Robustness of Stability through Necessary and Sufficient Lyapunov-like Conditions for Systems with a Continuum of Equilibria

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Abstract

The equivalence between robustness to perturbations and the existence of a continuous Lyapunov-like mapping is established in a setting of multivalued discrete-time dynamics for a property sometimes called semistability. This property involves a set consisting of Lyapunov stable equilibria and surrounded by points from which every solution converges to one of these equilibria. As a consequence of the main results, this property turns out to always be robust for continuous nonlinear dynamics and a compact set of equilibria. Preliminary results on reachable sets, limits of solutions, and set-valued Lyapunov mappings are included.

1. Introduction

For a discrete-time dynamical system, this paper deals with a set which consists of Lyapunov stable equilibria each of which is surrounded by points from where trajectories converge to one of these equilibria. Such equilibria were called *semistable* in [4]. Semistability in this sense was then studied in [17], [18], [19], [5], and more. The term "semistability", in a sense related to partial stability and different from [4], was discussed in Russian control literature; see the survey [30]. The term was also used by [26] and related works to represent a property of an equilibrium weaker than Lyapunov stability, in the setting of abstract and set-valued dynamical systems. Some results of [26] are related to preliminary results here; see Remark 2.11. This paper uses the term *pointwise asymptotic stability* to characterize a set consisting of semistable, in the sense of [4], equilibria. The goal is to establish robustness of pointwise asymptotic stability to perturbations in dynamics and characterize this robustness through regularity of appropriate Lyapunov-like mappings.

Sufficient conditions for pointwise asymptotic stability, for differential equations [4] and then for differential inclusions [18], were given in terms of classical Lyapunov functions and non-tangent, to the set of equilibria, behavior of trajectories. A converse Lyapunov result for differential equations was obtained in [17]; this converse result does

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not result in a sufficient Lyapunov condition. A different approach, inspired by [25] where the decrease of set-valued mappings was proposed as a sufficient condition for consensus, was pursued by the author in [11]. Necessary and sufficient Lyapunov-like conditions for pointwise asymptotic stability and a related invariance principle were given there in terms of a set-valued Lyapunov-like mapping. Some results of [11] are recalled in Section 3. Robustness of pointwise asymptotic stability has received limited treatment. The converse result of [17] was used in [20] to state robustness to higher-order perturbations under homogeneity assumptions. This robustness result included assumptions on Lyapunov stability of the equilibria for the perturbed, not just nominal, dynamics. A related result was given in [16] for a switching system.

For the classical concept of asymptotic stability, the equivalence of asymptotic stability to the existence of Lyapunov functions, with further relation between robustness of the asymptotic stability or regularity of the dynamics to the continuity or smoothness of Lyapunov functions, is well-appreciated. In particular, the equivalence of robustness of asymptotic stability of an equilibrium and the existence of a smooth Lyapunov function in the setting of nonlinear and multivalued dynamics was exhibited first in [8], in the setting of differential inclusions. This was later carried over to asymptotic stability of sets [29], to difference inclusions [21], and hybrid dynamics [7].

The contribution of this paper is showing the equivalence of robustness of pointwise asymptotic stability to the existence of continuous set-valued Lyapunov functions, in the setting of multivalued, but continuous in an appropriate sense, discrete-time dynamics and for a compact set of equilibria. This is shown in Theorem 4.3. Because continuous set-valued Lyapunov functions exist for pointwise asymptotically stable sets when the dynamics are continuous, for such dynamics the pointwise asymptotic stability of compact sets is always robust to sufficiently small perturbations. This is stated in Corollary 4.5 and appears to be a new result even in the single-valued case where the dynamics are given by a continuous function.

The relevance of pointwise asymptotic stability for analysis of consensus algorithms has been discussed, for example, in [17] and [20]. The issue of robustness of consensus algorithms has seen treatment in the literature, with the focus on convergence and not Lyapunov stability of consensus / equilibrium states and most often with robustness to changes of communication topology in time or to delays. Examples in Section 2 illustrate how state-dependent changes in communication topology fit in the framework of this paper and hint at applications of the robustness results to consensus problems. Furthermore, Example 2.5 suggest potential application of the results to analysis of optimization algorithms.

The paper is organized as follows. Section 2 introduces pointwise asymptotic stability and other basic concepts, and collects preliminary results on the behavior of solutions to a difference inclusion in the presence of a pointwise asymptotically stable closed set. In particular, results on continuous or semicontinuous dependence of the reachable set and of the limits of solutions on initial points are given. Section 3 introduces set-valued Lyapunov functions and employs them in necessary and sufficient conditions for pointwise asymptotic stability. A key observation here is that continuous set-valued dynamics lead to a continuous set-valued Lyapunov function. Section 4 states and proves the main result, Theorem 4.3.

2. Setting and basic results

Throughout the paper, a difference inclusion

$$x^+ \in F(x), \tag{1}$$

is considered, where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping, i.e., for every $x \in \mathbb{R}^n$, $F(x)$ is a subset of $\mathbb{R}^n$. The function $\phi : \mathbb{N}_0 \to \mathbb{R}^n$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, is a *solution to* (1) from the initial point $\xi \in \mathbb{R}^n$ if $\phi(0) = \xi$ and, for all $i \in \mathbb{N}$, $\phi(i) \in F(\phi(i-1))$. The set of all solutions to (1) from ξ is denoted $\mathcal{S}(\xi)$. Given a set $C \subset \mathbb{R}^n$, $\mathcal{S}(C)$ is the set of all solutions to (1) from points in C, $\mathcal{S}(C) = \bigcup_{\xi \in C} \mathcal{S}(\xi)$.

One motivation for considering set-valued dynamics, following Krasovskii [23] and Filippov [10], is the link between set-valued regularization of discontinuous dynamics and the effect on such dynamics of perturbations, as shown in [15], [14] for differential equations and in [28] for hybrid systems, which encompass difference equations and inclusions used here. The example below illustrates this.

Example 2.1. Let $x_1, x_2, \ldots, x_I \in \mathbb{R}^m$ represent states of I agents. Suppose that each agent changes its position following the rule: find the average state of all agents, including myself, whose states differ less than 1 from my state and move half-way towards this average. This and similar dynamics have seen treatment in the literature, with the origins going back to [13] and [24]; see the extensive discussion in [6]. In the simple case of two one-dimensional agents, dynamics are given by the function $f(x)$ equal to

$$\begin{cases} (x_1, x_2) & \text{if} \quad |x_1 - x_2| \geq 1 \\ \left(\frac{3x_1}{4} + \frac{x_2}{4}, \frac{x_1}{4}x_1 + \frac{3x_2}{4}\right) & \text{if} \quad |x_1 - x_2| < 1 \end{cases}.$$

The set-valued regularization of this discontinuous f is given by the set-valued mapping F whose graph is the closure of the graph of f. Alternatively, F is the "smallest" outer semicontinuous, as defined below, mapping such that $f(x) \in F(x)$ for all $x \in \mathbb{R}^n$. Explicitly, $F(x)$ is

$$\begin{cases} (x_1, x_2) & \text{if} \quad |x_1 - x_2| > 1 \\ \left\{x_1, \frac{3x_1}{4} + \frac{x_2}{4}\right\} \times \left\{x_2, \frac{x_1}{4} + \frac{3x_2}{4}\right\} & \text{if} \quad |x_1 - x_2| = 1 \\ \left(\frac{3x_1}{4} + \frac{x_2}{4}, \frac{x_1}{4} + \frac{3x_2}{4}\right) & \text{if} \quad |x_1 - x_2| < 1 \end{cases}.$$

The Cartesian product representing $F(x)$ when $|x_1 - x_2| = 1$ contains four points and represents the fact that small perturbations, or measurement error if this is cast as a feedback control problem, agent x_1 can either move or remain stationary, and so can x_2. $\triangle$

Let $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping. Let $x \in \mathbb{R}^n$. Then M has a *nonempty value* at x if $M(x) \neq \emptyset$. M is *outer semicontinuous (osc) at* x if for every sequence $x_i \to x$, every convergent sequence $y_i \in F(x_i)$, one has $\lim_{i\to\infty} y_i \in F(x)$. It is *continuous at* x if, additionally, for every $y \in F(x)$, every sequence $x_i \to x$, there exist $y_i \in F(x_i)$ such that the sequence of y_i converges and $\lim_{i\to\infty} y_i = y$. The mapping M is *locally bounded* at x if there exists a neighborhood U of x such that $F(U) = \bigcup_{z \in U} F(z)$ is bounded. If

M has compact values and is *locally bounded at* x, then osc at x is equivalently described by: for every $\varepsilon > 0$ there exists $\delta > 0$ such that $F(x + \delta I\!B) \subset F(x) + \varepsilon I\!B$, which means that for every $z \in x + \delta I\!B$, $F(z) \subset F(x) + \varepsilon I\!B$. Here $I\!B$ is the closed unit ball in $\mathbb{R}^n$, and so $z \in x + \delta I\!B$ means that z is in a closed ball of radius δ around x, i.e., $|z - x| \leq \delta$. The additional condition for continuity of such M at x is: for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $z \in x + \delta I\!B$, $F(x) \subset F(z) + \varepsilon I\!B$.

Throughout the paper, the following assumption is posed. It ensures, among other things, that solutions to (1) exist.

Assumption 2.2. *The set-valued mapping* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *has nonempty values and is locally bounded on* $\mathbb{R}^n$.

For $J \in \mathbb{N}_0$, consider the finite-horizon reachable set

$$\mathcal{R}_{\leq J}(\xi) := \{\phi(j) \,|\, \phi \in \mathcal{S}(\xi), j \in \{0, 1, \ldots, J\}\}.$$

When F is locally bounded, then $\mathcal{R}_{\leq J}$ is locally bounded, and then, if F is osc or continuous then so is $\mathcal{R}_{\leq J}$. This can be verified directly, but also follows from the representation $\mathcal{R}_{\leq J}(\xi) = \{\xi\} \cup F(\xi) \cup F^2(\xi) \cup \cdots \cup F^J(\xi)$ and results about unions and compositions of set-valued mappings, [27, 4.31, 5.52]. The infinite-horizon reachable set

$$\mathcal{R}(\xi) = \bigcup_{J \in \mathbb{N}} \mathcal{R}_{\leq J}(\xi)$$

does not inherit regularity properties from F, in fact, $\mathcal{R}(\xi)$ fails to have closed values even if F is a continuous function. Better regularity properties hold for the closure of the reachable set, i.e., the mapping $\overline{\mathcal{R}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by

$$\overline{\mathcal{R}}(\xi) = \overline{\mathcal{R}(\xi)}.$$

2.1. Pointwise asymptotic stability

A set consisting of equilibria which are semistable in the terminology of [4], i.e., a set consisting of Lyapunov stable equilibria and surrounded by points from which solutions converge to some equilibrium in the set, will be called pointwise asymptotically stable. Below, $\operatorname{rge}\phi$ denotes the range of the solution ϕ, so, for example, $\operatorname{rge}\phi \subset a + \varepsilon I\!B$ means that $\phi(j) \in a + \varepsilon I\!B$ for all $j \in \mathbb{N}_0$.

Definition 2.3. *The set* $A \subset \mathbb{R}^n$ *is* locally pointwise asymptotically stable *(PAS) for* (1) *if*

- *every* $a \in A$ *is* Lyapunov stable *for* (1)*, that is, for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $\operatorname{rge}\phi \subset a + \varepsilon I\!B$ *for every* $\phi \in \mathcal{S}(a + \delta I\!B)$*, and*
- (1) *is* locally convergent *to* A*, that is, there exists a neighborhood* V *of* A *such that, for every* $\phi \in \mathcal{S}(V)$*,* $\lim_{j\to\infty} \phi(j)$ *exists and belongs to* A*.*

The set of all ξ *such that, for every* $\phi \in \mathcal{S}(\xi)$*,* $\lim_{j\to\infty} \phi(j)$ *exists and belongs to* A *is the* basin of pointwise attraction *of* A*, and is denoted* $\mathcal{BPA}$*. If* $\mathcal{BPA} = \mathbb{R}^n$*, then* A *is* globally pointwise asymptotically stable

Example 2.4. The set

$$A = \{(x_1, x_2, \dots, x_I) \in \mathbb{R}^{Im} \mid x_1 = x_2 = \dots = x_I\}$$

is locally PAS for the dynamics outlined in Example 2.1. This can be verified directly. Alternative justification, for the two agent case, is suggested in Example 3.2. △

The next example illustrates that PAS sets naturally arise in many convex optimization algorithms. The notation $\|x\|_A$ represents the distance of x from the set A, so $\|x\|_A = \inf_{a\in A} \|x - a\|$ and $\|\cdot\|$ is the Euclidean norm.

Example 2.5. Let $A \subset \mathbb{R}^n$ be nonempty and closed and suppose $\lim_{j\to\infty} \|\phi(j)\|_A = 0$ for every solution ϕ to (1) and, for every $a \in A$, every $j \in \mathbb{N}$,

$$\|\phi(j) - a\| \leq \|\phi(j-1) - a\|. \tag{2}$$

Sequences $\phi(j)$ satisfying (2) are called *Fejér monotone* and result from many optimization algorithms, and appear in the analysis of nonexpansive mappings, variational inequalities, etc; see [9] and [2]. For example, let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function with the set of minimizers equal to A. Define continuous dynamics by letting $F(x)$ to be the unique minimizer of $y \mapsto f(y) + \frac{1}{2}\|y - x\|^2$, which is a simple case of the proximal point algorithm. Then the conditions above are met and A is pointwise asymptotically stable for (1). Indeed, (2) ensures Lyapunov stability of each $a \in A$ and convergence of every solution to a point in A follows from [2, Theorem 5.11]; or it can be concluded from Lyapunov stability of each $a \in A$ and the fact that the ω-limit of every solution is contained in A. △

Lyapunov stability of a point can be equivalently expressed in terms of neighborhoods. For Lyapunov stability of a set and for attractivity defined below, the two approaches may differ; see the discussion in [1], [3]. Below, a set A is *Lyapunov stable* if for every neighborhood U of A there exists a neighborhood V of A such that $\operatorname{rge}\phi \subset U$ for every $\phi \in \mathcal{S}(V)$. For a non-compact A this differs from (ε, δ)*-Lyapunov stability*, requiring that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\operatorname{rge}\phi \subset A + \varepsilon I\!B$ for every $\phi \in \mathcal{S}(A + \delta I\!B)$. Similarly, A is *locally attractive* if there is a neighborhood V of A such that $\lim_{j\to\infty} \|\phi(j)\|_A = 0$ for every $\phi \in \mathcal{S}(V)$, and A is *locally asymptotically stable* if it is Lyapunov stable and locally attractive. The *basin of attraction* of an asymptotically stable A, denoted $\mathcal{BA}$, is the set of all ξ such that for every $\phi \in \mathcal{S}(\xi)$, $\lim_{j\to\infty} \|\phi(j)\|_A = 0$.

It is clear that if A consists of a single point, $A = \{a\}$, then local pointwise asymptotic stability is the same as local asymptotic stability. If A is compact and every $a \in A$ is Lyapunov stable then A is Lyapunov stable. Furthermore:

Proposition 2.6. *Let $A \subset \mathbb{R}^n$ be closed.*

(a) If A is locally pointwise asymptotically stable then A is locally asymptotically stable, with $\mathcal{BPA} \subset \mathcal{BA}$. If, furthermore, A is compact then $\mathcal{BPA} = \mathcal{BA}$.

(b) If A is locally attractive and every $a \in A$ is Lyapunov stable, then A is locally pointwise asymptotically stable.

Proof. Suppose that the closed set A is PAS. Let U be a neighborhood of A. For each $a \in A$, let $\varepsilon_a > 0$ be such that $a + \varepsilon_a \mathbb{B} \subset U$ and then let $\delta_a > 0$ correspond to ε_a as required by Lyapunov stability of a. Also, let V come from local convergence of (1) to A. Then the set

$$V' = \bigcup_{a \in A} (a + \delta_a \mathbb{B}) \cap V \tag{3}$$

is such that every $\phi \in \mathcal{S}(V')$ satisfies $\operatorname{rge}\phi \subset U$ and $\lim_{j\to\infty} \|\phi(j)\|_A = 0$ because ϕ converges to a point in A. It is clear that $\mathcal{BPA} \subset \mathcal{BA}$. If A is compact, then every $\phi \in \mathcal{S}(\mathcal{BA})$ is bounded and has a nonempty ω-limit. Because $\lim_{j\to\infty} \|\phi(j)\|_A = 0$, every ω-limit point of ϕ is an element of A. Then each such ω-limit point is Lyapunov stable, and hence there can be only one such point. Thus ϕ is convergent to a point in A and so $\mathcal{BA} \subset \mathcal{BPA}$. This proves (a). Let V come from local attractivity of A and for $a \in A$ let $\delta_a > 0$ come from Lyapunov stability of a invoked with $\varepsilon = 1$, so that $\operatorname{rge}\phi \subset a + \mathbb{B}$ for every $\phi \in \mathcal{S}(a + \delta_a \mathbb{B})$. Let V' be the neighborhood of A in (3). Then every $\phi \in \mathcal{S}(V')$ is bounded, arguments as above show that ϕ converges to a point in A, and so V' verifies that (1) is locally convergent to A. This shows (b). ◻

To see that $\mathcal{BPA} \neq \mathcal{BA}$ in (a) above can happen, consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x,y) = \begin{cases} (x-1, y + e^{x-1} - e^x) & \text{if} \quad y > e^x \\ (x, y/e) & \text{if} \quad y \le e^x \end{cases}.$$

In particular, if $y = e^x + c$, $c > 0$ then $y^+ = e^{x^+} + c$. The x-axis is globally asymptotically stable, $\mathcal{BA} = \mathbb{R}^2$, and locally pointwise asymptotically stable, with $\mathcal{BPA} = \{(x,y) \,|\, y \le e^x\}$. Thus $\mathcal{BA} \neq \mathcal{BPA}$. Note that $\mathcal{BPA}$ is not open. Let $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be the set-valued regularization of the discontinuous f. Explicitly,

$$F(x,y) = \begin{cases} (x-1, y + e^{x-1} - e^x) & \text{if} \quad y > e^x \\ \{(x-1, y + e^{x-1} - e^x), (x, y/e)\} & \text{if} \quad y = e^x \\ (x, y/e) & \text{if} \quad y < e^x \end{cases}.$$

With dynamics given by F, the x-axis is still locally pointwise asymptotically stable, with $\mathcal{BPA} = \{(x,y) \,|\, y < e^x\}$, which is open. C.f. Proposition 2.10.

Note that local attractivity of a set A can be defined differently: there exists a neighborhood V of A such that every $\phi \in \mathcal{S}(V)$ converges to A in the sense that for every neighborhood U of A there exists j_0 such that $\phi(j) \in U$ for all $j > j_0$. For an unbounded A, this is a stronger condition than what was discussed above. With the new definition, no solution $\phi \in \mathcal{S}(V)$ can be divergent unless it converges to A in finite time: if $\lim_{j\to\infty} \|\phi(j)\| = \infty$ and $\operatorname{rge}\phi \cap A = \emptyset$ then $U = \bigcup_{a\in A} a + \frac{1}{2}\min_{j\in\mathbb{N}_0} \|\phi(j) - a\|\mathbb{B}$ is a neighborhood of A such that $\operatorname{rge}\phi \cap U = \emptyset$. Consequently, every $\phi \in \mathcal{S}(V)$ has an omega-limit point in A. Hence, if each $a \in A$ is Lyapunov stable and the closed set A is locally attractive according to the new definition, then A is locally pointwise asymptotically stable as ensured by Proposition 2.6 (b) and $\mathcal{BPA} = \mathcal{BA}$. Related results can be found in [3, Section 5].

2.2. Basic properties

This section collects several properties of solutions to (1) when there exists a PAS set. They lead up to Proposition 2.13 which is essential for Theorem 3.3 (c) which is then relied on in the main result. Versions of some of the results, for the continuous-time and single-valued setting, can be found in [4], [5]. Throughout the section, the following is assumed.

Assumption 2.7. *The set $A \subset \mathbb{R}^n$ is nonempty, closed, and pointwise asymptotically stable for* (1)*.*

This assumption implies that $F(x) = x$ for all $x \in A$ and also that F is continuous at each such x. Properties of solutions to (1) shown below require mild regularity of F not just at points of A but at all points of $\mathcal{BPA}$. The following assumption strengthens Assumption 2.2.

Assumption 2.8. *The set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is locally bounded, outer semi-continuous, and nonempty-valued on $\mathbb{R}^n$.*

The standard consequence of Assumption 2.8 for solutions to (1) is contained in the first part of (a) in Lemma 2.9. PAS strengthens this, as stated in the second part of (a). In (a), locally uniform convergence means uniform convergence on every compact subset of $\mathbb{N}_0$ and in (c), $\|\phi - \phi'\|_\infty = \sup_{j \in \mathbb{N}_0} \|\phi(j) - \phi'(j)\|$.

Lemma 2.9. *Pose Assumption 2.7 and 2.8. Then:*

(a) For every compact set $K \subset \mathbb{R}^n$ and every sequence $\phi_i \in \mathcal{S}(K)$, there exists a subsequence ϕ_{i_k} which converges pointwise and locally uniformly to some $\phi \in \mathcal{S}$. If $\phi(0) \in \mathcal{BPA}$, then the subsequence ϕ_{i_k} converges to ϕ uniformly.

(b) For any sequence of $\phi_i \in \mathcal{S}(\mathcal{BPA})$ uniformly convergent to $\phi \in \mathcal{S}(\mathcal{BPA})$, one has $\lim_{i\to\infty} (\lim_{j\to\infty} \phi_i(j)) = \lim_{j\to\infty} \phi(j)$.

(c) For every compact $K \subset \mathcal{BPA}$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $\phi \in \mathcal{S}(K + \delta \mathbb{B})$ there exists $\phi' \in \mathcal{S}(K)$ such that $\|\phi - \phi'\|_\infty < \varepsilon$.

Proof. For (a), the first conclusion is standard; local boundedness of F and a diagonalization argument yields pointwise convergence and it implies locally uniform convergence. For the second conclusion, let $a = \lim_{j\to\infty} \phi(j) \in A$. Pick an arbitrary $\varepsilon > 0$. Let $\delta > 0$ be related to $\varepsilon/2$ as required by Lyapunov stability of a. Let $J \in \mathbb{N}$ be such that $\phi(J) \in a + \delta/2\mathbb{B}$ and let $I \in \mathbb{N}$ be such that $\phi_i(J) \in \phi(J) + \delta/2\mathbb{B} \subset a + \delta\mathbb{B}$ for all $i \geq I$. Then, for all $i \geq I$, $j \geq J$, $\phi(j) \in a + \varepsilon/2\mathbb{B}$, $\phi_i(j) \in a + \varepsilon/2\mathbb{B}$ and consequently, $\phi_i(j) \in \phi(j) + \varepsilon\mathbb{B}$. Now, relying on local uniform convergence of ϕ_i to ϕ, pick $I' \geq I$ such that $\phi_i(j) \in \phi(j) + \varepsilon\mathbb{B}$ for all $i \geq I'$, $j \in \{0, 1, \ldots, J\}$. Then $\phi_i(j) \in \phi(j) + \varepsilon\mathbb{B}$ for all $i \geq I'$ and all $j \in \mathbb{N}_0$, which verifies uniform convergence. Conclusions (b) and (c) follow directly from (a). ◻

The *limit mapping* $L : \mathcal{BPA} \rightrightarrows A$ assigns to each $x \in \mathcal{BPA}$ the set of limits of all solutions from x.

$$L(\xi) = \left\{ \lim_{j\to\infty} \phi(j) \mid \phi \in \mathcal{S}(\xi) \right\}$$

In [5, Proposition 2.3], in a single-valued and continuous-time case, L was shown to be continuous. Proposition 2.10, which recalls some properties of $\overline{\mathcal{R}}$ and L from [11, Proposition 4.1], and Proposition 2.13 generalize this to the multivalued case.

Proposition 2.10. *Pose Assumption 2.7 and 2.8. Then:*

(a) $\mathcal{R}$, and thus $\overline{\mathcal{R}}$ and L, are locally bounded on $\mathcal{BPA}$;

(b) $\mathcal{BPA}$ is an open set;

(c) $\overline{\mathcal{R}}$ and L are outer semicontinuous on $\mathcal{BPA}$ and, for every $\xi \in \mathcal{BPA}$,

$$\overline{\mathcal{R}}(\xi) = \mathcal{R}(\xi) \cup L(\xi), \qquad L(\xi) = \overline{\mathcal{R}}(\xi) \cap A.$$

Proof. Only $L(\xi) = \overline{\mathcal{R}}(\xi) \cap A$ was not shown in [11, Proposition 4.1]. Take $a \in \overline{\mathcal{R}}(\xi) \cap A$. Let $\delta_i \in (0, 1/i)$ be related to $\varepsilon_i = 1/i$ as required by Lyapunov stability of a. For $i \in \mathbb{N}$, there exist $\phi_i \in \mathcal{S}(\xi)$ and $j_i \in \mathbb{N}_0$ such that $\phi_i(j_i) \in a + \delta_i/2\mathbb{B}$. Without relabeling, pass to a uniformly convergent subsequence of ϕ_i's, with the limit $\phi \in \mathcal{S}(\xi)$. Then $\lim_{j\to\infty} \phi(j) = a$, and hence $a \in L(\xi)$. $\square$

To see how L can be well-defined in absence of PAS but fail the properties in Proposition 2.10, consider $F : \mathbb{R} \to \mathbb{R}$ given by $F(x) = \sqrt{|x|}$. The interval $A = [0, 1]$ is the smallest compact and globally asymptotically stable set, with $\mathcal{BA} = \mathbb{R}$. Note that $L(x) = \{1\}$ if $x \neq 0$ while $L(0) = 0$. Consequently, L is not osc on $\mathcal{BA}$. In fact, it is not osc at 0, which is in the asymptotically stable set A.

Remark 2.11. In [26], semi-stability was defined to be a property of motions in a generalized dynamical system having sets as values. In this terminology, the mapping $j \to \mathcal{R}_{\leq j}(x)$ is semi-stable for every $x \in A$ when A is PAS. Indeed, the condition of [26] that for every $\varepsilon > 0$ there exist $\delta > 0$ and $J \in \mathbb{N}_0$ so that $x' \in x + \delta\mathbb{B}$ implies $\mathcal{R}_{\leq j+J}(x') \subset \mathcal{R}_{j+J}(x) + \varepsilon\mathbb{B}$ is satisfied, with $J = 0$, because $\mathcal{R}_{\leq j}(x) = \{x\}$ and x is Lyapunov stable. Hence, some results of [26] turn out closely related to this work. For example, [26, Theorem 3] suggests that $\overline{\mathcal{R}}(x) = \mathcal{R}(x) \cup \Lambda(x)$, where $\Lambda(x)$ is the ω-limit of the set $\{x\}$. Under the current assumptions, one can then argue that $\Lambda(x) = L(x)$, through Lemma 2.9, to recover the first equation in Proposition 2.10 (c). It should be noted that semi-stability of [26] does not imply pointwise asymptotic stability. $\triangle$

The next result shows uniform convergence of solutions from a compact set K to the compact set of their limits $L(K)$ and a finite-horizon approximation for $\mathcal{R}(K)$.

Lemma 2.12. *Pose Assumption 2.7 and 2.8. Then, for every compact set $K \subset \mathcal{BPA}$ and every $\varepsilon > 0$, there exists $J \in \mathbb{N}$ such that*

(a) for every $\phi \in \mathcal{S}(K)$, $\phi(j) \in L(K) + \varepsilon\mathbb{B}$ for every $j \geq J$;

(b) for every $\xi \in K$, $\mathcal{R}(\xi) \subset \mathcal{R}_{\leq J}(\xi) + \varepsilon\mathbb{B}$ and, consequently, $\mathcal{R}(K) \subset \mathcal{R}_{\leq J}(K) + \varepsilon\mathbb{B}$.

Proof. Suppose that (a) fails. Then there exist sequences of $\phi_i \in \mathcal{S}(K)$ and $j_i \in \mathbb{N}$ such that $\phi_i(j_i) \notin L(K) + \varepsilon\mathbb{B}$ and $\lim_{i\to\infty} j_i = \infty$. Without relabeling, pass to a uniformly convergent subsequence of ϕ_i, with limit $\phi \in \mathcal{S}(K)$. Let $a = \lim_{j\to\infty} \phi(j) \in L(K)$ and let $\delta > 0$ be related to ε as required by Lyapunov stability of a. There exists $J' \in \mathbb{N}$ such that $\phi(J') \in a + \delta/2\mathbb{B}$. There exists $I \in \mathbb{N}$ such that, for all $i > I$, $\phi_i(J') \in \phi(J') + \delta/2\mathbb{B}$

and thus $\phi_i(J') \in a + \delta I\!B$. Lyapunov stability of a now yields that $\phi_i(j) \in a + \varepsilon I\!B$ for all $i > I$, $j > J'$ which is a contradiction. For (b), let $M = L(K)$ and note that $M \subset A$ is compact thanks to outer semicontinuity and local boundedness of L on $\mathcal{BPA}$. For every $a \in M$, Lyapunov stability implies that there exists $\delta_a \in (0, \varepsilon/2)$ such that $\mathcal{R}(x) \subset a + \varepsilon/2 I\!B$ for every $x \in a + \delta_a I\!B$. The family of sets $a + \delta_a I\!B$, where $a \in M$, is an open cover of M. Let $a_i + \delta_{a_i} I\!B$, $i \in \{1, 2, \dots, I\}$ be a finite subcover and let $\delta = \min_{i \in I} \delta_{a_i}$. Then, for every $x \in M + \delta I\!B$, there exists $i \in I$ with $x \in a_i + \delta_{a_i} I\!B$, and hence

$$\mathcal{R}(x) \subset a_i + \delta_{a_i} I\!B \subset a_i + \varepsilon/2 I\!B \subset x + \varepsilon I\!B.$$

Take $J \in \mathbb{N}$ so that, for every $\phi \in \mathcal{S}(K)$, $\phi(J) \in M + \delta I\!B$. It exists because of (a). Then, for every $\phi \in \mathcal{S}(K)$, $\phi(j) \in \phi(J) + \varepsilon I\!B$ for every $j > J$ and this verifies the claim. □

Further regularity of the set-valued mappings $\overline{\mathcal{R}}$ and L can be concluded when F is continuous, not just osc.

Proposition 2.13. *Pose Assumption 2.7 and 2.8. If, furthermore, F is a continuous set-valued mapping, then $\overline{\mathcal{R}}$ and L are continuous on $\mathcal{BPA}$.*

Proof. Recall that $\overline{\mathcal{R}}$ and L locally bounded and osc. Pick $\varepsilon > 0$ and $\xi \in \mathcal{BA}$. By Lemma 2.12 there exists $J \in \mathbb{N}$ such that $\mathcal{R}(\xi) \subset \mathcal{R}_{\leq J}(\xi) + \varepsilon/2 I\!B$. Because both $\mathcal{R}_{\leq J}(\xi)$ and $\varepsilon/2 I\!B$ are closed and bounded, $\mathcal{R}_{\leq J}(\xi) + \varepsilon/2 I\!B$ is closed and thus $\overline{\mathcal{R}}(\xi) \subset \mathcal{R}_{\leq J}(\xi) + \varepsilon/2 I\!B$. Because $\mathcal{R}_{\leq J}$ is continuous, there exists $\delta > 0$ such that, for every $\xi' \in \xi + \delta I\!B$, $\mathcal{R}_{\leq J}(\xi) \subset \mathcal{R}_{\leq J}(\xi') + \varepsilon/2 I\!B$. For each such ξ',

$$\begin{array}{rcl} \overline{\mathcal{R}}(\xi) & \subset & \mathcal{R}_{\leq J}(\xi) + \varepsilon/2 I\!B \\ & \subset & \mathcal{R}_{\leq J}(\xi') + \varepsilon/2 I\!B + \varepsilon/2 I\!B \subset \overline{\mathcal{R}}(\xi') + \varepsilon I\!B. \end{array}$$

This verifies that $\overline{\mathcal{R}}$ is continuous. Now pick ξ and $\xi_i \to \xi$ in $\mathcal{BPA}$ and $y \in L(\xi)$. Use Lyapunov stability of y to pick $\delta_i > 0$ be such that $\phi(0) \in y + \delta_i I\!B$ implies $\operatorname{rge} \phi \subset y + i^{-1} I\!B$. Because $L(\xi) \subset \overline{\mathcal{R}}(\xi)$ and $\overline{\mathcal{R}}$ is continuous at ξ, there exist $\phi_i \in \phi(\xi)$ and $j_i \in \mathbb{N}_0$ so that $\phi_i(j_i) \in y + \delta_i I\!B$. Then $\lim_{j \to \infty} \phi_i(j) \in y + i^{-1} I\!B$, and because $\lim_{j \to \infty} \phi_i(j) \in L(\xi_i)$, this verifies continuity of L at ξ. □

The section concludes with an example of continuous F, related to the consensus dynamics in Example 2.1.

Example 2.14. Let $\phi : \mathbb{R}^m \to [0, 1]$ be a continuous function such that $\phi(0) = 1$, $\phi(z) = 0$ when $\|z\| \geq 1$. The j-th agent changes its position by moving half-way towards the weighted average of positions of other agents, given by

$$wa_j(x) = \sum_{i=1}^{J} x_i \phi(x_j - x_i) \Bigg/ \sum_{i=1}^{J} \phi(x_j - x_i) \, .$$

The function ϕ can represent the strength of the signal with which agents broadcast their positions or the certainty in observation of other agent positions. This yields continuous dynamics $x_i^+ = f_i(x) = \frac{1}{2}(x_i + wa_j(x))$ for which PAS of the set A from Example 2.4 can be analyzed. △

3. Set-valued Lyapunov functions

Set-valued Lyapunov functions provide necessary and sufficient conditions for pointwise asymptotic stability. Key results of [11] on this topic are recalled below in Theorem 3.3 and also strengthened for the case of continuous set-valued dynamics.

The following definition comes from [11, Definition 2.3], with a minor change allowing for lack of regularity of W at points not in A. Below, $\operatorname{dom} W = \{x \in \mathbb{R}^n \mid W(x) \neq \emptyset\}$ and forward invariance of a set $S \subset \mathbb{R}^n$ with respect to (1) means that every $\phi \in \mathcal{S}(S)$ satisfies $\operatorname{rge}\phi \subset S$.

Definition 3.1. *A set-valued mapping $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a* set-valued Lyapunov function *for* (1) *and a nonempty set $A \subset \mathbb{R}^n$ if*

(a) $W(x) = \{x\}$ *for every $x \in A$;*

(b) $x \in W(x)$ *for every $x \in \operatorname{dom} W$;*

(c) W *is locally bounded and, at every $x \in A$, it is outer semicontinuous;*

(d) $\operatorname{dom} W$ *is an open neighborhood of A forward invariant with respect to* (1) *and there exists a continuous and positive definite with respect to A function $\alpha : \operatorname{dom} W \to [0, \infty)$ such that, for any convergent sequence of points $x_i \in \operatorname{dom} W$ satisfying $\lim_{i\to\infty} \alpha(x_i) = 0$, one has $\lim_{i\to\infty} x_i \in A$ and*

$$W(F(x)) + \alpha(x)\mathbb{B} \subset W(x) \quad \forall x \in \operatorname{dom} W. \tag{4}$$

An outer semicontinuous (continuous) set-valued Lyapunov function W is a set-valued Lyapunov function W which is osc (continuous) at every $x \in \operatorname{dom} W$. Note that the existence of the continuous and positive definite with respect to A function α implies that A is closed.

Example 3.2. Consider the two agent dynamics from Example 2.1. For x with $|x_1 - x_2| < 1$, let

$$W(x) = \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right) + r(x)\mathbb{B},$$

where $r(x)$ is the distance of x from $(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2})$. For $|x_1 - x_2| \geq 1$, set $W(x) = \emptyset$. Then W is a set-valued Lyapunov function for (1) and $A = \{x \in \mathbb{R}^2 \mid x_1 = x_2\}$. In fact, W is continuous on $\operatorname{dom} W$ and (4) holds with $\alpha(x) = \frac{1}{2}r(x)$. Note that the set-valued mapping $W'(x) = x + r(x)\mathbb{B}$, is not a set-valued Lyapunov function because α satisfying (4) fails to exist: $\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}\right)$ is on the boundary of both $W'(x)$ and $W'(F(x))$, and in particular, $W'(F(x))$ is not in the interior of $W'(x)$. $\triangle$

Theorem 3.3.

(a) If there exists a set-valued Lyapunov function W for (1) *and A, then A is pointwise asymptotically stable and* $\operatorname{dom} W \subset \mathcal{BPA}$.

(b) *Assume that F satisfies Assumption 2.8 and the set $A \subset \mathbb{R}^n$ is nonempty and compact. If A is pointwise asymptotically stable then there exists an outer semicontinuous set-valued Lyapunov function W for* (1) *and A with* $\operatorname{dom} W = \mathcal{BPA}$.

(c) *Assume that F satisfies Assumption 2.8, is continuous, and the set $A \subset \mathbb{R}^n$ is nonempty and closed. If A is pointwise asymptotically stable then there exists a continuous set-valued Lyapunov function W for* (1) *and A with* $\operatorname{dom} W = \mathcal{BPA}$.

Proof. The sufficiency in (a) closely follows the idea of [25, Theorem 4] and was shown as [11, Theorem 3.1]. The assumptions in [11] involved osc of W but the proof of [11, Theorem 3.1] relied only on osc at points in A, as assumed here. The necessary condition in (b) was shown as [11, Theorem 4.7]. The necessary condition for pointwise asymptotic stability in (c) follows from the proof of [11, Theorem 4.4]. Indeed, while [11, Theorem 4.4] did not conclude continuity of W, the proof relied on setting $W(x) = \overline{\mathcal{R}}(x) + V(x)\mathbb{B}$, where V is a smooth function. Proposition 2.13 showed that continuity of F implies continuity for $\overline{\mathcal{R}}$, and then continuity of W follows from continuity and local boundedness of $\overline{\mathcal{R}}$ and of the mapping $x \mapsto V(x)\mathbb{B}$; see [27, Proposition 5.51]. □

The next result shows that the existence of a set-valued Lyapunov function implies the existence of a set-valued Lyapunov function with convex values; this will be used in Section 4 in the proof of the main result. For a set $S \subset \mathbb{R}^n$, $\operatorname{con} S$ denotes the convex hull of S, i.e., the smallest convex set containing S, equivalently, the intersection of all convex sets containing S.

Proposition 3.4. *If $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued Lyapunov function for* (1) *and a set $A \subset \mathbb{R}^n$ then so is* $\operatorname{con} W$. *If, furthermore, W is outer semicontinuous or continuous then so is* $\operatorname{con} W$.

Proof. Clearly, $\operatorname{con} W$ satisfies (a) and (b) of Definition 3.1. Local boundedness of W implies this property for $\operatorname{con} W$ and ensures that osc of W at every $x \in A$ implies osc of $\operatorname{con} W$ at each such x. (4) implies that $W(y) + \alpha(x)\mathbb{B} \subset W(x)$ for every $x \in \operatorname{dom} W$, $y \in F(x)$, and so

$$\operatorname{con}\left(W(y) + \alpha(x)\mathbb{B}\right) \subset \operatorname{con} W(x). \tag{5}$$

Now, $\operatorname{con}(W(y) + \alpha(x)\mathbb{B}) = \operatorname{con} W(y) + \alpha(x)\mathbb{B}$ because $\alpha(x)\mathbb{B}$ is a convex set. Indeed, $\operatorname{con}(W(y) + \alpha(x)\mathbb{B}) \subset \operatorname{con} W(y) + \alpha(x)\mathbb{B}$ because the convex set $\operatorname{con} W(y) + \alpha(x)\mathbb{B}$ contains $W(y) + \alpha(x)\mathbb{B}$ and $\operatorname{con}(W(y) + \alpha(x)\mathbb{B})$ is the smallest such set. On the other hand, take $z \in \operatorname{con} W(y) + \alpha(x)\mathbb{B}$. There exist finitely many $z_i \in W(y)$, $\lambda_i > 0$ with $\Sigma\lambda_i = 1$ and $b \in \mathbb{B}$ such that $z = \Sigma\lambda_i z_i + \alpha(x)b$. Then $z = \Sigma\lambda_i\,(z_i + \alpha(x)b)$, which verifies that $z \in \operatorname{con}(W(y) + \alpha(x)\mathbb{B})$. Consequently, (5) turns to

$$\operatorname{con} W(y) + \alpha(x)\mathbb{B} \subset \operatorname{con} W(x)$$

and this implies that (4) is satisfied by $\operatorname{con} W$. Outer semicontinuity or continuity of $\operatorname{con} W$ follows from [27, Proposition 4.30], in light of local boundedness of W. □

The remark below establishes the equivalence between the existence of a set-valued Lyapunov function as defined here and a set-valued mapping which satisfies (a), (b), and (c) of Definition 3.1 and (d'), below, used by [25] to characterize a decrease of a set-valued Lyapunov function.

Remark 3.5. If $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued Lyapunov function for (1) and a set $A \subset \mathbb{R}^n$ then

$$\operatorname{diam} W(F(x)) \leq \operatorname{diam} W(x) - 2\alpha(x) \quad \forall x \in \operatorname{dom} W. \tag{6}$$

On the other hand, suppose that $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfies (a), (b), and (c) of Definition 3.1 and

(d') there exists a function $\mu : \{W(x) \,|\, x \in U\} \to [0, \infty)$, where $U \subset \mathbb{R}^n$ is a neighborhood of A forward invariant with respect to (1), such that

$$W(F(x)) \subset W(x) \quad \forall x \in U, \tag{7}$$

and, moreover, such that $x \mapsto \mu(W(x))$ is locally bounded on U and for which there exists a lower semicontinuous function $\beta : U \to [0, \infty)$, positive definite with respect to A, and such that

$$\mu(W(F(x))) \leq \mu(W(x)) - \beta(x) \quad \forall x \in U. \tag{8}$$

Then the set-valued mapping $\Omega : U \rightrightarrows \mathbb{R}^n$ defined by

$$\Omega(x) = W(x) + \mu(W(x))\mathbb{B}, \quad \forall x \in U$$

is a set-valued Lyapunov function for (1) and A. Indeed, if $x \in U$, $y \in F(x)$, then $W(y) \subset W(x)$ thanks to (7) and $(\mu(W(y)) + \beta(x))\mathbb{B} \subset \mu(W(x))\mathbb{B}$ thanks to (8). Hence

$$\begin{aligned} \Omega(y) + \beta(x)\mathbb{B} &= W(y) + \mu(W(y))\mathbb{B} + \beta(x)\mathbb{B} \\ &\subset W(x) + \mu(W(x))\mathbb{B} = \Omega(x). \end{aligned}$$

This establishes (4) for Ω, with $\alpha = \beta$. Replacing β with a continuous and positive definite with respect to A function bounded above by β proves the claim. For example, one can consider $\beta'(x) = \inf_{u \in U} \{\beta(u) + |x - u|\}$, which has the needed properties and is Lipschitz continuous with constant 1; see [27, Example 9.11]. $\triangle$

4. Robustness

This section establishes the equivalence between the existence of a continuous set-valued Lyapunov function and robust pointwise asymptotic stability for a compact set A. The robustness definition below parallels what has been used for difference inclusions for example in [21] and [7]. When dynamics result from implementation of feedback in a control system, the perturbation (10) can model measurement error, numerical errors in feedback algorithm, external perturbations, etc.

Definition 4.1. *The set $A \subset \mathbb{R}^n$ is* robustly pointwise asymptotically stable *for* (1) *if it is pointwise asymptotically stable and there exists a continuous and positive definite with respect to A function $\rho : \mathbb{R}^n \to [0, \infty)$ such that A is pointwise asymptotically stable for*

$$x^+ \in F_\rho(x) \tag{9}$$

with the basin of pointwise attraction equal to $\mathcal{BPA}$, where $F_\rho : \mathbb{R} \rightrightarrows \mathbb{R}^n$ is the set-valued mapping defined by

$$F_\rho(x) = \bigcup_{y \in F(x+\rho(x)B)} y + \rho(y)\mathbb{B}. \tag{10}$$

The example below shows how pointwise asymptotic stability can fail under perturbations if those do not vanish sufficiently fast near A. While the convergence to a neighborhood of A may still occur, Lyapunov stability of points in A fails completely. [4, Example 1.1] is a related illustrative example.

Example 4.2. In $\mathbb{R}^2$ let $A = [-1,1] \times \{0\}$, let $P_A(x)$ be the nearest point to x in A, so that $\|x\|_A = \|x - P_A(x)\|$, let $\lambda(x) = \frac{\|x\|_A}{\|x\|_A+1}$, and define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x) = \lambda(x)x + (1-\lambda(x))P_A(x).$$

Then $\|F(x)\|_A = \lambda(x)\|x\|_A$ and $P_A(F(x)) = P_A(x)$. In short, (1) results in the decrease of the distance of x from A by a factor $\lambda(x)$. In particular, if $\|x\|_A = \frac{1}{n}$ then $\|x^+\|_A = \frac{1}{n+1}$, and the solution to (1) from $(0,1)$ is $\phi(j) = \left(0, \frac{1}{j+1}\right)$, $j \in \mathbb{N}_0$. Let $\rho(x) = \varepsilon\|x\|_A$ for any $\varepsilon > 0$. Then there exists a solution ϕ to (9), in fact to $x^+ \in F(x) + \rho(F(x))\mathbb{B}$, from $(0,1)$ such that $\phi_2(j) = \frac{1}{j+1}$ while $\phi_1(j)$ increases from 0 to 1 in finitely many steps, then decreases from 1 to -1 in finitely many steps, then increases from -1 to 1, etc. This is caused by the fact that the sequence $\|\phi(j)\|_A = \frac{1}{j+1}$ is not summable, and so neither is the sequence of perturbations $\rho(\phi(j))$. This solution converges to A, $\lim_{j\to\infty} \|\phi(j)\|_A = 0$, but does not converge to any point in A. In fact, it could be argued that the ω-limit of this solution equals A. This behavior is possible with an arbitrarily small $\varepsilon > 0$. However, small $\varepsilon > 0$ preserve semiglobal practical stability of A: for every compact set K and $\delta > 0$, there exists $\varepsilon > 0$ such that solutions to (9) from K converge to the δ-neighborhood of A. △

The main result of the paper is now stated and proved.

Theorem 4.3. *Let $A \subset \mathbb{R}^n$ be compact and pointwise asymptotically stable for* (1)*. The following are equivalent:*

(a) A is robustly pointwise asymptotically stable for (1)*.*

(b) There exists a continuous set-valued Lyapunov function W for (1) *and A.*

The idea behind the implication from (a) to (b) is this: pointwise asymptotic stability ensures the existence of an outer semicontinuous set-valued function and then robustness makes possible to construct from it a continuous set-valued Lyaupnov function. The construction is very similar to what was done in [8, Proposition 3.5], when using robustness of (the classical) asymptotic stability for a differential inclusion to construct a continuous differential inclusion for which asymptotic stability is still present. Here, the construction is applied to W, not to the dynamics. For readers convenience, the details are included. The idea behind the implication from (b) to (a) is that the decrease of a continuous set-valued Lyapunov function is preserved locally under small perturbations. Then, pointwise asymptotic stability implies (the classical) asymptotic stability, and the robustness of the latter property lets one combine the sizes of local perturbations to obtain a global robustness result for pointwise asymptotic stability.

Proof. (a) $\implies$ (b). First, note that if A is robustly PAS for (1) then A is robustly PAS for

$$x^+ \in \overline{F}(x), \tag{11}$$

where $\overline{F} : \mathbb{R} \rightrightarrows \mathbb{R}^n$ is the osc regularization of F, i.e., the set-valued mapping defined, at each $x \in \mathbb{R}^n$, by

$$\overline{F}(x) = \bigcap_{\delta > 0} \overline{F(x + \delta I\!B)}. \tag{12}$$

Indeed, $\overline{F}(x) \subset F_\rho(x)$ for every $x \in \mathcal{BPA}$, so A is PAS for (11). Furthermore, $\overline{F}_{\rho/2}(x) \subset F_\rho(x)$, where $\overline{F}_{\rho/2}$ is a perturbation of $\overline{F}$, and this verifies robustness of PAS of A for (11). Because $\overline{F}$ satisfies Assumption 2.8, without loss of generality, it is supposed below that F satisfies Assumption 2.8. Let ρ come from Definition 4.1 and, without loss of generality, let it be Lipschitz continuous with constant 1. (Otherwise, one considers $\rho'(x) = \min\{\inf_{z \in \mathcal{BPA}} \{\rho(z) + |x - z|\}, |x|_A\}$, which has the desired properties and satisfies $\rho'(x) \leq \rho(x)$.)

Theorem 3.3 yields an osc set-valued Lyapunov function W for (9) and A with $\operatorname{dom} W = \mathcal{BPA}$. This W can be assumed to be convex, by Proposition 3.4. Then, arguments as in the proof of [8, Proposition 3.5], with W replacing F and $\mathcal{BPA} \setminus A$ replacing $\mathbb{R}^n \setminus \{0\}$, yield the existence of a locally Lipschitz, hence continuous, set-valued mapping $W' : \mathcal{BPA} \setminus A \rightrightarrows \mathbb{R}^n$ such that, for every $x \in \mathcal{BPA} \setminus A$,

$$W(x) \subset W'(x) \subset \operatorname{con} W(x + \rho(x) I\!B). \tag{13}$$

Indeed, the interiors of $U_x = x + \frac{1}{3}\rho(x) I\!B$, $x \in \mathcal{BPA} \setminus A$, form a covering of $\mathcal{BPA} \setminus A$, and a locally finite open subcovering V_i, $i \in \mathbb{N}$, can be found, together with a subordinated smooth partition of unity ψ_i, $i \in \mathbb{N}$. For $i \in \mathbb{N}$, let x_i be such that $V_i \subset U_{x_i}$. For $x \in \mathcal{BPA} \setminus A$, let

$$W'(x) = \sum_{i=1}^{\infty} \psi_i(x) \operatorname{con} W\left(x_i + \frac{1}{3}\rho(x_i) I\!B\right). \tag{14}$$

Then W' is continuous at every $x \in \mathcal{BPA} \setminus A$. For such x, if $\psi_i(x) > 0$ then $x \in V_i \subset U_{x_i} = x_i + \frac{1}{3}\rho(x_i) I\!B$ and so $W(x) \subset W\left(x_i + \frac{1}{3}\rho(x_i) I\!B\right)$ and consequently $W(x) \subset W'(x)$. Furthermore, Lipschitz continuity of ρ with constant 1 and $x \in x_i + \frac{1}{3}\rho(x_i) I\!B$ imply $\frac{2}{3}\rho(x_i) < \rho(x)$ and $x_i + \frac{1}{3}\rho(x_i) I\!B \subset x + \frac{2}{3}\rho(x_i) I\!B \subset x + \rho(x) I\!B$. Then $W(x_i + \frac{1}{3}\rho(x_i) I\!B) \subset W(x + \rho(x) I\!B)$ and consequently $W'(x) \subset \operatorname{con} W(x + \rho(x) I\!B)$. This concludes the arguments borrowed from [8, Proposition 3.5].

Extend W' given by (14) to $\mathcal{BPA}$ by setting $W'(x) = \{x\}$ for $x \in A$. Then (13) holds for $x \in A$ (in fact, as an equality) and W' is continuous at each $x \in A$. Indeed, for every $a \in A$, $\varepsilon > 0$, by outer semicontinuity of W there exists $\delta > 0$ so that $W(a + \delta I\!B) \subset W(a) + \varepsilon I\!B$, and then by continuity of ρ and the fact that $\rho(a) = 0$, there exists $\delta' > 0$ such that $x+\rho(x)I\!B \subset a+\delta I\!B$ for every $x \in a+\delta' I\!B$. Hence, for $x \in a+\delta' I\!B$,

$$\begin{array}{rcl} W'(x) & \subset & \operatorname{con} W(x+\rho(x)I\!B) \subset \operatorname{con} W(a+\delta I\!B) \\ & \subset & W(a)+\varepsilon I\!B = W'(a)+\varepsilon I\!B. \end{array}$$

This shows W' is osc, and hence continuous because $W'(a)$ is a single point. It remains to show that W' is a set-valued Lyapunov function for (1). Because W is a set-valued Lyapunov function for (9) and $W(x)$ is convex, for every $x \in \mathcal{BPA}$, every $y \in F(x+\rho(x))$,

$$\operatorname{con} W(y+\rho(y)I\!B) + \alpha(x)I\!B \subset W(x).$$

Then, (13) implies $W'(y)+\alpha(x)I\!B \subset W'(x)$, for every $y \in F(x+\rho(x))$, in particular, for every $y \in F(x)$. Hence, the continuous set-valued mapping W' is a set-valued Lyapunov function for (1).

(b) $\implies$ (a). A preliminary result is needed.

Lemma 4.4. *Let W be a continuous set-valued Lyapunov function for* (1) *and A. Let α come from* (4)*. Then:*

- *(i) W is a set-valued Lyapunov function for* (11)*; in fact, it satisfies* (4) *with α.*
- *(ii) For every compact set $K \subset \operatorname{dom} W$ such that $K \cap A = \emptyset$ there exists $\delta > 0$ such that, for all $x \in K$,*

$$W(F(x+\delta I\!B)+\delta I\!B) + \frac{1}{2}\alpha(x)I\!B \subset W(x). \tag{15}$$

Proof. For (i), take $x \in \operatorname{dom} W$, $y \in \overline{F}(x)$ and note that there exist $x_i \to x$, $y_i \to y$, with $y_i \in F(x_i)$. Then $W(y_i)+\alpha(x_i)I\!B \subset W(x_i)$ and, for any $\varepsilon > 0$ and all large enough i, continuity of W implies that $W(y) \subset W(y_i)+\varepsilon I\!B$ and $W(x_i) \subset W(x)+\varepsilon I\!B$ while continuity of α implies that $\alpha(x) \le \alpha(x_i)+\varepsilon$. Then

$$\begin{array}{rcl} W(y)+\alpha(x)I\!B & \subset & W(y_i)+\alpha(x_i)I\!B + 2\varepsilon I\!B \\ & \subset & W(x_i)+2\varepsilon I\!B \subset W(x)+3\varepsilon I\!B. \end{array}$$

Hence $W(\overline{F}(x))+\alpha(x)I\!B \subset W(x)+3\varepsilon I\!B$, and because ε is arbitrary and $W(x)$ is closed, $W(\overline{F}(x))+\alpha(x)I\!B \subset W(x)$. This verifies (a).

If (ii) was false, there would exist a compact $K \subset \operatorname{dom} W \setminus A$, points $x_i \in K$, $x'_i \in x_i + \frac{1}{i}I\!B$, $y'_i \in F(x'_i)$, and $y_i \in y'_i + \frac{1}{i}I\!B$ so that $W(y_i)+\frac{1}{2}\alpha(x_i) \not\subset W(x_i)$. K is compact and F is locally bounded, so one can extract convergent subsequences, not relabeled, so that $x_i \to x \in K$, $y_i \to y \in \overline{F}(x)$, and, by continuity of W and α, so that $W(y)+\frac{1}{2}\alpha(x) \not\subset \operatorname{int} W(x)$. Here, $\operatorname{int} W(x)$ is the interior of $W(x)$. Because $x \notin A$, $\alpha(x) > 0$ and $W(y)+\frac{1}{2}\alpha(x) \not\subset \operatorname{int} W(x)$ contradicts (4). This proves (b). ◻

Thanks to Lemma 4.4, F can be replaced by $\overline{F}$, and so one can assume that F satisfies Assumption 2.8. PAS of the compact set A implies asymptotic stability, by Proposition 2.6, and $\mathcal{BPA} = \mathcal{BA}$. Then, Assumption 2.8 ensures that the asymptotic stability is robust; see [12, Theorem 7.21] or [22, Theorem 2.8]. That is, there exists a continuous and positive definite $\rho_0 : \mathcal{BPA} \to [0,\infty)$ such that A is asymptotically stable for $x^+ \in F_{\rho_0}(x)$, with F_{ρ_0} constructed from ρ_0 as in (10). Let $V : \mathcal{BPA} \to [0,\infty)$ be a smooth Lyapunov function verifying asymptotic stability of A for $x^+ \in F_{\rho_0}(x)$; it exists thanks to [22, Theorem 2.7]. For $i \in \mathbb{N}$, set $K_i = \{x \in \mathcal{BPA} \,|\, 1/(i+1) \leq V(x) \leq 1/i\}$. Then, let $\delta_i > 0$ be such that $\delta_i \leq \min_{x \in K_i} \rho_0(x)$ and such that (15) holds for all $x \in K_i$. Let $\rho : \mathcal{BPA} \to [0,\infty)$ be a continuous function such that $\rho \leq \rho_0$ and $\rho(x) \leq \min\{\delta_1, \delta_2, \dots, \delta_i\}$ for all $x \in K_i$. Then, for every $x \in A \cup \bigcup_{i=1}^{\infty} K_i$, every $y' \in F(x + \rho(x)\mathbb{B})$, and every $y \in y' + \rho(y')\mathbb{B}$,

$$W(y) + \frac{1}{2}\alpha(x)\mathbb{B} \subset W(x). \tag{16}$$

Indeed, if $x \in K_i$ then $y' \in A$ or $y' \in K_j$ with $j \geq i$, because sublevel sets of V are forward invariant for $x^+ \in F_{\rho_0}(x)$ and so for $x^+ \in F_\rho(x)$. If $y' \in K_j$ then $\rho(y') \leq \delta_i$ by construction, while if $y' \in A$ then $\rho(y') = 0$. Then, (15) with δ_i in place of δ implies (16). If $x \in A$, then $\rho(x) = 0$, $y' = x$, and (16) holds.

Having $W(F_\rho(x)) + \frac{1}{2}\alpha\mathbb{B} \subset W(x)$ on a neighborhood of A where $V(x) \leq 1$ gives that A is locally PAS for (9). Because A is also asymptotically stable for (9), with basin of attraction $\mathcal{BPA}$, Proposition 2.6 ensures that A is PAS for (9), with basin of pointwise attraction equal to $\mathcal{BPA}$. In short, A is robustly PAS for (9). ◻

Combining Theorem 4.3 with Theorem 3.3 yields the following consequence, which applies in particular when the dynamics are given by a continuous function.

Corollary 4.5. *Suppose that F in* (1) *satisfies Assumption 2.8 and is continuous. If a nonempty and compact set $A \subset \mathbb{R}^n$ is pointwise asymptotically stable for* (1) *then A is robustly pointwise asymptotically stable for* (1).

5. Conclusions

The paper has shown that pointwise asymptotic stability of a closed set for a continuous difference inclusion is equivalent to the existence of a continuous set-valued Lyapunov function, and when the set is compact, the existence of such a set-valued Lyapunov function is equivalent to robustness of pointwise asymptotic stability. In particular, pointwise asymptotic stability of a compact set is robust for continuous dynamics. An interesting theoretical question is how to define a set-valued Lyapunov function for continuous-time dynamics given by a differential equation or inclusion, and whether similar robustness results are possible. A practical issue to be addresses is whether the robustness results can help in the analysis and design of consensus algorithms.

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