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LYAPUNOV FUNCTIONS AND DUALITY FOR CONVEX PROCESSES*

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Abstract. The paper studies convex Lyapunov functions for differential and difference inclusions with right-hand sides given by convex processes, that is, by set-valued mappings the graphs of which are convex cones. Convex conjugacy between weak Lyapunov functions for such inclusions and Lyapunov functions for adjoint inclusions is established. Asymptotic stability concepts are compared, and the existence of convex Lyapunov functions for classes of convex processes is shown. The relevance of the results for the study of asymptotic controllability or stabilizability and of detectability for linear control systems with conical control and state constraints is underlined.

Key words. convex process, adjoint process, Lyapunov function, duality, convex conjugacy, asymptotic stability, stabilizability, detectability

AMS subject classifications. 34A60, 49J53, 52A41, 37N35, 93D20, 93B05, 93B07

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1. Introduction.

A convex process is a set-valued mapping, the graph of which is a convex cone. Differential and difference inclusions for which the right-hand side is a convex process and their asymptotic stability and controllability properties have been studied, for example, by [3], [38], [39], [2], [8], [34], [17], and [37], for two main reasons. One reason is that linear control systems with conical constraints, for example, nonnegativity constraints on control as in [7], [29], [24], [26] or on states [6], leading to so-called positive systems [14], [5], can be modeled by convex processes. Another reason is that convex processes can be used to approximate less regular dynamics [16], [8], [37], and through such approximations, properties like controllability or stability can be analyzed.

For linear dynamical and control systems, there exist various equivalences between properties of a system and properties of the adjoint, or dual, system. For a convex process, there exist some characterizations of asymptotic stability through eigenvalues of the adjoint process [37] recalled here as Theorem 1.1; the duality between controllability and observability cones for a process and its adjoint has been shown [3], generalizing some results on linear control system with constraints [7], [29]; and there is a duality between concepts of viability and invariance [2].

The contribution of this paper is to show that convex Lyapunov functions, through the basic operation of convex conjugacy, can be used to establish implications between certain asymptotic stability properties of a convex process and other asymptotic stability properties of the adjoint process. Asymptotic stability properties and the existence of convex Lyapunov functions is also studied here in more detail than before in [8], [37], and the constructions of Lyapunov functions, following the standard approaches in systems theory, are elementary. Methods used in this paper, convex conjugacy of Lyapunov functions in particular, are very different from methods used in the works mentioned above.

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The next two sections discuss the relevance of convex processes for systems theory and some related results on duality of dynamical systems. Formal definitions of a convex process, its adjoint, and other notions used in the discussion and in the rest of the paper are in section 1.3.

1.1. Motivation and relevance for systems theory. The basic example of a convex process is provided by a linear mapping, $F(x) = Ax$ for a matrix A, in which case the adjoint convex process is just $F^*(y) = A^T y$, where A^T is the transpose of A. Asymptotic stability of the linear differential equation

$$\dot{x} = Ax \tag{1.1}$$

is equivalent to asymptotic stability of the dual linear differential equation

$$\dot{y} = A^T y. \tag{1.2}$$

This equivalence holds because the eigenvalues of A and of A^T are the same, and asymptotic stability here is equivalent to the eigenvalues having negative real parts. This equivalence can be also verified by a pair of convex functions conjugate to one another. Lyapunov [27] showed that asymptotic stability of (1.1) is equivalent to the existence of a symmetric and positive definite matrix P such that the quadratic and convex function $V(x) = \frac{1}{2}x \cdot Px$ decreases along every solution to (1.1). The decrease means that $PA + A^T P$ is negative definite, which is equivalent to $P^{-1}A^T + AP^{-1}$ being negative definite, which in turn means that $V^*(y) = \frac{1}{2}y \cdot P^{-1}y$, which happens to be the convex conjugate of V, decreases along every solution to (1.2). This is a basic case of a pair of convex conjugate Lyapunov functions establishing the equivalence of asymptotic stability for a convex process and its adjoint.

Another example of a convex process F and its adjoint F^* is provided by

$$F(x) = Ax + K, \quad F^*(y) = \begin{cases} A^T y & \text{if} \quad y \in K^*, \\ \emptyset & \text{if} \quad y \notin K^*, \end{cases} \tag{1.3}$$

where A is a matrix, K is a closed convex cone, and K^* is the cone polar to K. When K is a linear subspace given as the range of a matrix B, weak asymptotic stability of $\dot{x} \in F(x)$ is exactly the *asymptotic controllability* or *stabilizability* of the control system

$$\dot{x} = Ax + Bu, \tag{1.4}$$

i.e., the property that for every initial point ξ, there exists a control function u on $[0, \infty)$ such that the resulting solution to (1.4) from ξ converges to 0 as $t \to \infty$. From linear systems theory, see [25], this stabilizability is known to be equivalent to *detectability* of the dual linear system

$$\begin{cases} \dot{y} = A^T y, \\ o = B^T y, \end{cases} \tag{1.5}$$

i.e., the property that every solution to (1.5) for which the output $o(t)$ is always 0 is convergent to 0 as $t \to \infty$. When K is the range of B, the dual cone K^* is the kernel of B^T, and thus detectability of (1.5) is exactly asymptotic stability of $\dot{y} \in F^*(y)$ with F^* as in (1.3).

Convex processes provide a framework for handling conical constraints and establishing equivalences as above. For example, consider K to be the image under B of the nonnegative orthant, in which case the dual cone K^* consists of all vectors y for which $B^T y$ is in the nonnegative orthant. Then, the object dual to

$$\dot{x} = Ax + Bu,\ u \geq 0,$$

is (1.5) with the constraint that $o(t)$ be nonnegative. The duality between controllability and observability in such cases was dealt with in [7], [29], [3]. This paper lays a foundation for analyzing stabilizability and detectability in a format which lets one treat control and state constraints and continuous and discrete-time dynamics.

1.2. Related instances of duality results. One way of generalizing the duality between asymptotic stability for a linear system and its adjoint is to consider a linear differential inclusion. In its basic form, it is

$$\dot{x} \in \mathcal{A}(x) = \operatorname{con} A_i x, \tag{1.6}$$

where $A_i,\ i = 1, 2, \ldots, k$ are matrices and con stands for the convex hull: $\operatorname{con} A_i x$ is the smallest convex set containing $A_1x, A_2x, \ldots, A_kx$. The dual linear differential inclusion is

$$\dot{y} \in \mathcal{A}^T(y) = \operatorname{con} A_i^T y. \tag{1.7}$$

Asymptotic stability of (1.6) was shown to be equivalent to asymptotic stability of (1.7) by [4]. Asymptotic stability here can also be characterized by a convex Lyapunov function [28], in fact a smooth one [13]. The equivalence of asymptotic stability of (1.6) and of (1.7) can then be verified through the use of convex conjugates of Lyapunov functions [23]. This carries over to discrete-time systems, and for both discrete-time and continuous-time cases, the Lyapunov function conjugacy leads to numerical methods of establishing asymptotic stability [20]. It must be noted that the right-hand sides of (1.6), (1.7) are not convex processes. For example, the mapping $x \mapsto \operatorname{con}\{x, 2x\}$, where $x \in \mathbb{R}$, does not have a convex graph.

A different duality was established for a strict and closed convex process F and its adjoint F^* in [3], generalizing [7], [29], which dealt with linear control systems with nonnegative controls. When translated to the setting of this paper, where convergence to 0 is of interest, the duality result of [3] states that

$$\left(C_T^F\right)^* = -D_T^{F^*}, \quad \left(C^F\right)^* = -D^{F^*}, \tag{1.8}$$

where

$$C_T^F = \left\{ \phi(0) \,|\, \phi \text{ satisfies } \dot{\phi}(t) \in F(\phi(t)) \text{ on } [0, T] \text{ and } \phi(T) = 0 \right\},$$

$$D_T^{F^*} = \left\{ \psi(T) \,|\, \psi \text{ satisfies } \dot{\psi}(t) \in F^*(\psi(t)) \text{ on } [0, T] \right\},$$

and $C^F = \bigcup_{T>0} C_T^F,\ D^{F^*} = \bigcap_{T>0} D_T^{F^*}$. The results of [7], [29], and [3] were not shown using Lyapunov technology, but through analysis of eigenvectors and invariant subspaces. An extension beyond the strict and autonomous case was obtained in [34] through the use of conjugate duality for optimal control problems.

Weak asymptotic stability of convex processes has been studied in [38], [39], [8], [37], [17], mostly through eigenvalue and higher-order spectral analysis and limited use of Lyapunov functions. Dual characterizations of weak asymptotic stability of a convex process, through properties of its adjoint, originally derived in [38], [39], can be found in [37, Theorem 8.10] and are summarized below.

THEOREM 1.1. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a closed and strict convex process. Then the following are equivalent:*

(a) $\dot{x} \in F(x)$ *is weakly asymptotically stable.*
(b) *The only solution to* $\dot{y} \in -F^*(y)$ *with a nonnegative Lyapunov exponent is the trivial solution* $y(t) \equiv 0$.
(c) *The restriction of* F^* *to the maximal subspace invariant under* F^* *is an asymptotically stable linear operator, and either the maximal eigenvalue of* F^* *is negative or* F^* *has no eigenvalues.*

1.3. Convex processes: Preliminaries. Given a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, i.e., a mapping that to each $x \in \mathbb{R}^n$ assigns a set $F(x) \subset \mathbb{R}^m$, the *domain* is $\operatorname{dom} F = \{x \in \mathbb{R}^n \,|\, F(x) \neq \emptyset\}$, the *range* is $\operatorname{rge} F = \bigcup_{x\in\mathbb{R}^n} F(x)$, and the *graph* is $\operatorname{gph} F = \{(x, y) \in \mathbb{R}^{n+m} \,|\, y \in F(x)\}$. A nonempty set $K \subset \mathbb{R}^n$ is a *cone* if for every $x \in K$, every $\lambda \geq 0$, one has $\lambda x \in K$. Convexity of a cone can thus be characterized as follows: for every $x, x' \in K$, $x + x' \in K$. Given a cone $K \subset \mathbb{R}^n$, its *dual cone* is

$$K^* = \{y \in \mathbb{R}^n \,|\, x \cdot y \geq 0 \ \forall x \in K\},$$

where $x\cdot y$ stands for the dot product. K^* is always a closed convex cone, and $(K^*)^* = K$ if K is closed and convex. For example, in $\mathbb{R}^n$, $\{0\}^* = \mathbb{R}^n$ and $(\mathbb{R}^n)^* = \{0\}$; if K is the nonnegative orthant $\mathbb{R}^n_+$, then $K^* = \mathbb{R}^n_+$ as well; if $K = \operatorname{rge} B = B\mathbb{R}^k$ for a $n \times k$ matrix B, then $K^* = \ker B^T$, the kernel of B^T; and, more generally, if $K = BU$ for a $n \times k$ matrix B and a cone $U \subset \mathbb{R}^k$, then $K^* = \{y \,|\, B^T y \in U^*\}$. The set-valued analysis concepts used here follow [31].

A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a *convex process* if $\operatorname{gph} F$ is a convex cone. A convex process F is called *strict* if $\operatorname{dom} F = \mathbb{R}^n$ and *closed* if $\operatorname{gph} F$ is closed, equivalently, if F is outer semicontinuous. Every linear mapping is a strict and closed convex process. The mapping F in (1.3) is a convex process if K is a cone; it is then strict, and it is closed if K is a closed cone. The mapping

$$F(x) = \begin{cases} Ax & \text{if} \quad x \in X, \\ \emptyset & \text{if} \quad x \notin X \end{cases} \tag{1.9}$$

is a convex process if X is a cone, it is not strict unless X is the whole space, and it is closed if and only if X is closed.

The *adjoint convex process* of a convex process $F : \mathbb{R}^n \to \mathbb{R}^n$ is the closed convex process $F^* : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by

$$F^*(y) = \{w \in \mathbb{R}^n \mid w \cdot x \leq y \cdot v \ \ \forall (x, v) \in \operatorname{gph} F\}.$$

That this defines a closed convex process follows from an alternative definition of F^*:

$$(y, w) \in \operatorname{gph} F^* \ \equiv \ (-w, y) \in (\operatorname{gph} F)^*$$

and from properties of convex cones mentioned before. Examples of convex processes and their adjoints include $F(x) = Ax$, $F^*(y) = A^T y$; the pair F, F^* in (1.3); and, more generally,

$$F(x) = \begin{cases} Ax + K & \text{if} \quad x \in X, \\ \quad\emptyset & \text{if} \quad x \notin X, \end{cases} \qquad F^*(y) = \begin{cases} A^T y - X^* & \text{if} \quad y \in K^*, \\ \quad\emptyset & \text{if} \quad y \notin K^*, \end{cases} \tag{1.10}$$

where K and X are convex cones. If F is a closed convex process, then $F^{**} = (F^*)^* = -F(-\cdot)$. Consequently, if $F(x) = Ax$, then $F^{**} = F$; and for F in (1.3) with K closed, $F^{**}(x) = -(A(-x) + K) = Ax - K$.

For more background on convex processes, consult the books [30], where convex processes were introduced; [2], where extensions to spaces beyond $\mathbb{R}^n$ are included; [37], which includes elements of the theory of differential inclusions given by convex processes; and [8], where the convex processes were used to approximate nonlinear control systems with control constraints.

1.4. Standing assumptions. Throughout the paper, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is always a convex process. Thus if F given by (1.3), then K is a convex cone, and if given by (1.10), then K and X are convex cones. F^* is the convex process adjoint to F.

2. Convex conjugacy of Lyapunov functions.

The paper is concerned with stability properties of differential inclusions

$$\dot{x} \in F(x), \tag{2.1}$$

respectively, difference inclusions

$$x^+ \in F(x), \tag{2.2}$$

and their relationship to stability properties of the adjoint differential inclusions

$$\dot{y} \in F^*(y), \tag{2.3}$$

respectively, adjoint difference inclusions

$$y^+ \in F^*(y). \tag{2.4}$$

The standard notation $\dot{x}$ in (2.1) represents $\frac{dx}{dt}$, while x^+ and (2.2) is to be understood as requiring that $x(j+1) \in F(x(j))$.

This section establishes duality between convex weak Lyapunov functions for (2.1) or (2.2) and convex Lyapunov functions for (2.2) or (2.4). The existence of such Lyapunov functions is addressed in section 4 . Let $\mathcal{H}$ be the set of all functions $f : \mathbb{R}^n \to [0, \infty)$ which are positive definite, i.e., $f(0) = 0$ and $f(x) > 0$ if $x \neq 0$, positively homogeneous of degree 2, and for which there exist $a, b > 0$ such that $a\|x\|^2 \le f(x) \le b\|x\|^2$ for all $x \in \mathbb{R}^n$. The lower and upper bounds hold automatically for every positive definite and homogeneous function if it is continuous, as it always is if it is convex. Examples of functions in $\mathcal{H}$ include quadratic functions $f(x) = \frac{1}{2}x \cdot Px$ with symmetric and positive definite P, pointwise minima and maxima of finitely many such functions, and, more generally, squares of Minkowski functionals (also called gauges) of compact sets containing 0 in their interiors. Let $\mathcal{HS}$ be the set of all $f \in \mathcal{H}$ which are differentiable.

A function $V \in \mathcal{HS}$ is a *weak Lyapunov function* for (2.1) if, for some $\gamma > 0$,

$$\forall x \in \operatorname{dom} F \ \exists v \in F(x) \text{ such that } \nabla V(x) \cdot v \le -\gamma V(x). \tag{2.5}$$

A function $V \in \mathcal{H}$ is a *weak Lyapunov function* for (2.2) if, for some $\gamma \in (0, 1)$,

$$\forall x \in \operatorname{dom} F \ \exists v \in F(x) \text{ such that } V(v) \le \gamma V(x). \tag{2.6}$$

A function $V \in \mathcal{HS}$ is a *Lyapunov function* for (2.1) if, for some $\gamma > 0$,

$$\forall x \in \operatorname{dom} F \ \forall v \in F(x) \ \ \nabla V(x) \cdot v \le -\gamma V(x). \tag{2.7}$$

A function $V \in \mathcal{H}$ is a *Lyapunov function* for (2.2) if, for some $\gamma \in (0,1)$,

$$\forall x \in \operatorname{dom} F \ \forall v \in F(x) \quad V(v) \leq \gamma V(x). \tag{2.8}$$

For a function $f \in \mathcal{H}$, its convex conjugate function f^*, defined by

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \{ y \cdot x - f(x) \},$$

belongs to $\mathcal{H}$ and is convex. If $f \in \mathcal{H}$ is convex, then the convex conjugate of f^* is f. For example, if P is a symmetric and positive definite matrix, then the convex conjugate of $f(x) = \frac{1}{2} x \cdot Px$ is $f^*(y) = \frac{1}{2} y \cdot P^{-1} y$; and if P_i, $i = 1, 2, \ldots, I$, are such matrices then the convex conjugate of $f(x) = \frac{1}{2} \max_{i=1,2,\ldots,I} x \cdot P_i x$ is the convex hull of, i.e., the greatest convex function bounded above by $\min_{i=1,2,\ldots,I} \frac{1}{2} y \cdot P_i^{-1} y$. For a general exposition, see [30] or [31], and for a discussion oriented toward conjugacy of Lyapunov functions, see [23].

Some properties of convex functions are reflected in the same properties of their conjugates. As noted above, $f \in \mathcal{HC}$ if and only $f^* \in \mathcal{HC}$, where $\mathcal{HC}$ is the set of all $f \in \mathcal{H}$ which are convex. Furthermore, some properties of convex functions are reflected in other properties of their conjugates. Here, it is relevant that a convex $f \in \mathcal{H}$ is strictly convex if and only if f^* is differentiable. Thus, $f \in \mathcal{HCS}$ if and only if $f^* \in \mathcal{HCS}$, where $\mathcal{HCS}$ is the set of all $f \in \mathcal{HS} \cap \mathcal{HC}$ which are strictly convex. For $f \in \mathcal{HCS}$, $y = \nabla f(x)$ is equivalent to $x = \nabla f^*(y)$ and to $f^*(y) = y \cdot x - f(x)$, and if, furthermore, $f(x) = 1/2$, then $f^*(y) = 1/2$.

Homogeneity of $V \in \mathcal{HS}$ and of a convex process imply that it is enough to check the inequality in (2.5) along a single level set of V. This is stated formally in the lemma below. Similar equivalent conditions exist for (2.6), (2.7), and (2.8).

Lemma 2.1. *Let $V \in \mathcal{HS}$. Then* (2.5) *is equivalent to*

$$\forall x \in \operatorname{dom} F \textit{ with } V(x) = \frac{1}{2} \ \exists v \in F(x) \textit{ such that } \nabla V(x) \cdot v \leq -\frac{1}{2}\gamma. \tag{2.9}$$

Proof. Clearly, (2.5) implies (2.9). For the opposite implication, take any $x \in \operatorname{dom} F$, $x \neq 0$. Note that $V(x) > 0$, set $\lambda = \sqrt{2V(x)}$, and let $x' = x/\lambda$. Then $V(x') = V(x/\lambda) = V(x)/\lambda^2 = 1/2$, and (2.9) implies that there exists $v' \in F(x')$ such that $\nabla V(x') \cdot v' \leq -\gamma/2$. Positive homogeneity of V of degree 2, and thus positive homogeneity of ∇V of degree 1, turns $\nabla V(x/\lambda) \cdot v' \leq -\gamma/2$ to $\nabla V(x) \cdot (\lambda v') \leq -\gamma V(x)$. Because F is a convex process, its graph is a convex cone, $F(x) = F(\lambda x') = \lambda F(x')$, and $v = \lambda v' \in F(x)$. Hence (2.9) implies (2.5). ∎

The main results on the duality between Lyapunov and weak Lyapunov functions are now stated and proved.

Theorem 2.2. *Let F be strict. If $V \in \mathcal{HCS}$ is a weak Lyapunov function for* (2.1)*, then V^* is a Lyapunov function for* (2.3)*. If $V \in \mathcal{HC}$ is a weak Lyapunov function for* (2.2)*, then V^* is a Lyapunov function for* (2.4)*.*

Proof. Consider the continuous-time case first. Let $V \in \mathcal{HCS}$ satisfy (2.5) with some $\gamma > 0$. Take any $y \in \operatorname{dom} F^*$ with $V^*(y) = 1/2$ and let $x = \nabla V^*(y)$, so that $y = \nabla V(x)$ and $V(x) = 1/2$. By the definition of F^*, for every $w \in F^*(y)$, $x \cdot w \leq y \cdot v$ for every $v \in F(x)$. By assumption, there exists $v \in F(x)$ such that $\nabla V(x) \cdot v \leq -\gamma/2$. Thus, for every $y \in \operatorname{dom} F^*$ with $V^*(y) = 1/2$ and every $w \in F^*(y)$, $x \cdot w = \nabla V^*(y) \cdot w \leq -\gamma/2$. Homogeneity implies that V^* is a Lyapunov function for (2.3).

For the discrete-time case, let $V \in \mathcal{HC}$ satisfy (2.6) with some $\gamma \in (0,1)$. Then, for every $y \in \operatorname{dom} F^*$ and every $w \in F^*(y)$,

$$\begin{aligned} V^*(w) &= \sup_{x\in\mathbb{R}^n} \{w\cdot x - V(x)\} \le \sup_{x\in\mathbb{R}^n}\left\{w\cdot x - \frac{1}{\gamma}\inf_{v\in F(x)} V(v)\right\} \\ &= \sup_{x\in\mathbb{R}^n}\sup_{v\in F(x)}\left\{w\cdot x - \frac{1}{\gamma}V(v)\right\} \le \sup_{x\in\mathbb{R}^n}\sup_{v\in F(x)}\left\{y\cdot v - \frac{1}{\gamma}V(v)\right\} \\ &= \frac{1}{\gamma}\sup_{x\in\mathbb{R}^n}\sup_{v\in F(x)}\{(\gamma y)\cdot v - V(v)\} \le \frac{1}{\gamma}\sup_{v\in\mathbb{R}^n}\{(\gamma y)\cdot v - V(v)\} \\ &= \frac{1}{\gamma}V^*(\gamma y) = \gamma V^*(y), \end{aligned}$$

where the first inequality holds because $-V(x) \le -\frac{1}{\gamma}\inf_{v\in F(x)} V(v)$, as implied by (2.6); the second inequality comes from the definition of F^*; and the third inequality holds because $\operatorname{rge} F \subset \mathbb{R}^n$. This finishes the proof. ◻

Theorem 2.2 relied on strictness of F, which cannot be omitted without other technical assumptions on the domains and ranges of F or ∇V; see Example 3.2. It did not rely on regularity of F. Theorem 2.4 does not directly rely on regularity of F either but invokes (2.10) at every $x \in \operatorname{dom} F$, $y \in \operatorname{dom} F^*$. This holds in particular when F is strict and closed. The lemma below can be found in [37, Theorem 2.9].

LEMMA 2.3. *Let F be closed. Then, for every $x \in \operatorname{int}\operatorname{dom} F$ and every $y \in \operatorname{dom} F^*$,*

$$\inf_{v\in F(x)} y\cdot v = \sup_{w\in F^*(y)} w\cdot x. \tag{2.10}$$

Beyond the strict and closed case, the equality (2.10) holds for all $x \in \operatorname{dom} F$, not just $\operatorname{int}\operatorname{dom} F$, for example, when F is given by (1.10). Indeed, a simple calculation reveals that (2.10) holds for all $x \in X$, $y \in K^*$ and both sides of (2.10) equal $y \cdot Ax$.

THEOREM 2.4. *Suppose that* (2.10) *holds for all $x \in \operatorname{dom} F$, $y \in \operatorname{dom} F^*$. If $V^* \in \mathcal{HCS}$ is a Lyapunov function for* (2.3)*, then V is a weak Lyapunov function for* (2.1)*. If $V^* \in \mathcal{HC}$ is a Lyapunov function for* (2.4)*, then V is a weak Lyapunov function for* (2.2).

Proof. Let $V^* \in \mathcal{HCS}$ be a Lyapunov function for (2.3), which thanks to homogeneity implies that for every $y \in \operatorname{dom} F^*$ with $V^*(y) = 1/2$, $\sup_{w\in F^*(y)} \nabla V^*(y)\cdot w \le -\gamma/2$. Take any $x \in \operatorname{dom} F$ with $V(x) = 1/2$ and let $y = \nabla V(x)$, which implies $x = \nabla V^*(y)$ and $V^*(y) = 1/2$. If $y \in \operatorname{dom} F^*$, then, by assumption,

$$\inf_{v\in F(x)} \nabla V(x)\cdot v = \sup_{w\in F^*(y)} \nabla V^*(y)\cdot w \le -\gamma/2.$$

Suppose now that $y \notin \operatorname{dom} F^*$. Because $\operatorname{gph} F$ is a cone, $F(0)$ is a cone, and furthermore, because $(0, F(0)) \subset \operatorname{gph} F$ and $\operatorname{gph} F$ is a convex cone, for any $v \in F(x)$ one has $(x, v + F(0)) \subset \operatorname{gph} F$. Recall now that $\operatorname{dom} F^* = (F(0))^*$, and hence $y \notin \operatorname{dom} F^*$ implies that there exists $v_0 \in F(0)$ such that $y \cdot v_0 < 0$. Pick an arbitrary $v_1 \in F(x)$ and note that

$$\inf_{v\in F(x)} \nabla V(x)\cdot v \le \inf_{v\in v_1+F(0)} y\cdot v = y\cdot v_1 + \inf_{v\in F(0)} y\cdot v \le y\cdot v_1 + \inf_{\lambda>0} y\cdot(\lambda v_0) = -\infty.$$

Thus $\inf_{v\in F(x)} \nabla V(x)\cdot v \le -\gamma/2$ if $V(x) = 1/2$. Homogeneity extends this to (2.5).

For the discrete-time case, let $V^* \in \mathcal{HC}$ satisfy (2.8) with some $\gamma \in (0,1)$. Then

$$\inf_{v\in F(x)} V(v) = \inf_{v\in F(x)} \sup_{y\in\mathbb{R}^n} \{v\cdot y - V^*(y)\} = \sup_{y\in\mathbb{R}^n} \inf_{v\in F(x)} \{v\cdot y - V^*(y)\}$$

for every $x \in \mathbb{R}^n$. Indeed, switching the inf and sup is possible thanks to [30, Theorem 37.3] because V^* is coercive (as it is positive definite and homogeneous of degree 2), equivalently, such that V is finite-valued. If $y \notin \operatorname{dom} F^*$, then arguments as above show that $\inf_{v\in F(x)} v\cdot y = -\infty$, and thus, for $y \notin \operatorname{dom} F^*$, one has $\inf_{v\in F(x)} \{v\cdot y - V^*(y)\} = -\infty$. Thus

$$\begin{aligned}\inf_{v\in F(x)} V(v) &= \sup_{y\in\operatorname{dom} F^*} \inf_{v\in F(x)} \{v\cdot y - V^*(y)\} = \sup_{y\in\operatorname{dom} F^*} \sup_{w\in F^*(y)} \{x\cdot w - V^*(y)\}\\ &\le \sup_{y\in\operatorname{dom} F^*} \sup_{w\in F^*(y)} \left\{x\cdot w - \frac{1}{\gamma}V^*(w)\right\} \le \sup_{w\in\mathbb{R}^n} \left\{x\cdot w - \frac{1}{\gamma}V^*(w)\right\}\\ &= \gamma \sup_{w\in\mathbb{R}^n} \left\{x\cdot\left(\frac{w}{\gamma}\right) - V^*\left(\frac{w}{\gamma}\right)\right\} = \gamma V(x),\end{aligned}$$

where the second equality comes from (2.10) and the first inequality holds because for every $y \in \operatorname{dom} F^*$ and every $w \in F^*(y)$, $x\cdot w - V(y) \le x\cdot w - \frac{1}{\gamma}V(w)$ as implied by (2.8). This finishes the proof. □

3. Asymptotic stability concepts.

Solutions to (2.1) are locally absolutely continuous functions ϕ defined on some subinterval I of $\mathbb{R}_+$ that satisfy $\dot\phi(t) \in F(\phi(t))$ for almost all $t \in I$. Solutions to (2.2) are functions ϕ on some subinterval I of $\{0,1,2,\dots\}$ such that $\phi(j+1) \in F(\phi(j))$ for all $j \in I$ with $j+1 \in I$. A solution ϕ to (2.1) or (2.2) is *complete* if the interval on which it is defined, $\operatorname{dom}\phi$, is unbounded. A solution ϕ to (2.1) or (2.2) is *maximal* if there does not exist another solution ψ such that $\operatorname{dom}\phi$ is a strict subset of $\operatorname{dom}\psi$ and on $\operatorname{dom}\phi$ the solutions are equal. Homogeneity of F ensures that if ϕ is a solution to (2.1) or (2.2), then so is $\lambda\phi$ for all $\lambda > 0$.

The differential inclusion (2.1), respectively, the difference inclusion (2.2), is *asymptotically stable* if it is

- (Lyapunov) *stable*, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that every solution ϕ with $\phi(0) \in \delta\mathbb{B}$ is such that $\operatorname{rge}\phi \subset \varepsilon\mathbb{B}$, and
- *attractive*, that is, every maximal solution ϕ is complete and $\lim_{t\to\infty}\phi(t) = 0$, respectively, $\lim_{j\to\infty}\phi(j) = 0$.

The differential inclusion (2.1), respectively, the difference inclusion (2.2), is *weakly asymptotically stable* if it is

- *weakly attractive*, i.e., for every $\xi \in \operatorname{dom} F$ there exists a complete solution ϕ_ξ with $\phi_\xi(0) = \xi$ and $\lim_{t\to\infty}\phi_\xi(t) = 0$, respectively, $\lim_{j\to\infty}\phi_\xi(j) = 0$, and
- considering only solutions ϕ_ξ as above results in stability, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\xi \in \delta\mathbb{B}$, then $\operatorname{rge}\phi_\xi \subset \varepsilon\mathbb{B}$.

When F represents a control system $\dot x = f(x,u)$, $u \in U$, i.e., when $F(x) = \bigcup_{u\in U} f(x,u)$, weak asymptotic stability as defined above resembles the notion of asymptotic controllability, as introduced by [40]. (Asymptotic controllability sometimes includes a bounded control property which corresponds to velocities of ϕ_ξ from ξ near 0 being uniformly bounded.) Also, weak Lyapunov functions as defined in this paper can be related to control Lyapunov functions; see, for example, [35]. In some of the literature on dynamics generated by convex processes, for example, [37], [17], weak asymptotic stability is used to describe what here is called weak attractivity. Note though that Proposition 3.1 ensures that for strict F, weak asymptotic stability as defined here and as used in [37] is the same. In particular, this ensures that Theorem 1.1 holds here. Weak attractivity too has seen different interpretations, for example, in [1], and an attractive system is sometimes called convergent [36]. Note also that every convex process can be formulated as a control system $\dot{x} = u$, $u \in F(x)$.

Homogeneity of F ensures that stability is uniform for (2.1) and (2.2). In fact, if for some $\varepsilon, \delta > 0$ one has that every solution ϕ with $|\phi(0)| = \delta$ satisfies $|\phi(t)| \le \varepsilon$, then every solution ϕ satisfies $|\phi(t)| \le \varepsilon |\phi(0)|/\delta$. In the discrete-time case, but not in the continuous-time case, stability implies local boundedness of F. Indeed, if (2.2) is stable, then $F(0) = \{0\}$, and consequently, if F is given by (1.10), then $K = \{0\}$ and F has the form (1.9). If, furthermore, F is closed, then $F(0) = \{0\}$ implies that F is locally bounded.

Homogeneity and convexity of $\operatorname{gph} F$ ensures that weak attractivity is sufficient for weak asymptotic stability of (2.1), (2.2). In fact, a stronger stability property follows when $\operatorname{dom} F$ is polyhedral. (For strict F and continuous-time dynamics, this was established in [37, Lemma 8.4], generalizing [8, Lemma 4.1] which dealt with F as in (1.3); discrete-time (1.3) is done in [8, Lemma 8.9]. The proof below is similar.) The inclusion (2.1) or (2.2) is *weakly exponentially stable* if it is weakly asymptotically stable and the solutions ϕ_ξ from the definition of weak attractivity satisfy, for some $M, k > 0$, the bound

$$\|\phi(t)\| \le M e^{-kt} \|\phi(0)\| \quad \forall t \in \operatorname{dom} \phi. \tag{3.1}$$

For general homogeneous systems, asymptotic stability under some further uniformity assumptions, but not attractivity, ensures exponential stability; this is valid beyond classical systems, for example, in nonlinear switching systems [36].

PROPOSITION 3.1. *Let* $\operatorname{dom} F$ *be polyhedral. If* (2.1)*, respectively,* (2.2)*, is weakly attractive, then* (2.1)*, respectively,* (2.2)*, is weakly exponentially stable.*

Proof. Consider a weakly attractive (2.2). Because $\operatorname{dom} F$ is polyhedral, it is generated by finitely many $x_i \in \operatorname{dom} F$, $i = 1, 2, \dots, l$, i.e., every $x \in \operatorname{dom} F$ can be represented by $x = \sum_{i=1}^{l} \lambda_i x_i$, $\lambda_i \ge 0$. Without loss of generality, $\|x_i\| = 2$. Furthermore, there exist points $x_i \in \operatorname{dom} F$, $\|x_i\| = 2$, $i = l+1, l+2, \dots, m$, such that $\operatorname{con}\{0, x_1, x_2, \dots, x_m\}$ contains $\mathbb{B} \cap \operatorname{dom} F$, where $\mathbb{B}$ is the unit ball centered at 0. Weak attractivity implies that there exists $J > 0$ and, for each $i = 1, 2, \dots, m$, a solution ϕ_i with $\phi_i(0) = x_i$ such that $\|\phi_i(J)\| \le 1/2$. Then, for every $x \in \mathbb{B} \cap \operatorname{dom} F$ there exists a solution ϕ such that $\phi(J) \in \mathbb{B}/2$: indeed, $x = \sum_{i=1}^{m} \lambda_i x_i$ with $\lambda_i \ge 0$, $\sum_{i=1}^{m} \lambda_i \le 1$, the function $\phi(j) = \sum_{i=1}^{m} \lambda_i \phi_i(j)$ is a solution to (2.2) with $\phi(0) = x$, and

$$\|\phi(J)\| = \left\| \sum_{i=1}^{m} \lambda_i \phi_i(J) \right\| \le \sum_{i=1}^{m} \lambda_i \|\phi_i(J)\| \le \sum_{i=1}^{m} \lambda_i \frac{1}{2} \le \frac{1}{2}.$$

In fact, for ϕ as above, $\|\phi(j)\| \le R$ for $j = 0, 1, \dots, J-1$, where $R = \max\{\|\phi_i(j)\| \mid i = 1, 2, \dots, m,\ j = 0, 1, \dots, J-1\}$; the calculation is just as the one displayed above. Homogeneity now implies that for every $x \in \mathbb{B}/2 \cap \operatorname{dom} F$ there exists a solution ϕ with $\phi(0) = x$, $\phi(J) \in \mathbb{B}/4$, and $|\phi(j)| \le R/2$ for $j = 0, 1, \dots, J-1$. Similar existence follows for $x \in \mathbb{B}/4 \cap \operatorname{dom} F$, $x \in \mathbb{B}/8 \cap \operatorname{dom} F$, etc. Concatenation then yields, for every $x \in \mathbb{B} \cap \operatorname{dom} F$, a solution that satisfies (3.1) with $M = 2R$, $k = (\ln 2)/J$, and homogeneity implies the existence of such a solution for every $x \in \operatorname{dom} F$. The proof for (2.1) follows exactly the same approach; cf. [37, Lemma 8.4]. ∎

Example 3.2. Consider a case of (1.10):

$$F(x) = \begin{cases} x + \mathbb{R}_- & \text{if} \quad x \in \mathbb{R}_+, \\ \emptyset & \text{if} \quad x \notin \mathbb{R}_+, \end{cases} \qquad F^*(y) = \begin{cases} y - \mathbb{R}_+ & \text{if} \quad y \in \mathbb{R}_-, \\ \emptyset & \text{if} \quad y \notin \mathbb{R}_-, \end{cases}$$

Then (2.1) is weakly asymptotically, and exponentially, stable, because for $x \in \mathbb{R}_+$ one has $-2x \in \mathbb{R}_-$, thus $-x \in F(x)$, and the solution to $\dot{x} = -x$ converges to 0. In fact, because for $x \in \mathbb{R}_+$ one has $-\sqrt{x} \in F(x)$, for every initial point $\xi \geq 0$ there exists a solution to (2.1) which reaches 0 in finite-time: $x(t) = \left(\sqrt{\xi} - t/2\right)^2$. Furthermore, because $-a\sqrt{x} \in F(x)$ for arbitrarily large $a > 0$, 0 can be reached from any initial point $\xi > 0$ in arbitrarily short time. Thus, in the notation of section 1.2, $C^F = C^F_T = \mathbb{R}_+$.

On the other hand, (2.3) is not asymptotically stable. Indeed, for every initial point ξ in $\operatorname{dom} F^* = \mathbb{R}_-$, every solution $y(t) \leq \xi$ for all t in its domain. Thus weak asymptotic stability for (2.2) does not imply asymptotic stability for (2.3). However, it can be verified that $D^{F^*}_T = D^{F^*} = \mathbb{R}_-$, and so the duality (1.8) holds. Note also that $V \in \mathcal{HCS}$ given by $V(x) = \frac{1}{2}x^2$ is a weak Lyapunov function for (2.1) but $V^* \in \mathcal{HCS}$ given by $V(y) = \frac{1}{2}y^2$ is not a Lyapunov function for (2.3).

The same observation is true for the discrete-time case. There exist solutions to (2.2) that reach 0 in one jump, which verifies weak asymptotic stability, but all solutions to (2.4) remain bounded away from 0 by the initial condition, which violates asymptotic stability. Note also that every $y < 0$ is an eigenvector F^* with any eigenvalue $\lambda \geq 1$, because for $y \leq 0$, $F^*(y) = (-\infty, y]$, one has $\lambda y \in F^*(y)$. For strict F, the existence of such an eigenvector for F^* precludes asymptotic stability of (2.2); see [17, Theorem 5.3]. This example shows that the result does not carry over to nonstrict F. (In [17], asymptotic stability required existence of solutions converging to 0 even from points not in the domain of F; hence, in that terminology and for nonstrict and closed F, asymptotic stability always fails.)

Attractivity, and then asymptotic stability, relies on asymptotic properties of solutions when $t \to \infty$ or $j \to \infty$. When existence of complete solutions is not guaranteed, a natural property to consider is preasymptotic stability. The differential inclusion (2.1), respectively, the difference inclusion (2.2), is *preasymptotically stable* if it is stable and

- *preattractive*, that is, every maximal solution is bounded and every complete solution ϕ satisfies $\lim_{t\to\infty} \phi(t) = 0$, respectively, $\lim_{j\to\infty} \phi(j) = 0$.

For sufficiently regular differential and difference inclusions, when domains are not equal to $\mathbb{R}^n$ and, hence, when complete solutions need not exist, preasymptotic stability turns out equivalent to the existence of smooth Lyapunov functions. For some convex processes, this is explicitly shown in the next section. Furthermore, it is preasymptotic stability, and not asymptotic stability, that is naturally inherited by a differential or difference inclusion from a linear/conical approximation to it. For further discussion, see [21].

Example 3.3. Consider a case of (1.3), obtained with $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $K = \{0\} \times \mathbb{R}_+$:

$$F(x) = (x_2, -x_1) + \{0\} \times \mathbb{R}_+, \quad F^*(y) = \begin{cases} (-y_2, y_1) & \text{if} \quad y_2 \geq 0, \\ \emptyset & \text{if} \quad y_2 < 0. \end{cases}$$

Then (2.1) is weakly asymptotically stable. This can be seen by considering

$$f(x) = \begin{cases} (x_2, 0) & \text{if } x_2 < 0,\ 0 < x_1 < -x_2, \\ (x_2, -x_1) & \text{otherwise,} \end{cases}$$

which is a selection from $F(x)$, i.e., $f(x) \in F(x)$ for all x, and observing that every solution to $\dot{x} = f(x)$ converges to 0. It is not asymptotically stable because there exist periodic solutions (rotating around circles centered at the origin). On the other hand, (2.3) is preasymptotically stable. The differential equation $(\dot{y}_1, \dot{y}_2) = (-y_2, y_1)$ represents counterclockwise motion along circles, and so every solution ψ satisfies $\|\psi(t)\| = \|\psi(0)\|$ and so is bounded, as well as periodic with period 2π. The only complete solution to this differential equation that satisfies the constraint $y_2 \geq 0$ is identically 0. All other maximal solutions have domains at most π long and are bounded. Clearly, (2.3) is not asymptotically stable.

Example 3.4. Consider a case of (1.10) with $A = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $K = X = \mathbb{R} \times \{0\}$. Then (2.1) is preasymptotically stable. Having $\dot{x}_2 = x_1$ and x_2 constrained to be 0 suggests that the only maximal solution with a nontrivial domain is the complete solution $x(t) \equiv 0$, and it is bounded and converges to 0. All other maximal solutions have domains consisting of a single point and are bounded. Hence, preasymptotic stability for (2.1) with F as in (1.10) is possible even though $K \cap X \neq \emptyset$. One can show though that if $\operatorname{int} X \neq \emptyset$, it must be that $K \cap X = \emptyset$.

A stronger version of preasymptotic stability is uniform preasymptotic stability. The inclusion (2.1), respectively, (2.2), is *uniformly preasymptotically stable* if, for every solution,

$$\|\phi(t)\| \leq \beta\left(\|\phi(0)\|, t\right) \quad \forall t \in \operatorname{dom} \phi,$$

where $\beta : [0, \infty) \times [0, \infty)$, respectively, $\beta : [0, \infty) \times \{0, 1, 2, \dots\}$, is a continuous function such that $r \mapsto \beta(r, t)$ is 0 at 0 and nondecreasing and $t \mapsto \beta(r, t)$ is nonincreasing and $\beta(r, t) \to 0$ when $t \to \infty$. Furthermore, (2.1), respectively, (2.2), is *preexponentially stable* if every solution ϕ to (2.1), respectively, (2.2), satisfies (3.1). If (2.1), respectively, (2.2) is preexponentially stable and every maximal solution is complete, then it is *exponentially stable*.

LEMMA 3.5. *If* (2.1)*, respectively,* (2.2)*, is uniformly preasymptotically stable, then* (2.1)*, respectively,* (2.2)*, is preexponentially stable.*

The proof is similar to proofs of related results for different kinds of homogeneous dynamics, for example, [1, Theorem 2]. Furthermore, homogeneity implies that uniform bound on, and uniform convergence of, solutions ϕ with $\|\phi(0)\| = 1$ ensures uniform, and then exponential, preasymptotic stability. A general result covering both continuous-time and discrete-time dynamics is in [22, Proposition 4.3].

A principle established in [10] implies that asymptotic stability for set-valued but regular and locally bounded dynamics is uniform. Here, homogeneity leads further, to an exponential bound, and in the discrete-time case, local boundedness comes automatically from asymptotic stability.

LEMMA 3.6. *If F is closed and locally bounded and* (2.1) *is preasymptotically stable, then* (2.1) *is preexponentially stable. If F is closed and* (2.2) *is preasymptotically stable, then* (2.2) *is preexponentially stable.*

Proof. In both cases, outer semicontinuity and local boundedness of F imply uniform preasymptotic stability; see [21, Theorem 7.12]. Lemma 3.5 then gives preexponential stability. For the discrete-time case, preasymptotic stability of (2.2) implies that $F(0) = \{0\}$; indeed, otherwise Lyapunov stability is violated by solutions with $\phi(0) = 0$, $\phi(1) \neq 0$. Because F is closed, $F(0) = \{0\}$ implies that F is locally bounded; indeed, because $F(0) = (\operatorname{dom} F^*)^*$ by [37, Lemma 2.11], $\operatorname{dom} F^* = \mathbb{R}^n$ and F^* is strict, and then local boundedness of F follows from [37, Theorem 2.12]. ∎

One is naturally led to consider another property: the differential inclusion (2.1), respectively, the difference inclusion (2.2), is weakly preasymptotically stable if, for every $\xi \in \operatorname{dom} F$, there exists a maximal solution ϕ with $\phi(0) = \xi$ which is bounded, and if it is also complete, then $\lim_{t\to\infty} \phi(t) = 0$, respectively, $\lim_{j\to\infty} \phi(j) = 0$, and considering only solutions ϕ_ξ as above results in stability. Weak preasymptotic stability will not play a role below.

4. Existence of convex Lyapunov functions.

This section discusses the existence of weak Lyapunov functions for weakly asymptotically stable convex processes, and Lyapunov functions for classes of asymptotically and preasymptotically stable convex processes.

4.1. Weak asymptotic stability and weak Lyapunov functions.

The existence of a smooth weak Lyapunov function $V \in \mathcal{HCS}$ for (2.1) or (2.2) with strict and closed F implies weak exponential stability. In fact, exponential stability can be achieved through continuous selections from F that rely on V. For the case of (2.1), let $V \in \mathcal{HCS}$ be a weak Lyapuov function. The set $S(x) = \{v \in \mathbb{R}^n \mid \nabla V(x) \cdot v \le -\frac{\gamma}{2} V(x)\}$ depends continuously on x, as follows from [31, Example 5.10] and the fact that for every x there exists $v \in S(x)$ with $\nabla V(x) \cdot v \le -\frac{2\gamma}{3} V(x)$. Furthermore, the existence of such v implies that the convex sets $S(x)$ and $F(x)$ cannot be separated. This, and continuity of F, imply that $S(x) \cap F(x)$ depends continuously on x; see [31, Theorem 4.32c]. Now, [31, Proposition 4.9] implies that the projection of 0 onto the nonempty convex set $S(x) \cap F(x)$ depends continuously on x. (This is the same as picking the minimum norm element of $S(x) \cap F(x)$.) This projection defines the needed f. For (2.2), one picks $f(x)$ by projecting 0 onto $\{v \in F(x) \mid V(v) \le -\gamma V(x)\}$, and arguments justifying continuity are very similar to those given above. The case of a nonstrict F is not as simple and is not addressed here. For general control systems, not related to convex processes, construction of feedbacks from control Lyapunov functions are given in [11], [12].

The existence of a convex and homogeneous of degree 1 weak Lyapunov function W for (2.1) was shown in [37, Theorem 9.1] for the case of a strict F, generalizing [8, Theorem 4.1], which dealt with F as in (1.3). Squaring W and smoothing it appropriately can yield a function in $\mathcal{HCS}$ which is a weak Lyapunov function for (2.1). Proposition 4.2 shows an alternative construction through optimal control. Related optimal control problems for positive systems, with nonnegativity constraints on the states, are studied in [5]. The existence of a convex and homogeneous of degree 1 weak Lyapunov function for (2.2) for the case of strict F was claimed in [37, Chapter 8, Problem 4] and shown for (1.3) in [8, Theorem 8.8]. Proposition 4.3 verifies this claim, and the proof does not rely on the spectral techniques suggested in [37]. Propositions 4.2 and 4.3 rely on smoothing of a nonsmooth convex function; one approach to this is now recalled.

LEMMA 4.1. *Let $f \in \mathcal{HC}$, and for $\lambda > 0$ consider $s_\lambda f$, where s_λ represents the self-dual smoothing operator* [19, *Definition* 2.1]. *Then for every $\lambda > 0$, $s_\lambda f \in \mathcal{HCS}$ and $s_\lambda f \to f$ when $\lambda \searrow 0$ uniformly on compact subsets of $\mathbb{R}^n$.*

Proof. Strictly convexity and differentiability is verified by [19, Lemma 2.3] and so is uniform convergence, in light of continuity of f and $s_\lambda f$. Positive homogeneity of degree 2 can be verified directly. Indeed, computing $s_\lambda f$ amounts to taking a convex conjugate of f, adding a multiple of norm squared to the conjugate, taking a conjugate of the sum, and adding a multiple of norm squared to the result. Each of these four operations preserves homogeneity. □

PROPOSITION 4.2. *Let F be strict and closed. If* (2.1) *is weakly asymptotically stable, then there exists a weak Lyapunov function $V \in \mathcal{HCS}$ for* (2.1).

Proof. Let $L(x,v) = \|x\|^2 + \|v\|^2 + \delta_{\mathrm{gph}\,F}(x,v)$, where the indicator function $\delta_{\mathrm{gph}\,F}$ is given by $\delta_{\mathrm{gph}\,F}(x,v) = 0$ if $(x,v) \in \mathrm{gph}\,F$, equivalently, if $v \in F(x)$, and $\delta_{\mathrm{gph}\,F}(x,v) < \infty$ if $v \notin F(x)$. This L fits the framework of [18]. Indeed, because $\mathrm{gph}\,F$ is closed and convex, L is convex and lower semicontinuous; because $\mathrm{gph}\,F$ is a convex cone, the distance from 0 of the set $\{v \in \mathbb{R}^n \mid L(x,v) \neq \infty\}$ grows with at most linear rate in x; and because $L(x,v) \geq \|v\|^2$, L is bounded below by a coercive function of v, and so L satisfies the basic assumption used in [32]. Furthermore, L is positive semidefinite, and so it meets all assumptions of [18]. For $x \in \mathbb{R}^n$, let

$$V_0(x) = \inf\left\{ \int_0^\infty L\big(\phi(t), \dot{\phi}(t)\big)\, dt \mid \phi(0) = x \right\},$$

where the minimization is over all locally absolutely continuous $\phi : [0,\infty) \to \mathbb{R}^n$. The indicator term in L ensures that ϕ for which the integral above is finite is in fact a solution to (2.1). The function V_0 is convex, positive definite, positively homogeneous of degree 2, and by the existence of solutions with exponentially decaying $x(t)$ and $\dot{x}(t)$ [37, Theorem 9.2], it is finite-valued. By [18, Corollary 2.5], V_0 satisfies a Hamilton–Jacobi equation which states that for every $x \in \mathbb{R}^n$ and every $y \in \partial V_0(x)$ (here ∂V_0 is the convex subdifferential of V), one has $\sup_{v \in \mathbb{R}^n} \{(-y) \cdot v - L(x,v)\} = 0$. Consequently, for every such x and y there exists $v \in F(x)$ such that $-y \cdot v - \|x\|^2 - \|v\|^2 = 0$, and thus $y \cdot v \leq -\|x\|^2$.

Now recall Lemma 4.1 and set $V = s_\lambda(V_0)$ with small enough $\lambda > 0$. Then $V \in \mathcal{HCS}$. Because $\nabla s_\lambda(V_0)$ converge to ∂V graphically when $\lambda \searrow 0$ [31, Theorem 12.35], and because ∂V is locally bounded thanks to finiteness of V_0 [31, Exercise 12.40], taking small enough λ ensures that for every x with $\|x\| = 1$ there exists $v \in F(x)$ such that $\nabla V(x) \cdot v \leq -\|x\|^2/2$. Homogeneity implies that V is a weak Lyapunov function. ∎

PROPOSITION 4.3. *Let F be strict. If* (2.2) *is weakly asymptotically stable, then there exists a weak Lyapunov function $V \in \mathcal{HCS}$ for* (2.2).

Proof. For $x \in \mathbb{R}^n$, let

$$V_0(x) = \inf\left\{ \sum_{j=0}^{\infty} \|\phi(j)\|^2 \mid \phi \text{ is a solution to (2.2) with } \phi(0) = x \right\}.$$

Then V_0 is positively homogeneous of degree 2 and convex on $\mathbb{R}^n$. Homogeneity follows from homogeneity of F, and convexity is easily deduced from the fact that a convex combination of two solutions to (2.2) is a solution to (2.2). Furthermore,

$$\|x\|^2 \leq V(x) \leq M^2\|x\|^2/(1 - e^{-2k}), \tag{4.1}$$

where the second inequality follows from the existence, for every $x \in \mathrm{dom}\,F$, of a solution ϕ_ξ satisfying (3.1), as guaranteed by Proposition 3.1, and M and k come from (3.1). Finiteness and convexity of V_0 ensure that it is continuous.

For $\varepsilon \in (0, \|x\|^2/2)$, let ϕ be such that $\sum_{j=0}^{\infty} \|\phi(j)\|^2 \leq V_0(x) + \varepsilon$. Then

$$V_0(x) + \varepsilon \geq \sum_{j=0}^{\infty} \|\phi(j)\|^2 = \|\phi(0)\|^2 + \sum_{j=1}^{\infty} \|\phi(j)\|^2 \geq \|x\|^2 + V_0(\phi(1)),$$

and because $\phi(1) \in F(x)$, for every $x \neq 0$, there exists $v \in F(x)$ such that

$$V_0(v) \leq V_0(x) - \|x\|^2/2 \leq \gamma' V_0(x),$$

where $\gamma' = \left(1 - \frac{1-e^{-2k}}{2M^2}\right)$ and the second inequality comes from (4.1).

Let $V = s_\lambda(V_0)$ as in Lemma 4.1. Then $V \in \mathcal{HCS}$ and, because $s_\lambda(V_0)$ converge to V uniformly on compact sets when $\lambda \searrow 0$, for every $\gamma \in [0, \gamma')$ there exists $\lambda > 0$ such that $V_0(v) \le \gamma' V_0(x)$ implies $V(v) \le \gamma V(x)$ for x with $\|x\| = 1$. The conclusion follows from homogeneity. ∎

The construction can be attempted if $\operatorname{dom} F$ is polyhedral and, more generally, when (2.2) is weakly exponentially stable. More care needs to be given then to continuity of V_0.

4.2. Asymptotic stability and Lyapunov functions. It is standard that if there exists Lyapunov function $V \in \mathcal{HS}$ for (2.1), respectively, (2.2), then (2.1), respectively, (2.2), is preexponentially stable. For example, for (2.2), one has

$$a\|\phi(j)\|^2 \le V(\phi(j)) \le \gamma^j V(\phi(0)) \le \gamma^j b \|\phi(0)\|^2$$

for every solution ϕ, and thus $\|\phi(j)\| \le M e^{-kj} \|\phi(0)\|$ for $k = (\ln \gamma)/2$, $M = b/a$, where a and b are such that $a\|x\|^2 \le V(x) \le b\|x\|^2$.

The existence of a weak Lyapunov function for a preasymptotically stable (2.1) immediately follows from the duality result of [37], as summarized in Theorem 1.1, and from duality of weak Lyapunov and Lyapunov functions in Theorem 2.2.

COROLLARY 4.4. *Let F be closed and locally bounded. If* (2.1) *is preasymptotically stable, then there exists a Lyapunov function $V \in \mathcal{HCS}$ for* (2.1).

Proof. By Lemma 3.6, (2.1) is preexponentially stable. The exponential bound (3.1) on solutions to (2.1) implies that the Lyapunov exponent of nonzero solutions ψ to $x \in -F(x)$, namely, $-\limsup_{t\to\infty} \frac{1}{t} \ln |\psi(t)|$, is bounded above by $-k$, where $k > 0$. Let $G(x) = -F^*(-x)$, so that $G^* = F$. Note that G is strict and closed. Theorem 1.1 implies that $\dot{y} \in G(y)$ is weakly asymptotically stable. Proposition 4.2 yields a weak Lyapunov function $W \in \mathcal{HCS}$ for $\dot{y} \in G(y)$. Theorem 2.2 implies that $V = W^*$ is the desired Lyapunov function for $\dot{x} \in G^*(x) = F(x)$. ∎

Problem 4 in [37, section 8.6] suggests that this approach is possible in the discrete-time case, as it claims that Theorem 1.1 is then valid too, and Theorem 2.2 covers the discrete-time case as well.

The remainder of this section shows how the existence of a Lyapunov function can be obtained in some cases without the duality in Theorem 1.1, which then provides an alternative approach to the equivalence of weak asymptotic stability and asymptotic stability for a pair of adjoint convex processes. First, the existence of smooth—but not convex—Lyapunov functions is obtained, under a local boundedness assumption, as a consequence of [41, Theorem 2], which, essentially, combined a converse Lyapunov result for hybrid inclusions (subsuming differential and difference inclusions) in [9, Theorem 3.14] and a technique of [33] to produce a homogeneous Lyapunov function from a smooth Lyapunov function.

LEMMA 4.5. *Let F be closed and locally bounded. If* (2.1)*, respectively,* (2.2)*, is preasymptotically stable, then there exists a Lyapunov function $V \in \mathcal{HS}$ for* (2.1)*, respectively,* (2.2).

Proof. Either (2.1) or (2.2) can be viewed as a hybrid inclusion in the framework of [21]. The hybrid inclusion is then preasymptotically stable and can be augmented to yield an asymptotically stable hybrid inclusion; see [9, Lemma 7.12]. For the augmented hybrid inclusion, the data satisfies the standing assumptions in [9]—this is where outer semicontinuity and local boundedness of F matters. Then, [9, Theorem 7.9] implies that the hybrid inclusion is robustly $\mathcal{KLL}$-stable, which in turn ensures that [41, Theorem 2] applies. Having $\delta = 0$ and picking $\kappa = 2$ in that theorem proves the current result. ∎

The next result shows that convexification of (weak) Lyapunov functions leads, in some cases, to convex (weak) Lyapunov functions.

LEMMA 4.6. *Suppose that $W : \mathbb{R}^n \to [0, \infty)$ is differentiable, positive definite, and coercive. Then $V = \operatorname{con} W$ is differentiable, positive definite, and coercive, and the following implications hold:*

(a) *If W is a weak Lyapunov function for* (2.1) *or* (2.2)*, then so is V.*

(b) *If F is as in* (1.10) *and W is a Lyapunov function for* (2.1) *or* (2.2)*, then so is V.*

If, furthermore, W is positively homogeneous with degree $p > 0$, then so is V.

Proof. The claims about differentiability, coercivity, etc. are clear. Because W is coercive, for every $x \in \mathbb{R}^n$,

$$V(x) = \min \left\{ \sum_{i=0}^{n} \lambda_i W(x_i) \mid \sum_{i=0}^{n} \lambda_i x_i = x, \sum_{i=0}^{n} \lambda_i = 1, \lambda_i \geq 0 \right\}.$$

Let λ_i, x_i, for $i = 0, 1, \ldots, n$ be the minimizers in the formula above. To prove (a), in the continuous-time case, let $v_i \in F(x_i)$ be such that $\nabla W(x_i) \cdot v_i \leq -\gamma W(x_i)$ and note that $v = \sum_{i=0}^{n} \lambda_i v_i \in F(x)$ as the graph of F is convex. Because $\nabla W(x_i) = \nabla V(x)$, one obtains

$$\nabla V(x) \cdot v = \nabla V(x) \cdot \sum_{i=0}^{n} \lambda_i v_i = \sum_{i=0}^{n} \nabla W(x_i) \cdot \lambda_i v_i \leq \sum_{i=0}^{n} (-\gamma \lambda_i W(x_i)) = -\gamma V(x).$$

In the discrete-time case, let $v_i \in F(x_i)$ be such that $W(v_i) \leq \gamma W(x_i)$ and note that $W(x_i) = V(x_i)$ and $v = \sum_{i=0}^{n} \lambda_i v_i \in F(x)$. Then

$$\begin{aligned} V(v) = V\left(\sum_{i=0}^{n} \lambda_i v_i\right) &\leq \sum_{i=0}^{n} \lambda_i V(v_i) \leq \sum_{i=0}^{n} \lambda_i W(v_i) \\ &\leq \sum_{i=0}^{n} \lambda_i \gamma W(x_i) = \gamma \sum_{i=0}^{n} \lambda_i W(x_i) = \gamma V(x). \end{aligned}$$

To prove (b), take $x \in X$, $k \in K$ and use the notation above. In the continuous-time case,

$$\begin{aligned} \nabla V(x) \cdot (Ax + k) = \nabla V(x) \cdot \sum_{i=0}^{n} \lambda_i (Ax_i + v) &= \sum_{i=0}^{n} \lambda_i \nabla W(x_i) \cdot (Ax_i + v) \\ &\leq \sum_{i=0}^{n} \lambda_i (-\gamma W(x_i)) = -\gamma V(x). \end{aligned}$$

In the discrete-time case,

$$\begin{aligned} V(Ax + k) = V\left(\sum_{i=0}^{n} \lambda_i (Ax_i + v)\right) &\leq \sum_{i=0}^{n} \lambda_i V(Ax_i + v) \leq \sum_{i=0}^{n} \lambda_i W(Ax_i + v) \\ &\leq \sum_{i=0}^{n} \lambda_i (\gamma W(x_i)) = -\gamma V(x). \qquad \square \end{aligned}$$

Note that the conical structure of K and X is not relevant in the proof of (b) above; hence the result holds if K and X are just convex sets.

To combine Lemma 4.5 and Lemma 4.6 to obtain a Lyapunov function for (2.1) or (2.2), assumptions of closedness, local boundedness, and single-valuedness must be posed. This requires F to have the form (1.9).

COROLLARY 4.7. *Let F be as in* (1.9) *with X closed. If* (2.1)*, respectively,* (2.2)*, is preasymptotically stable, then there exists a Lyapunov function $V \in \mathcal{HCS}$ for* (2.1)*, respectively,* (2.2)*.*

Proof. Lemma 4.5 implies the existence of a Lyapunov function $V_1 \in \mathcal{HS}$ for (2.1), respectively, (2.2). Because F is single-valued, the concepts of a weak Lyapunov function and of a Lyapunov function for (2.1), (2.2) are the same. Thus, Lemma 4.6 implies that the convex function $V_2 = \operatorname{con} V_1$ is also a Lyapunov function for (2.1), respectively, (2.2), and $V_2 \in \mathcal{HS}$. In particular, (2.7), respectively, (2.8), holds with V_2 for every $x \in \operatorname{dom} F$ with $\|x\| = 1$. Then (2.7), respectively, (2.8), holds for such x with $V = V_2 + \alpha\|\cdot\|^2$ and γ replaced by $\gamma' \in (0, \gamma)$ for sufficiently small $\alpha > 0$. Then $V \in \mathcal{HCS}$ and homogeneity imply that (2.7), respectively, (2.8), holds also for x with $\|x\| \neq 1$. ∎

This is a weaker result than Corollary 4.4. Both results are different than related results on the existence of Lyapunov functions for positive linear and switching systems; see [15] and the references therein. Here, invariance is not assumed and general cones are considered, and the conclusions are weaker.

4.3. Duality of asymptotic stability concepts. Based on the existence of Lyapunov functions established in the previous sections and on the conjugacy between Lyapunov and weak Lyapunov functions from section 2, but without invoking Theorem 1.1, one obtains some implications between stability properties of (2.1), (2.2) and of (2.3), (2.4).

COROLLARY 4.8. *If F is strict and* (2.1)*, respectively,* (2.2)*, is weakly attractive, then* (2.3)*, respectively,* (2.4)*, is preexponentially stable.*

Proof. For a strict F, Proposition 3.1 showed that weak attractivity ensures weak exponential stability. Then, Proposition 4.2, respectively, 4.3, gives the existence of a Lyapunov function $V \in \mathcal{HCS}$ for (2.1), respectively, (2.2). Theorem 2.2 shows that $V^* \in \mathcal{HCS}$ is a Lyapunov function for (2.3), respectively, (2.4), and consequently, (2.3), respectively, (2.4), is preexponentially stable. ∎

If F and F^* are as in (1.3), and so F^* has the form as in (1.9), stronger conclusions hold.

COROLLARY 4.9. *Let F and F^* be given by* (1.3)*. The following are equivalent:*

(a) (2.1)*, respectively,* (2.2)*, is weakly attractive.*
(b) (2.1)*, respectively,* (2.2)*, is weakly exponentially stable.*
(c) (2.3)*, respectively,* (2.4)*, is preexponentially stable.*
(d) (2.3)*, respectively,* (2.4)*, is preasymptotically stable.*

Proof. Proposition 3.1 shows that (a) implies (b), because $\operatorname{dom} F = \mathbb{R}^n$ is polyhedral, and the converse implication is obvious. Corollary 4.8 shows that (a) implies (c) because F is strict, and (c) implies (d). If (2.3), respectively, (2.4), is preasymptotically stable, then Corollary 4.7, applied to F^* in which K^* is closed, gives a Lyapunov function $V^* \in \mathcal{HCS}$ for (2.3), respectively, (2.4). Theorem 2.4 then shows that $V \in \mathcal{HCS}$ is a weak Lyapunov function for (2.1), respectively, (2.2). Weak exponential stability for (2.1), respectively, (2.2) follows, and so (d) implies (b). ∎

5. Conclusions and future work. The main results of this paper showed, roughly, that a convex function V is a weak Lyapunov function for dynamics given by a convex process F if and only if the conjugate function V^* is a Lyapunov function for dynamics given by the adjoint convex process F^*. Further results related various asymptotic stability properties of dynamics given by convex processes to the existence of convex Lyapunov and weak Lyapunov functions.

An important application of the results is the analysis of linear time-invariant dynamical systems with conical constraints. Let U be a closed cone. Suppose that

(i) the linear control system $\dot{x} = Ax + Bu$ is asymptotically controllable with locally integrable controls $u : [0, \infty) \to U$.

Then

(ii) the dual linear system $\dot{y} = A^T y$, $o = B^T y$ is detectable through U^*,

in the sense that every solution $y : [0, \infty) \to \mathbb{R}^n$ to $\dot{y} = A^T y$ which satisfies $o(t) = B^T y(t) \in U^*$ is such that $\lim_{t \to \infty} y(t) = 0$. Indeed, (i) implies that (2.1) is weakly attractive with $F(x) = Ax + K$, where $K = BU$, and Corollary 4.9 implies that (2.3) is preasymptotically/exponentially stable. This entails convergence to 0 of every complete solution to (2.3) with $F^*(y)$ given in (1.3). But $y \in K^*$ is then equivalent to $B^T y \in U^*$, and the conclusion about detectability follows. To show the reverse implication through results of this paper, one needs to check when detectability in (ii) implies preasymptotic stability for (1.5). This and treatment of state constraints are topics for future work. Regarding state constraints, note that Example 3.2 showed that in their presence, weak asymptotic stability need not dualize to preasymptotic stability. On the other hand, Lyapunov functions for the adjoint dynamics (2.3), (2.4) in the case of (1.10) dualize to weak Lyapunov functions for (2.1), (2.2) as seen in Theorem 2.4. Hence, results of this paper will lead to some relations between appropriately understood asymptotic controllability and detectability for linear systems with conical control and state constraints.

Developments in this paper are different from, and complement, Theorem 1.1, where dual characterization of weak asymptotic stability was given in terms of eigenvalues, invariant subspaces, and Lyapunov exponents rather than Lyapunov functions. The implication from (a) to (b) and (c) in Theorem 1.1 can be established through Proposition 4.2, which gives a weak Lyapunov function V for $\dot{x} \in F(x)$, and Theorem 2.2, which states that V^* is a Lyapunov function for the adjoint inclusion. The resulting preexponential stability of the adjoint inclusion then implies (b) and (c). On the other hand, Theorem 1.1 was essential in the converse Lyapunov result for preasymptotic stability given here in Corollary 4.4. Further comparison of the current work to Theorem 1.1 hinge upon relating (b) and (c) of this theorem to preasymptotic stability, which is an interesting topic for future work.

In general, a complete and symmetric theory of conjugate Lyapunov functions for convex processes requires resolving the issue of the existence of a convex Lyapunov function for a general preasymptotically stable convex process. Section 4.2 gives an affirmative answer for locally bounded processes, with a direct, and not relying on duality, construction of the Lyapunov function given only for special cases. One should expect that a direct construction, not relying on [21], [9], or [41], should be possible. A related and interesting question is whether the duality result (1.8) of [3] can be derived by relying on duality between Lyapunov inequalities that characterize finite-time convergence.

REFERENCES

[1] D. Angeli, P. De Leenheer, and E. D. Sontag, *Chemical networks with inflows and outflows: A positive linear differential inclusions approach*, Biotechnology Progress, 25 (2009), pp. 632–642.

[2] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Basel, Switzerland, 1990.

[3] J.-P. Aubin, H. Frankowska, and C. Olech, *Controllability of convex processes*, SIAM J. Control Optim., 24 (1986), pp. 1192–1211.

[4] N. E. Barabanov, *Stability of inclusions of linear type*, in Proceedings of the 14th American Control Conference, Vol. 5, 1995, pp. 3366–3370.
[5] C. Beauthier and J. J. Winkin, *LQ-optimal control of positive linear systems*, Optimal Control Appl. Methods, 31 (2010), pp. 547–566.
[6] A. Berman, M. Neumann, and R. J. Stern, *Nonnegative matrices in dynamic systems*, Pure and Applied Mathematics, John Wiley & Sons, New York, 1989.
[7] R. F. Brammer, *Controllability in linear autonomous systems with positive controllers*, SIAM J. Control, 10 (1972), pp. 339–353.
[8] V. A. Bushenkov and G. V. Smirnov, *Stabilization Problems with Constraints: Analysis and Computational Aspects*, Gordon and Breach Science Publishers, Amsterdam, 1997.
[9] C. Cai, A. R. Teel, and R. Goebel, *Smooth Lyapunov functions for hybrid systems. Part II: (Pre-)asymptotically stable compact sets*, IEEE Trans. Automat. Control, 53 (2008), pp. 734–748.
[10] F. H. Clarke, Y. S. Ledyaev, and R. J. Stern, *Asymptotic stability and smooth Lyapunov functions*, J. Differential Equations, 149 (1998), pp. 69–114.
[11] F. H. Clarke and R. J. Stern, *State constrained feedback stabilization*, SIAM J. Control Optim., 42 (2003), pp. 422–441.
[12] F. H. Clarke and R. J. Stern, *Lyapunov and feedback characterizations of state constrained controllability and stabilization*, Systems Control Lett., 54 (2005), pp. 747–752.
[13] W. P. Dayawansa and C. F. Martin, *A converse Lyapunov theorem for a class of dynamical systems which undergo switching*, IEEE Trans. Automat. Control, 44 (1999), pp. 751–760.
[14] L. Farina and S. Rinaldi, *Positive Linear Systems Theory and Applications*, Pure Appl. Math., Wiley-Interscience, New York, 2000.
[15] E. Fornasini and M. E. Valcher, *Linear copositive Lyapunov functions for continuous-time positive switched systems*, IEEE Trans. Automat. Control, 55 (2010), pp. 1933–1937.
[16] H. Frankowska, *Local controllability and infinitesimal generators of semigroups of set-valued maps*, SIAM J. Control Optim., 25 (1987), pp. 412–432.
[17] P. Gajardo and A. Seeger, *Higher-order spectral analysis and weak asymptotic stability of convex processes*, J. Math. Anal. Appl., 318 (2006), pp. 155–174.
[18] R. Goebel, *Duality and uniqueness of convex solutions to stationary Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 357 (2005), pp. 2187–2203.
[19] R. Goebel, *Self-dual smoothing of convex and saddle functions*, J. Convex Anal., 15 (2008), pp. 179–190.
[20] R. Goebel, T. Hu, and A. R. Teel, *Dual matrix inequalities in stability and performance analysis of linear differential/difference inclusions*, in Current Trends in Nonlinear Systems and Control, Birkhäuser, Basel, Switzerland, 2006, pp. 103–122.
[21] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*, Princeton University Press, Princeton, NJ, 2012.
[22] R. Goebel and A. R. Teel, *Pre-asymptotic stability and homogeneous approximations of hybrid dynamical systems*, SIAM Rev., 52 (2010), pp. 87–109.
[23] R. Goebel, A. R. Teel, T. Hu, and Z. Lin, *Conjugate convex Lyapunov functions for dual linear differential inclusions*, IEEE Trans. Automat. Control, 51 (2006), pp. 661–666.
[24] M. Heymann and R. J. Stern, *Controllability of linear systems with positive controls: geometric considerations*, J. Math. Anal. Appl., 52 (1975), pp. 36–41.
[25] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1972.
[26] A. Lewis, R. Henrion, and A. Seeger, *Distance to uncontrollability for convex processes*, SIAM J. Control Optim., 45 (2006), pp. 26–50.
[27] A. M. Lyapunov, *The general problem of the stability of motion*, Internat. J. Control, 55 (1992), pp. 521–790 (in English).
[28] A. P. Molchanov and Ye. S. Pyatnitskiy, *Criteria of asymptotic stability of differential and difference inclusions endountered in control theory*, Systems Control Lett., 13 (1989), pp. 59–64.
[29] Y. Ohta, H. Maeda, and S. Kodama, *Reachability, observability, and realizability of continuous-time positive systems*, SIAM J. Control Optim., 22 (1984), pp. 171–180.
[30] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.
[31] R. T. Rockafellar and R. J-B. Wets, *Variational Analysis*, Springer, New York, 1998.
[32] R. T. Rockafellar and P. R. Wolenski, *Convexity in Hamilton-Jacobi theory,* 1*: Dynamics and duality*, SIAM J. Control Optim., 39 (2000), pp. 1323–1350.
[33] L. Rosier, *Homogeneous Lyapunov function for homogeneous continuous vector fields*, Systems Control Lett., 19 (1992), pp. 467–473.

[34] A. Seeger, *A duality result for the reachable cone of a nonstrict convex process*, J. Nonlinear Convex Anal., 2 (2001), pp. 363–368.

[35] R. Sepulchre, M. Jankovic, and P. V. Kokotović, *Constructive nonlinear control*, Comm. Control Engrg. Ser., Springer, New York, 1997.

[36] J. Shen and J. Hu, *Stability of discrete-time switched homogeneous systems on cones and conewise homogeneous inclusions*, SIAM J. Control Optim., 50 (2012), pp. 2216–2253.

[37] G. V. Smirnov, *Introduction to the Theory of Differential Inclusions*, Grad. Stud. Math. 41, American Mathematical Society, Providence, RI, 2002.

[38] G. V. Smirnov, *Weak asymptotic stability of differential inclusions.* I, Avtomat. i Telemekh., 7 (1990), pp. 57–64.

[39] G. V. Smirnov, *Weak asymptotic stability of differential inclusions.* II, Avtomat. i Telemekh., 8 (1990), pp. 56–63.

[40] E. D. Sontag, *A Lyapunov-like characterization of asymptotic controllability*, SIAM J. Control Optim., 21 (1983), pp. 462–471.

[41] S. E. Tuna and A. R. Teel, *Homogeneous hybrid systems and a converse Lyapunov theorem*, in Proceedings of the 45th IEEE Conference on Decision and Control, 2006, pp. 6235–6240.