4.2. *Under Assumption 3.2, let $A \subset \mathbb{R}^n$ be a nonempty closed set that is locally pointwise asymptotically stable for (1). Then, there exist continuous functions $\underline{\omega}, \bar{\omega} : \mathcal{B}(A) \rightarrow \mathbb{R}_{\geq 0}$, positive definite with respect to A on $\mathcal{B}(A)$, and \mathcal{K}_∞ functions α, β such that*

$$\alpha(\underline{\omega}(\phi(j))) \leq \beta(\bar{\omega}(\phi(0))) e^{-j} \quad (6)$$

for all $\phi \in \mathcal{S}(\mathcal{B}(A))$, $j \in \mathbb{N}$.

Proof. Let A_i , $i = 1, 2, \dots$ be nonempty compact subsets of A that form a locally finite cover of A . Let $C_i = L^{-1}(A_i) \cap \mathcal{B}(A)$, $i = 1, 2, \dots$. Outer semicontinuity of L implies that every C_i is relatively closed in $\mathcal{B}(A)$. Furthermore, the sets C_i form a locally finite cover of $\mathcal{B}(A)$. The set A_i is asymptotically stable for (1) constrained to C_i , in the sense that every $x \in A_i$ is stable for (1) and all solutions to (1) from C_i remain in C_i and converge to A_i . Let ω_i be any proper indicator of A_i on $\mathcal{B}(A)$. A consequence of (Cai et al., 2008, Theorem 3.14) is that there exist class \mathcal{K}_∞ functions α_i and β_i such that, for every $\phi \in \mathcal{S}(C_i)$ and $j \in \mathbb{N}$,

$$\alpha_i(\omega_i(\phi(j))) \leq \beta_i(\omega_i(\phi(0))) e^{-j}.$$

Then, for every $i = 2, 3, \dots$, $\phi \in \mathcal{S}(C_i)$, and $j \in \mathbb{N}$,

$$\alpha_1(\alpha_1^{-1}(\alpha_i(\omega_i(\phi(j)))))) \leq \beta_1(\beta_1^{-1}(\beta_i(\omega_i(\phi(0)))))) e^{-j}.$$

For each $x \in \mathcal{B}(A)$, let

$$\underline{w}(x) = \min_{\{i | x \in C_i\}} \alpha_1^{-1}(\alpha_i(\omega_i(x))),$$

$$\overline{w}(x) = \max_{\{i | x \in C_i\}} \beta_1^{-1}(\beta_i(\omega_i(x))).$$

Then, since the sets C_i form a locally finite cover of $\mathcal{B}(A)$, \underline{w} is positive definite with respect to A and \overline{w} is finite. Furthermore, \underline{w} is lower semicontinuous, and \overline{w} is upper semicontinuous. Now, pick any continuous functions $\underline{\omega}$ and $\overline{\omega}$ on $\mathcal{B}(A)$, positive definite with respect to A , such that $\underline{\omega} \leq \underline{w}$ and $\overline{\omega} \leq \overline{w}$. Then (6) holds with $\alpha = \alpha_1$, $\beta = \beta_1$. \square

4.1. A converse result for a continuous F

When F is a continuous set-valued mapping, one can show, using methods similar to what was used in the proof of Proposition 4.1, that \overline{R} is a set-valued mapping that is continuous at every $x \in \mathcal{B}(A)$. Then \overline{R} can be used to show that the sufficient condition in Theorem 3.3 is in fact necessary for global pointwise asymptotic stability:

Remark 4.3. *Let A be closed and globally pointwise asymptotically stable for (1), where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfies Assumption 3.2 and is continuous. Then the set-valued mapping $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by*

$$W(x) = \overline{R}(x) \quad \text{for every } x \in \mathbb{R}^n$$

is a set-valued Lyapunov function candidate for A on \mathbb{R}^n which is continuous, satisfies $W(F(x)) \subset W(x)$ for all $x \in \mathbb{R}^n$, and $W(F(x)) = W(x)$ implies that $x \in A$.

To see that $W(F(x)) = W(x)$ implies that $x \in A$, let $F(x) = y$ and note that $W(y) = W(x)$ implies that either $x = L(y)$, in which case $x \in A$, or that $x \in R(y) \subset R(x)$, which, by global pointwise asymptotic stability, is only possible if $x \in A$.

Below, a stronger converse result is shown. When F is continuous, the bound (6) established in Proposition 4.2, is equivalent to the existence of a smooth Lyapunov function; see (Kellett and Teel, 2005, Theorem 2.7, Theorem 2.10). That is, there exists a smooth function $V : \mathcal{B}(A) \rightarrow \mathbb{R}_{\geq 0}$ such that, for some $\gamma \in (0, 1)$,

$$V(F(x)) \leq \gamma V(x) \quad \text{for all } x \in \mathcal{B}(A), \quad (7)$$

and, for a pair \mathcal{K}_∞ functions $\overline{\alpha}$, $\underline{\alpha}$,

$$\underline{\alpha}(\underline{w}(x)) \leq V(x) \leq \overline{\alpha}(\overline{w}(x)) \quad \text{for all } x \in \mathbb{R}^n. \quad (8)$$

Such V will be used in the construction of a set-valued Lyapunov function.

Theorem 4.4. *Suppose $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a continuous set-valued mapping that satisfies Assumption 3.2. Let $A \subset \mathbb{R}^n$ be a nonempty closed set that is globally pointwise asymptotically stable for (1). Then there exists a set-valued Lyapunov function candidate W for A on \mathbb{R}^n such that*

there exists a continuous and positive definite with respect to A function $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that

$$W(F(x)) + \alpha(x)\mathbb{B} \subset W(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Proof. For any $x \in \mathcal{B}(A)$, let

$$W(x) = \overline{R}(x) + V(x)\mathbb{B},$$

where $\overline{R}(x)$ is the closure of the reachable set, as defined in Example 2.6. Since $x \mapsto V(x)\mathbb{B}$ is outer semicontinuous (in fact continuous) and locally bounded, and by Proposition 4.1, the mapping W is outer semicontinuous and locally bounded. Clearly, $W(x) = \{x\}$ for every $x \in A$ and $\text{int } W(x) \neq \emptyset$ if $x \notin A$. One has $W(F(x)) = \overline{R}(F(x)) + V(F(x))\mathbb{B} \subset \overline{R}(x) + V(F(x))\mathbb{B} \subset \overline{R}(x) + \gamma V(x)\mathbb{B}$, and thus

$$\begin{aligned} W(F(x)) + (1 - \gamma)V(x)\mathbb{B} &\subset \overline{R}(x) + \gamma V(x)\mathbb{B} + (1 - \gamma)V(x)\mathbb{B} \\ &= \overline{R}(x) + V(x)\mathbb{B} \\ &= W(x). \end{aligned}$$

Hence (4) is satisfied, with $\alpha(x) = (1 - \gamma)V(x)$. \square

The same result holds for local pointwise asymptotic stability, if \mathbb{R}^n is replaced by $\mathcal{B}(A)$. (Then, additional properties of either α or W are needed for sufficiency; recall Theorem 3.1.) A similar result should be possible for more general F . As long as \mathcal{KL} stability of A can be established, as in Proposition 4.2, and it can be shown to be robust, then a smooth Lyapunov function $V : \mathcal{B}(A) \rightarrow \mathbb{R}$ satisfying (7) for all $x \in \mathcal{B}(A)$ exists, see Kellett and Teel (2005). Then, the construction of a set-valued Lyapunov function can be repeated.

With the use of a Lyapunov function V one can also easily construct functions characterizing the decrease of a set-valued Lyapunov function $W = \overline{R}$ in (3), as used in Moreau (2005), Angeli and Bliman (2006).

Remark 4.5. *Let A be closed and pointwise asymptotically stable for (1), and V be a continuous and positive definite with respect to A function satisfying (7) for some $\gamma \in (0, 1)$. Take $W(x) = \overline{R}(x)$ and set $\mu(W(x)) = \sup_{y \in W(x)} V(x)$, $\beta(x) = (1 - \gamma)\mu(W(x))$. Then, the decrease condition (3) holds.*

Remark 4.6. *Let W be a function as in the conclusions of Theorem 4.4. If F satisfies Assumption 3.2, then one can verify that there exists a continuous function $\rho : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, positive definite with respect to A , such that*

$$W(F_\rho(x)) + (\alpha(x)/2)\mathbb{B} \subset W(x) \quad \text{for all } x \in \mathbb{R}^n, \quad (9)$$

where, for all x , $F_\rho(x)$ is the set

$$\{y \in \mathbb{R}^n | \exists z \in F((x + \rho(x)\mathbb{B})) \text{ s.t. } y \in z + \rho(z)\mathbb{B}\}.$$

Such perturbations of F are considered in robustness analysis of difference inclusions, see, for example Kellett and

Teel (2005). In turn, (9) implies that A is globally pointwise asymptotically stable for the difference inclusion

$$x^+ \in F_\rho(x).$$

In this sense, the existence of W as in the conclusions of Theorem 4.4 ensures that pointwise asymptotic stability of A is robust.

4.2. A converse result for a compact A

Another converse result is now stated, applicable to the case where dynamics are constrained to a closed set $C \subset \mathbb{R}^n$ such that $F(x) \cap C \neq \emptyset$ for all $x \in C$, and where A , the pointwise asymptotically stable subset of C , is compact. For example, in the setting of Example 2.1, one may be interested in $C = S^{m-1} \times \dots \times S^{m-1}$, where $S^{m-1} \subset \mathbb{R}^m$ is the sphere in \mathbb{R}^m , and $A = C \cap \mathcal{A}$. This results from questions about consensus on the sphere.

Consider the constrained difference inclusion

$$x^+ \in F(x), \quad x \in C. \quad (10)$$

Solutions to (10) are functions $\phi : \mathbb{N} \rightarrow \mathbb{R}^n$ such that, for every $j \in \mathbb{N}$, $\phi(j+1) \in F(\phi(j))$ and $\phi(j) \in C$; equivalently, they are solutions ϕ to $x^+ \in F(x) \cap C$ with $\phi(0) \in C$. Pointwise asymptotic stability of A for (10) is understood as in Definition 2.2, but with only solutions to (10) taken into account. Arguments as in Proposition 4.1 show that:

- the *basin of attraction*, denoted $\mathcal{B}(A)$, of A for (10), i.e., the set of all $x \in C$ such that for every solution ϕ to (10) with $\phi(0) = x$, the limit $\lim_{j \rightarrow \infty} \phi(j)$ exists and $\lim_{j \rightarrow \infty} \phi(j) \in A$, is relatively open in C ; in other words, $\mathcal{B}(A) \cup (\mathbb{R}^n \setminus C)$ is open;
- the mapping $L : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, where $L(x)$ for $x \in C$ is the set of $y \in C$ such that there exists a solution ϕ to (10) with $\phi(0) = x$ and $\lim_{j \rightarrow \infty} \phi(j) = y$ while $L(x) = \emptyset$ for $x \notin C$, is outer semicontinuous and locally bounded at every point of $\mathcal{B}(A)$.

Let $O = \mathcal{B}(A) \cup (\mathbb{R}^n \setminus C)$ and ω be a proper indicator of A on O . By (Cai et al., 2008, Theorem 3.14), there exists a smooth $V : O \rightarrow \mathbb{R}_{\geq 0}$ such that, for some $\gamma \in (0, 1)$, $V(F(x)) \leq \gamma V(x)$ for all $x \in C \cap O = \mathcal{B}(A)$, and (8) holds, with $\bar{\omega}$ and $\underline{\omega}$ being proper indicators of A on O .

Theorem 4.7. *Let $C \subset \mathbb{R}^n$ be closed and such that $F(x) \cap C \neq \emptyset$ for all $x \in C$, $A \subset C$ be a nonempty compact set that is pointwise asymptotically stable for (10), and $\mathcal{B}(A)$ be the basin of attraction of A for (10). Under Assumption (3.2) there exists a set-valued Lyapunov function candidate W for A on $\mathcal{B}(A)$ and a continuous function $\alpha : \mathcal{B}(A) \rightarrow \mathbb{R}_{\geq 0}$, which is a proper indicator of A on $\mathcal{B}(A)$, such that*

$$W(F(x)) + \alpha(x)\mathbb{B} \subset W(x) \quad \text{for all } x \in \mathcal{B}(A).$$

A proof similar to that of Theorem 4.4 can be given, involving a construction based on \bar{R} . An alternative construction is used below, based on the mapping L and its inverse. That is, the construction relies on the set $L^{-1}(L(x))$, which consists of all points from which some solution has the same limit as one of the solutions from x . The construction also involves the set $V^{-1}([0, V(x)])$, which, in other words, is the set $\{y \in O \mid V(y) \leq V(x)\}$.

Proof. Let $V : O \rightarrow \mathbb{R}_{\geq 0}$ be a smooth function as in (7), (8), with $\bar{\omega}$ being a proper indicator of A on O . For any $x \in \mathcal{B}(A)$, let

$$W_0(x) = V^{-1}([0, V(x)]) \cap L^{-1}(L(x)) \quad (11)$$

Pick any $x \in \mathcal{B}(A)$. By definition, $x \in W_0(x)$. For all x' sufficiently close to x , $V(x') < V(x) + 1$, so $V^{-1}([0, V(x')]) \subset V^{-1}([0, V(x) + 1])$, and the latter set is compact. Thus W_0 is locally bounded at x . Outer semicontinuity of $x \mapsto V^{-1}([0, V(x)])$ comes from continuity of V . The comments preceding the theorem give outer semicontinuity and local boundedness of L , and (Rockafellar and Wets, 1998, Theorem 5.7) gives outer semicontinuity of L^{-1} . Then (Rockafellar and Wets, 1998, Proposition 5.52) gives outer semicontinuity of $x \mapsto L^{-1}(L(x))$. Outer semicontinuity of W_0 , defined as in (11), is then straightforward. If $x \in A$, then $V(x) = 0$ and consequently $W_0(x) = \{x\}$. For every $x \in \mathcal{B}(A)$ and every $y \in F(x)$ one has $L(y) \subset L(x)$, and

$$\begin{aligned} W_0(y) &= V^{-1}([0, V(y)]) \cap L^{-1}(L(y)) \\ &\subset V^{-1}([0, \gamma V(x)]) \cap L^{-1}(L(x)) \\ &\subset V^{-1}([0, V(x)]) \cap L^{-1}(L(x)) \\ &= W_0(x). \end{aligned}$$

Thus $W_0(F(x)) \subset W_0(x)$. Then, for W given by $W(x) = W_0(x) + V(x)\mathbb{B}$, and for any $x \in \mathcal{B}(A)$, any $y \in F(x)$, $W(y) = W_0(y) + V(y)\mathbb{B} \subset W_0(x) + V(y)\mathbb{B} \subset W_0(x) + \gamma V(x)\mathbb{B}$, and thus

$$\begin{aligned} W(y) + (1 - \gamma)V(x)\mathbb{B} &\subset W_0(x) + \gamma V(x)\mathbb{B} + (1 - \gamma)V(x)\mathbb{B} \\ &= W_0(x) + V(x)\mathbb{B} \\ &= W(x). \end{aligned}$$

This W meets the desired condition, with $\alpha(x) = (1 - \gamma)V(x)$. \square

In the case of A being closed, but not compact, and set-valued F , a converse result can be obtained for compact subsets of A . Indeed, suppose that $A \subset \mathbb{R}^n$ is a closed set that is locally pointwise asymptotically stable; $\mathcal{B}(A)$ is its basin of attraction; $A_0 \subset A$ is compact; and $\mathcal{X} = L^{-1}(A_0) \cap \mathcal{B}(A)$. Then $\mathcal{B}(A)$ is open, \mathcal{X} is relatively closed in $\mathcal{B}(A)$, and A_0 is asymptotically stable for the inclusion (1) constrained to \mathcal{X} . Using a Lyapunov function for this stability, one can construct a set-valued Lyapunov function candidate W for A_0 on \mathcal{X} and a continuous function $\alpha : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$, which is a proper indicator of A_0 on \mathcal{X} , so that $W(F(x)) + \alpha(x)\mathbb{B} \subset W(x)$ for all $x \in \mathcal{X}$.

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