Continuous Time Linear Quadratic Regulator With Control Constraints via Convex Duality

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***Abstract*—A continuous time infinite horizon linear quadratic regulator with input constraints is studied. Optimality conditions, both in the open loop and feedback form, and continuity and differentiability properties of the optimal value function and of the optimal feedback are shown. Arguments rely on basic ideas of convex conjugacy, and in particular, use a dual optimal control problem.**

***Index Terms*—Convex conjugacy, dual optimal control problem, input constraints, linear-quadratic regulator, nonlinear feedback.**

I. Introduction

Despite its role in construction of stabilizing feedbacks, the continuous time constrained linear quadratic regulator problem has not, to our knowledge, seen a thorough analysis that included open-loop and feedback optimality conditions, regularity analysis of the optimal value function, its Hamilton–Jacobi description, and a characterization of its gradient via a Hamiltonian system. We provide it here, focusing on a problem with input constraints only.

Our analysis of the continuous time linear quadratic regulator with control constraints ($\mathcal{CLQR}$) benefits from two techniques previously used (but not simultaneously) to study this, and other optimal control problems on infinite time horizon: duality and reduction to a finite time horizon.

The use of dual convex optimal control problems was proposed in [1] and [2], and first applied to the infinite time horizon case in [3]; see also [4] and [5]. Under some strict convexity assumptions not compatible with control applications, [3] gave open-loop optimality conditions and a characterization of the gradient of the value function in terms of the Hamiltonian system. The extension in [6] to the control setting gave only local results.

With no reference to duality, open-loop optimality conditions, transversality conditions at infinity, and regularity of the optimal value function and of optimal policies for convex problems have seen treatment in theoretical economics, see [7]–[10]. These works often use barrier functions rather than hard constraints, assume nonnegativity of the state (representing the capital), or place interiority conditions on the control; these are not compatible with $\mathcal{CLQR}$. Problems closely related to $\mathcal{CLQR}$ were analyzed in [11], and [12] (which dealt with nonnegative controls) via some convex analysis tools, but not duality. Most of the mentioned works, and the general open-loop necessary conditions in [13], do not address feedback at all.

When 0 is in the interior of the feasible control set, near the origin $\mathcal{CLQR}$ is just the classical linear quadratic regulator, the theory of which is well-known; see [14]. Relying on the Principle of Optimality, one can then restate $\mathcal{CLQR}$ as a finite time horizon problem with quadratic terminal cost. This is often done in receding horizon control (see [15] and the references therein) and in direct approaches to computation of the optimal feedback, most of which focus on discrete time problems. A distinguishing feature of discrete time is that, for both finite and infinite horizon problems, the value function is piecewise quadratic and the optimal feedback is piecewise affine. This allows for their efficient computation, see [16] and [17]. The recent work of [18], in continuous time but for a finite horizon, explicitly finds the optimal feedback via an offline computation, and observes the differentiability of the value function and continuity and piecewise differentiability of the feedback, based on sensitivity analysis for parametric optimization (see [19]).

In this note, we study $\mathcal{CLQR}$ in the duality framework as proposed by [3] and with the dual control problem as suggested by [2] while also taking advantage, when necessary, of the reduction to a finite time horizon. We choose to work "from scratch" rather than relying on some general duality results for control problems on finite or infinite time horizons (see [5], [20], and [21]) or on parametric optimization. For the most part, we base our work on few results from convex analysis, and a single application of the (finite time horizon) Maximum Principle.

II. Preliminaries and the Dual Problem

The continuous time infinite horizon linear quadratic regulator with control constraints ($\mathcal{CLQR}$) is the following problem: Minimize

$$\frac{1}{2}\int_0^{+\infty} y(t)^T Q y(t) + u(t)^T R u(t) dt \tag{1}$$

subject to linear dynamics

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \ x(0) = \xi \\ y(t) = Cx(t) \end{cases} \tag{2}$$

and a constraint on the input

$$u(t) \in U, \quad \text{for all } t \in [0, +\infty). \tag{3}$$

Here, the state $x : [0, +\infty) \to I\!R^n$ is locally absolutely continuous, $y : [0, +\infty) \to I\!R^m$ is the output, and the minimization is carried out over all locally integrable controls $u : [0, +\infty) \to I\!R^k$. The *optimal value function* $V : I\!R^n \to [0, +\infty]$ is the infimum of (1) subject to (2), (3), parameterized by the initial condition $\xi \in I\!R^n$. Throughout the note, we assume the following.

Assumption 2.1: (Standing Assumption):

i) Matrices Q and R are symmetric and positive definite.
ii) The pair (A, B) is controllable. The pair (A, C) is observable.
iii) The set U is closed, convex, and $0 \in \operatorname{int} U$.

For the unconstrained problem (1), (2), $V(\xi) = (1/2)\xi^T P \xi$, where P is the unique symmetric and positive definite solution of the Riccati equation, and the optimal feedback is linear: given the state x, the optimal control is $-R^{-1}B^T P x$; see [14].

In presence of the constraint (3), V is a positive definite convex function that may have infinite values: $V(\xi) = +\infty$ if no feasible process exists. (We call a pair $x(\cdot), u(\cdot)$ *admissible* if it satisfies (2), (3), and *feasible* if additionally the cost (1) is finite.) There exists a neighborhood N of 0 on which $V(\xi) = (1/2)\xi^T P \xi$; in fact, one can take $N = \{x \in I\!R^n | \xi^T P \xi \le r\}$ where r is small enough so that $-R^{-1}B^T P \xi \in U$ for all $\xi \in N$. (Such r exists since $0 \in \operatorname{int} U$.) In general, $V(\xi) \ge (1/2)\xi^T P \xi$. Any feasible process is such that $x(t) \to 0$ as $t \to +\infty$. Indeed, $V(x(t)) \le \int_t^{+\infty} y(s)^T Q y(s) + u(s)^T R u(s) ds$ by the definition of V, the integral tends to 0 as $t \to \infty$ as the process has finite cost, and thus $x(t)^T P x(t) \to 0$ as $t \to \infty$. The *effective domain* of V, i.e. the set $\operatorname{dom} V = \{\xi \in I\!R^n | V(\xi) < +\infty\}$, is open. Indeed, let $x(\cdot), u(\cdot)$ be any feasible process with $x(0) = \xi$ and let $\bar{t}$ be such that $x(\bar{t}) \in \operatorname{int} N$, where N is the neighborhood mentioned

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above. Then for any ξ' sufficiently close to ξ, the solution $x'(\cdot)$ with $x'(0) = \xi'$ generated by $u(t)$ on $[0, \bar{t}]$ is such that $x'(\bar{t}) \in N$. Using the unconstrained linear control from then on leads to a feasible process for ξ. Lower semicontinuity of V, and the existence of optimal processes for $\mathcal{CLQR}$ can be shown via standard arguments that involve picking weakly convergent subsequences from bounded in $L^2[0, \infty)$ sequences of controls (see [22, Th. 1.3]), ensuring that the limit satisfies the constraints (see the example following [22, Th. 1.6]), and relying on lower semicontinuity of continuous convex functionals with respect to weak convergence, see the Corollary to [22, Th. 1.6]. Finally, since any convex function that is finite on an open set is continuous on that set, V is continuous on dom V.

In defining a dual problem to $\mathcal{CLQR}$, we will use the concept of a convex conjugate function. For a proper, lower semicontinuous and convex $f : I\!R^n \to (-\infty, +\infty]$, its *convex conjugate* $f^* : I\!R^n \to (-\infty, +\infty]$ is defined by

$$f^*(p) = \sup_{x \in I\!R^n} \left\{ p^T x - f(x) \right\}.$$

It is a proper, lower semicontinuous and convex function itself, and $(f^*)^* = f$ (the conjugacy gives a one to one correspondence between convex functions and their conjugates). For example, for $f(x) = (1/2)x^T M x$ for a symmetric and positive definite matrix M, we have $f^*(p) = (1/2)p^T M^{-1} p$. Another example will be provided by the function ρ defined by (6). The standard reference for this, and other convex analysis material we use, is [23].

Following [2] and [3], the dual problem to $\mathcal{CLQR}$ is the following optimal control problem: Minimize

$$\int_0^{+\infty} \rho\left(q(t)\right) + \frac{1}{2} w(t)^T Q^{-1} w(t) dt \tag{4}$$

subject to

$$\begin{cases} \dot{p}(t) = -A^T p(t) - C^T w(t), \ p(0) = \eta \\ q(t) = B^T p(t) \end{cases} \tag{5}$$

where $p : [0, +\infty) \to I\!R^n$ is a locally absolutely continuous arc describing the dual state, $q : [0, +\infty) \to I\!R^k$ is the dual output, and the minimization is carried out over all locally integrable (dual) control functions $w : [0, \infty) \to I\!R^m$. In (4), the function $\rho : I\!R^k \to I\!R$ is the convex conjugate of the function given by $(1/2)u^T R u$ for $u \in U$ and by $+\infty$ for $u \notin U$. That is

$$\rho(q) = \sup_{u \in U} \left\{ q^T u - \frac{1}{2} u^T R u \right\}. \tag{6}$$

This function is finite-valued everywhere, convex, nonnegative, bounded above by $(1/2)q^T R^{-1} q$, and equal to $(1/2)q^T R^{-1} q$ on a neighborhood of 0. It is also differentiable, with $\nabla \rho$ Lipschitz continuous. Furthermore, if U is polyhedral, ρ is piecewise linear-quadratic; see [24, Ex. 11.18].

Example 2.2: (Standard Saturation): In case of standard saturation of single-input systems (where $U = [-1, 1]$) and with $R = 1$, one obtains $\rho(q) = -q - (1/2)$ if $q < -1$, $\rho(q) = (1/2)q^2$ if $-1 \le q \le 1$, and $\rho(q) = q - (1/2)$ if $1 < q$. Then, $\nabla \rho$ is exactly the standard saturation function.

The *optimal value function* $W : I\!R^n \mapsto [0, +\infty)$ for the dual problem is the infimum of (4) subject to (5), parameterized by the initial condition η. W is a positive definite, finite everywhere, and convex (and hence continuous). Also, W is quadratic near 0, optimal processes exist, and for each dual feasible process we have $p(t) \to 0$. (We call a pair $p(\cdot)$, $w(\cdot)$ *dual admissible* if it satisfies (5) and *dual feasible* if additionally, it has finite cost, i.e. (4) is finite.)

Example 2.3: (Duality in the Unconstrained Case): The value function for the unconstrained linear quadratic regulator (1), (2) is given by $V_u(\xi) = (1/2)\xi^T P \xi$, where P is the unique symmetric and positive definite solution of the Riccati equation

$$PA + A^T P - C^T Q C + P B R^{-1} B^T P = 0. \tag{7}$$

This is equivalent to

$$-P^{-1} A^T - A P^{-1} - B R^{-1} B^T + P^{-1} C^T Q C P^{-1} = 0.$$

Just as (7) corresponds to the problem (1), (2), the equivalent version corresponds to a dual linear quadratic regulator (4), (5) with $\rho(q) = (1/2)q^T R^{-1} q$. The function $W_u(\eta) = (1/2)\eta^T P^{-1} \eta$ is the value function for this problem. Indeed, as (5) is stabilizable and detectable, the matrix describing the value function is the unique positive definite solution of the second equation above. In particular, the value functions V_u, W_u are convex functions conjugate to each other.

III. Main Results

To shorten the notation, we will use a subscript to denote time dependence. Instead of $x(t)$ we write x_t, etc. The discussion below, leading up to one of our main results, Theorem 3.1, also shows motivation for considering the dual problem in the form (4), (5).

For all y, w, we have that

$$\frac{1}{2} y^T Q y + \frac{1}{2} w^T Q^{-1} w \ge y^T w \tag{8}$$

and this is an equality if and only if w maximizes $y^T w - (1/2)w^T Q^{-1} w$, equivalently, if $w = Qy$. One observes this by rewriting (8) as $y^T Q y \ge y^T w - (1/2) w^T Q^{-1} w$. Similarly, for all $u \in U$, q

$$\frac{1}{2} u^T R u + \rho(q) \ge u^T q \tag{9}$$

and this holds as an equality if and only if q maximizes $u^T q - \rho(q)$, equivalently, if $u = \nabla \rho(q)$. Furthermore, by the definition (6) of ρ, the equality in (9), rewritten as $\rho(q) = u^T q - (1/2)u^T R u$, shows that u maximizes $u^T q - (1/2)u^T R u$ over the set U. (Cf. [23, Theorem 23.5].)

Consider any admissible process (x_t, u_t) and any dual admissible process (p_t, w_t). Then

$$\begin{aligned} \frac{d}{dt}(x_t^T p_t) &= (A x_t + B u_t)^T p_t + x_t^T(-A^T p_t - C^T w_t) \\ &= u_t^T B^T p_t - (C x_t)^T w_t \end{aligned}.$$

Combining this, (8) with $y = -C x_t$, (9), and the discussion following (9), gives

$$\begin{aligned} \frac{1}{2} x_t^T C^T Q C x_t + \frac{1}{2} u_t^T R u_t + \rho(B^T p_t) + \frac{1}{2} w_t^T Q^{-1} w_t \\ \ge \frac{d}{dt}\left(x_t^T p_t\right). \end{aligned} \tag{10}$$

Now, (10) turns into an equality if and only if $u_t = \nabla \rho(B^T p_t)$, which is equivalent to

$$u_t \text{ maximizes } u^T B^T p_t - \frac{1}{2} u^T R u \text{ over } u \in U \tag{11}$$

and if $w_t = -Q C x_t$, which is equivalent to

$$w_t \text{ maximizes } -w^T C x_t - \frac{1}{2} w^T Q^{-1} w. \tag{12}$$

Integrating (10) yields: for any feasible (x_t, u_t) and dual feasible (p_t, w_t)

$$\int_0^\tau \frac{1}{2}x_t^T C^T QCx_t + \frac{1}{2}u_t^T Ru_t dt + \int_0^\tau \rho(B^T p_t) + l\frac{1}{2}w_t^T Q^{-1} w_t dt \geq x_\tau^T p_\tau - \xi^T \eta \quad (13)$$

and this holds as an equality if and only if (x_t, u_t) and (p_t, w_t) satisfy (11) and (12) on $[0, \tau]$.

Before stating the open-loop optimality conditions, we need to introduce the following objects. The (maximized) Hamiltonian associated with $\mathcal{CLQR}$ is

$$H(x,p) = p^T Ax - \frac{1}{2}x^T C^T QCx + \rho(B^T p). \quad (14)$$

The Hamiltonian differential system $\dot{x}_t = \nabla_p H(x_t, p_t)$, $\dot{p}_t = -\nabla_x H(x_t, p_t)$ takes the form

$$\dot{x}_t = Ax_t + B\nabla\rho(B^T p_t) \quad \dot{p}_t = -A^T p_t + C^T QCx_t. \quad (15)$$

Theorem 3.1: (Open Loop Optimality):

a) If a feasible process $(\bar{x}_t, \bar{u}_t)$ is optimal for $\mathcal{CLQR}$ then there exists an arc $\bar{p}_t$ such that (11) and (15) hold and $\bar{p}_t \to 0$ as $t \to \infty$. On the other hand, if for some admissible process $(\bar{x}_t, \bar{u}_t)$, there exists an arc $\bar{p}_t$ such that (11), (15), and $\lim_{t\to\infty} \bar{x}_t^T \bar{p}_t = 0$ hold, then $(\bar{x}_t, \bar{u}_t)$ is optimal for $\mathcal{CLQR}$.

b) If a dual feasible process $(\bar{p}_t, \bar{w}_t)$ is optimal for the dual problem (4), (5) then there exists an arc $\bar{x}_t$ such that (12) and (15) hold and $\bar{x}_t \to 0$ as $t \to \infty$. On the other hand, if for some dual admissible process $(\bar{p}_t, \bar{w}_t)$, there exists an arc $\bar{x}_t$ such that (12), (15) hold and $\lim_{t\to\infty} \bar{x}_t^T \bar{p}_t = 0$, then $(\bar{p}_t, \bar{w}_t)$ is optimal for the dual problem (4), (5).

Proof: We show a), the proof of b) is similar. We show sufficiency first. Given an arc $\bar{p}_t$ as assumed, let $\bar{w}_t = QC\bar{x}_t$, and let $\tau \to \infty$ in (13) (noting that (13) holds as an equality for $(\bar{x}_t, \bar{u}_t), (\bar{p}_t, \bar{w}_t)$) to obtain

$$\int_0^\infty \frac{1}{2}\bar{x}_t^T C^T QC\bar{x}_t + \frac{1}{2}\bar{u}_t^T R\bar{u}_t dt + \int_0^\infty \rho(B^T \bar{p}_t) + \frac{1}{2}\bar{w}_t^T Q^{-1} \bar{w}_t dt = -\xi^T \eta. \quad (16)$$

In particular, this implies that $(\bar{x}_t, \bar{u}_t)$ has finite cost. On the other hand, for any other feasible process (x_t, u_t) (and, hence, such that $x_t \to 0$ as $t \to \infty$) we also have $x_t^T \bar{p}_t \to 0$, and then letting $\tau \to \infty$ in (13) yields

$$\int_0^\infty \frac{1}{2}x_t^T C^T QCx_t + \frac{1}{2}u_t^T Ru_t dt + \int_0^\infty \rho(B^T \bar{p}_t) + \frac{1}{2}\bar{w}_t^T Q^{-1} \bar{w}_t dt \geq -\xi^T \eta. \quad (17)$$

This combined with (16) implies that $(\bar{x}_t, \bar{u}_t)$ is optimal.

Necessity is shown via reduction of the infinite horizon problem to a finite horizon formulation. For an optimal process $(\bar{x}_t, \bar{u}_t)$, and so a process with a finite cost, we know that $\bar{x}_t \to 0$ as $t \to \infty$. Thus, there exists $\tau \geq 0$ such that for all $t \geq \tau$, $\bar{x}_t$ is in the neighborhood of the origin on which $V(\xi) = (1/2)\xi^T P\xi$, $\bar{u}_t = -R^{-1}B^T P\bar{x}_t \in U$, and $\nabla\rho(-B^T P\bar{x}_t) = -R^{-1}B^T P\bar{x}_t$. Optimality of $(\bar{x}_t, \bar{u}_t)$ dictates that $(\bar{x}_t, \bar{u}_t)$ restricted to $[0, \tau]$ minimizes

$$\int_0^\tau \frac{1}{2}x_t^T C^T QCx_t + \frac{1}{2}u_t^T Ru_t dt + \frac{1}{2}x_\tau^T Px_\tau$$

where P is the solution to the Riccati equation for the unconstrained regulator. The Maximum Principle (see, for example, [25, Th. 44]) yields the existence of $\bar{p}_t$ on $[0, \tau]$ such that (11) and (15) hold on this time interval, and $\bar{p}_\tau = -P\bar{x}_\tau$. On $[\tau, \infty)$, $(\bar{x}_t, \bar{u}_t)$ is optimal for the unconstrained problem, and $\bar{p}_t$ can be extended to $[0, \infty)$ by setting $\bar{p}_t = -P\bar{x}_t$ [on $[\tau, \infty)$, the second equation in (15) is a consequence of the Riccati equation (7)]. ■

The adjoint arc $\bar{p}_t$, verifying optimality of $(\bar{x}_t, \bar{u}_t)$ for $\mathcal{CLQR}$ is optimal (together with $\bar{w}_t = QC\bar{x}_t$) for the dual problem (and its optimality can be verified by x_t). We add that if one knows that a candidate for a minimum in $\mathcal{CLQR}$, say (x_t, u_t), is such that x_t is bounded (which is the case if (x_t, u_t) has finite cost), then the existence of $\bar{p}_t$ as described in the necessary conditions in a) of Theorem 3.1 is also sufficient for optimality. Indeed, if x_t is bounded, then $\bar{p}_t \to 0$ implies $x_t^T \bar{p}_t \to 0$ as $t \to \infty$. We now show that $W(\eta) = V^*(-\eta)$, or equivalently, $V(\xi) = W^*(-\xi)$.

Theorem 3.2: (Value Function Conjugacy): The (equivalent to each other) formulas hold:

$$W(\eta) = \sup_{x\in\mathbb{R}^n} \left\{-\eta^T x - V(x)\right\}$$
$$V(\xi) = \sup_{p\in\mathbb{R}^n} \left\{-\xi^T p - W(p)\right\}. \quad (18)$$

The first supremum is attained for every η, the second is attained for every $\xi \in \operatorname{dom} V$.

Proof: The formulas are equivalent by [23, Th. 12.2]. Inequality (13) implies that for all ξ, η, $V(\xi) + W(\eta) \geq -\xi^T \eta$. For a given η, let $(\bar{p}_t, \bar{w}_t)$ be the optimal process for the dual problem, $\bar{x}_t$ an adjoint arc as guaranteed by Theorem 3.1 b), finally define $\bar{u}_t$ via (11). For such processes, (13) turns into an equality and together leads to (16). This, together with (17), implies that $(\bar{x}_t, \bar{u}_t)$ is optimal for $\mathcal{CLQR}$ with the initial condition ξ, and (16) turns to $V(\xi) + W(\eta) = -\xi^T \eta$. Combined with $V(\xi) + W(\eta) \geq -\xi^T \eta$, this yields $W(\eta) = \max_{x\in\mathbb{R}^n}\{-\eta \cdot x - V(x)\}$. Symmetric arguments show that the second supremum is attained when $V(\xi) < \infty$. ■

Lemma 3.3: (Strict Convexity): V is strictly convex on $\operatorname{dom} V$. W is strictly convex.

Proof: We only show the statement for W. Strict convexity of a function f is the property that $(1-\lambda)f(z') + \lambda f(z'') > f((1-\lambda)z' + \lambda z'')$ unless $z' = z''$. (This is present, for example, for any positive definite quadratic function.) Pick any η', η'' and let (p_t', w_t'), $(p_t'', w_t' \prime)$ be respective optimal processes. If $w_t' \neq= w_t''$, since Q^{-1} is positive definite and ρ is convex, we have

$$(1-\lambda)W(\eta') + \lambda W(\eta'') > W\left((1-\lambda)\eta' + \lambda\eta''\right). \quad (19)$$

Previously, we used the fact that a convex combination of optimal processes for η', η'' is feasible for $(1-\lambda)\eta' + \lambda\eta''$. Similarly, (19) occurs if $-B^T p_t' \neq -B^T p_t''$ for large t (since close to the origin, ρ is a positive definite quadratic). However, if $B^T p_t' = B^T p_t''$ for all large enough t and $w_t' = w_t''$, then observability of $(-A^T, B^T)$ implies that $p_0' = p_0''$. Thus W is strictly convex. ■

The equivalence of strict convexity of a convex function and of (appropriately understood) differentiability of its conjugate, see [23, Th. 26.3], shows the following corollary. (Continuous differentiability is automatic for differentiable convex functions; see [23, Th. 25.5].)

Corollary 3.4: (Differentiability of Value Functions): V is continuously differentiable at every point of $\mathrm{dom}\, V$ and $\|\nabla V(x_i)\| \to +\infty$ for any sequence of points $x_i \in \mathrm{dom}\, V$ converging to a point not in $\mathrm{dom}\, V$. W is continuously differentiable.

Corollary 3.5: (Hamiltonian System): The following are equivalent.

a) $\eta = -\nabla V(\xi)$.

b) $\xi = -\nabla W(\eta)$.

c) There exist arcs x_t, p_t on $[0, +\infty)$, from ξ, η, such that (15) holds and $(x_t, p_t) \to (0, 0)$.

Proof: Equivalence of a) and (b) is a general property of convex functions. Also, either a) or b) is equivalent to $V(\xi) + W(\eta) = -\xi^T\eta$, and implies that $V(\xi)$ is finite. Then optimal processes (x_t, u_t), (p_t, w_t) lead to an equality in (13) what implies that x_t, p_t satisfy (15) (and each converges to 0). On the other hand, if c) holds then (13) turns to an equality with u_t given by (11) and w_t given by (12). But this implies that (x_t, u_t), (p_t, w_t) are optimal for, respectively, $\mathcal{CLQR}$ and the dual. Then, (13) turns to $V(\xi) + W(\eta) = -\xi^T\eta$, what implies a) and b). ■

Suppose x_t and p_t satisfy (15). Then, $(d/dt)H(x_t, p_t) = 0$, were H is the Hamiltonian (14). (The equality can be verified directly.) If $\eta = -\nabla V(\xi)$ then $(x_t, p_t) \to (0, 0)$. Since $H(0, 0) = 0$ and H is continuous, it must be that $H(x_t, p_t) = 0$ for all t. In light of Corollary 3.5, we obtain the following.

Corollary 3.6: (Hamilton-Jacobi Equations): For all $x \in \mathrm{dom}\, V$, $H(x, -\nabla V(x)) = 0$. For all $p \in \mathbb{R}^n$, $H(-\nabla W(p), p) = 0$.

In fact, V and W are the unique convex functions solving the Hamilton-Jacobi equations above; see [5]. The said equations make it easy to find V for one-dimensional problems.

Example 3.7: (Lack of Piecewise Quadratic Structure): Consider minimizing $(1/2)\int_0^\infty x^2(t) + u^2(t)dt$ subject to $\dot{x}(t) = u(t)$ and $u(t) \in [-1, 1]$. The Hamiltonian is $H(x, p) = -(1/2)x^2 + \bar{\rho}(p)$, where $\bar{\rho}$ is the function described in Example 2.2. Since by convexity of V, ∇V is a nondecreasing function, we obtain that $V(x)$ equals $-(1/2)(x^2 + 1)$ if $x < -1$, x if $-1 \le x \le 1$, and $(1/2)(x^2 + 1)$ if $x > 1$. Note that ∇V—and not V—is piecewise quadratic.

Theorem 3.8: (Feedback Optimality):

a) The process $(\bar{x}_t, \bar{u}_t)$ is optimal for $\mathcal{CLQR}$ if and only if $\bar{x}_0 = \xi$, $\dot{\bar{x}}_t = A\bar{x}_t + B\bar{u}_t$ and $\bar{u}_t$ maximizes $-u^T B^T \nabla V(\bar{x}_t) - (1/2)u^T Ru$ over all $u \in U$.

b) The process $(\bar{p}_t, \bar{w}_t)$ is optimal for the dual problem (4), (5) if and only if $\bar{p}_0 = \eta$, $\dot{\bar{p}}_t = -A^T\bar{p}_t - C^T\bar{w}_t$ and $\bar{w}_t$ maximizes $-w^T C\nabla W(p_t) - (1/2)w^T Q^{-1}w$ over all $w \in \mathbb{R}^n$.

The maximum conditions can be written as $\bar{u}_t = \nabla\rho(-B^T\nabla V(\bar{x}_t))$ and $\bar{w}_t = -QC\nabla W(\bar{p}_t)$.

Proof: If $(\bar{x}_t, \bar{u}_t)$ is optimal, then by Theorem 3.1 there exists $\bar{p}_t$ such that $\bar{x}_t, \bar{p}_t$ satisfy (15) and $\bar{p}_t \to 0$. Since by optimality, $\bar{x}_t \to 0$, Corollary 3.5 implies that $\bar{p}_0 = -\nabla V(\bar{x}_0)$. But the existence of $\bar{x}_t$, $\bar{p}_t$ as described also implies that there exist a convergent to (0,0) solution to (15) from the point $(\bar{x}_\tau, \bar{p}_\tau)$, for any $\tau \ge 0$. (Indeed, one just considers the truncation of arcs $\bar{x}_t$ and $\bar{p}_t$ to $[\tau, \infty)$.) Then, Corollary 3.5 yields $\bar{p}_\tau = -\nabla V(\bar{x}_\tau)$. This and (11) show that the desired formula for $\bar{u}_t$ holds. Now, suppose $\bar{u}_t$ maximizes $-u^T B^T \nabla V(\bar{x}_t) - (1/2)u^T Ru$, equivalently, $\bar{u}_t = \nabla\rho(-B^T\nabla V(\bar{x}_t))$. Near any point where $V(\bar{x}_t)$ is finite, ∇V is bounded. As $\nabla\rho$ is continuous, $\bar{u}_t$ is locally bounded and, hence, $\bar{x}_t$ is locally Lipschitz. By convexity, V is also locally Lipschitz (where finite), and thus $t \mapsto V(x(t))$ is locally Lipschitz. Consequently, $(d/dt)V(\bar{x}_t) = \nabla V(\bar{x}_t)^T\dot{\bar{x}}_t$ almost everywhere, which, by the first Hamilton–Jacobi equation in Corollary 3.6, becomes

$$\frac{d}{dt}V(\bar{x}_t) = -\frac{1}{2}\bar{x}_t C^T QC\bar{x}_t - \frac{1}{2}\bar{u}_t^T R\bar{u}_t. \tag{20}$$

Thus, $\bar{x}_t \to 0$ as $t \to \infty$. Integrating yields $V(\xi) = V(\bar{x}_\tau) + (1/2)\int_0^\tau \bar{x}_t^T C^T QC\bar{x}_t + \bar{u}_t^T R\bar{u}_t dt$. Letting $\tau \to \infty$ and noting that $V(\bar{x}_\tau) \to 0$, implies optimality. Proof of b) is similar. ■

Most of the results stated so far hold for more general convex optimal control problems (see [5] and [26]) but there they require a less direct approach. Here, further use of the quadratic structure of V near 0 and a result of [21] lead to stronger regularity of V.

Theorem 3.9: (Locally Lipschitz Gradients): The mappings ∇V, respectively, ∇W, are locally Lipschitz continuous on $\mathrm{dom}\, V$, respectively, on $\mathbb{R}^n$.

Proof: We show the statement for ∇V. Pick a compact subset $K \subset \mathrm{dom}\, V$. Convergence to 0 of optimal arcs for $\mathcal{CLQR}$ is uniform from compact sets and thus there exists t_0 such that, for all $t > t_0$ and any $\xi \in K$, the optimal process (x_t, u_t) for $V(\xi)$ satisfies $u_t \in \mathrm{int}\, U$. Thus, on K, V is the minimum of $\int_0^{t_0}(1/2)x_t^T C^T QCx_t + (1/2)u_t^T Ru_t dt + (1/2)x_{t_0}^T Px_{t_0}$. The terminal cost and the Hamiltonian (14) of this problem are differentiable with globally Lipschitz gradients. Now, [21, Th. 3.3] states that so is the value function of this problem (the Lipschitz constant may depend on t_0). But this value function, on K, equals V. ■

A stronger statement can be made about ∇W. First, as a continuous time counterpart of a discrete time result [27, Lemma 3], one can show that the function $f : \mathbb{R}^n \mapsto \overline{\mathbb{R}}$ given by $f(\xi) = V(\xi) - V_u(\xi)$, where $V_u(\xi) = (1/2)\xi^T P\xi$ is the solution to the unconstrained linear-quadratic regulator (recall Example 2.3), is positive definite and convex. Thus, $V(\xi) = f(\xi) + 12\xi^T P\xi$ and

$$W(\eta) = \inf_{s \in \mathbb{R}^n}\left\{f^*(s) + \frac{1}{2}(\eta - s)^T P^{-1}(\eta - s)\right\}$$

see [24, Th. 11.23(a), Prop. 12.60]. In particular, ∇W is Lipschitz continuous with the constant K, where K^{-1} is the smallest eigenvalue of P.

IV. Computation

Theorem 3.8 and Corollary 3.5 suggest a procedure for computing the optimal feedback for $\mathcal{CLQR}$. Given the state x, the optimal control is $\nabla\rho(B^T p)$, where p is such that there exist a solution x_t, p_t to (15), originating at (x, p) and converging to (0,0). As near (0,0) we have $p = -Px$, integrating (15) backwards from points of the form $(x, -Px)$ leads to values of the adjoint arc p corresponding to any state in $\mathrm{dom}\, V$. Thus, the idea is as follows.

1) Find the matrix P by solving the Riccati equation (7), and the corresponding optimal feedback matrix for the unconstrained problem $F_u = -R^{-1}B^T P$.
2) Find a neighborhood N of 0 so that for all $\xi \in N$ one has $F_u\xi \in U$ and such that N is invariant under $\dot{x} = (A + BF_u)x$.
3) For each x on the boundary of N, find the solution of the backward Hamiltonian system

$$\begin{aligned}\dot{x}(t) &= -Ax(t) - B\nabla\rho\left(B^T p(t)\right)\\ \dot{p}(t) &= A^T p(t) - C^T QCx(t)\end{aligned} \tag{21}$$

on $[0, +\infty)$, originating from $(x, -Px)$.

Some comments are in order. On the set N of step 2), the optimal feedback for $\mathcal{CLQR}$ is the linear feedback F_u. For single-input systems, if $U = [-1, 1]$ and $R = 1$, one can choose $N = \{x | x^T Px \le (B^T PB)^{-1}\}$. In fact, this set is the largest ellipse given by P that meets the condition that $F_u x \in U$ for all $x \in N$. The (backwards) Hamiltonian system (21) involves $\nabla\rho$. As ρ is given by (6), $\nabla\rho$ can be found without computing ρ itself (see [24, Ex. 11.18])

$$\nabla\rho(q) = \arg\max_u\left\{q^T u - \frac{1}{2}u^T Ru | u \in U\right\}. \tag{22}$$

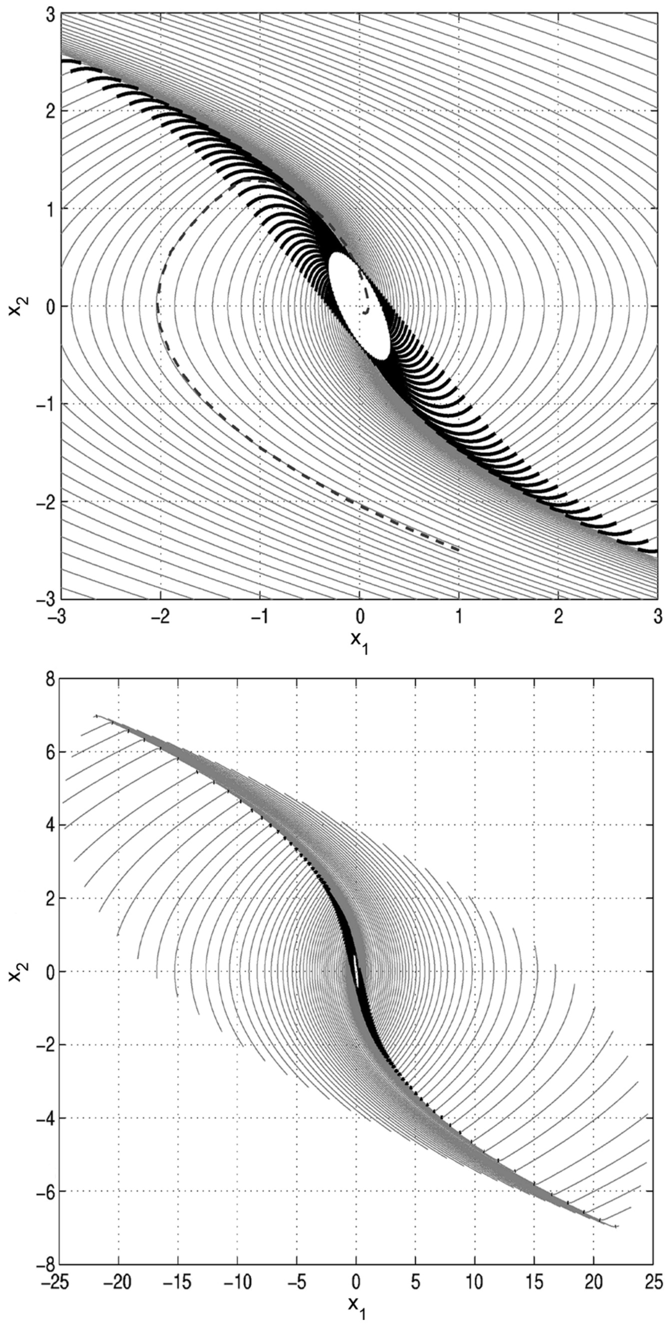

Fig. 1. Double integrator: Optimal trajectories.

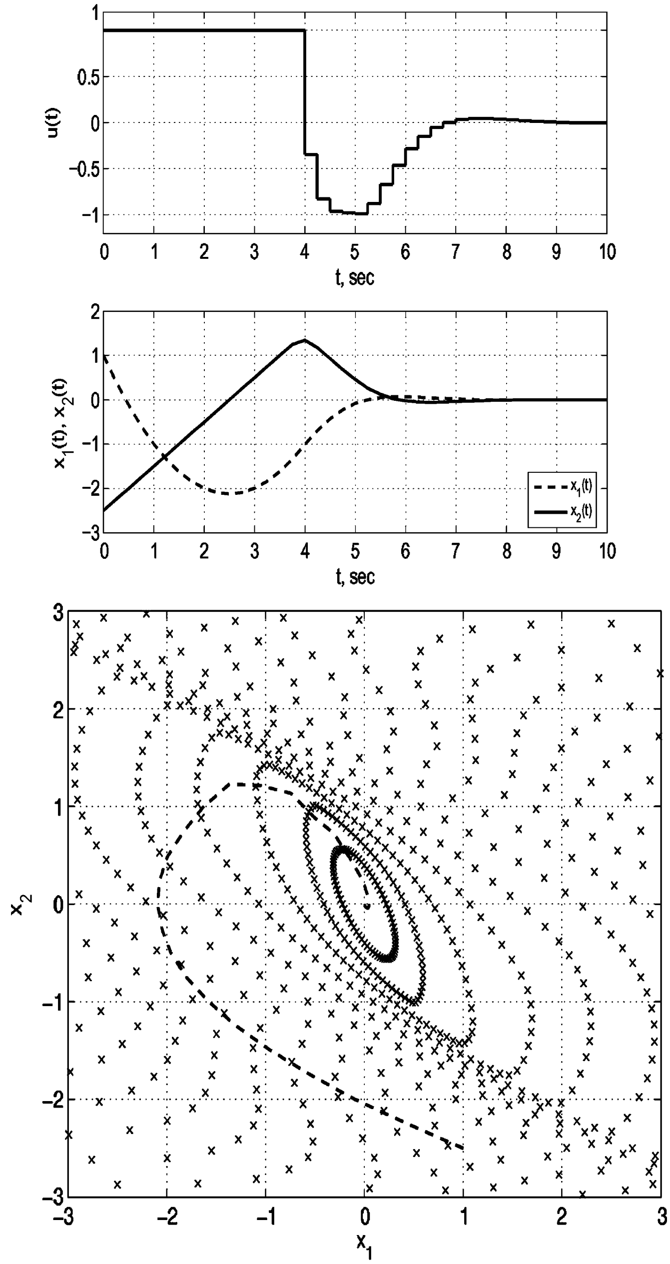

Fig. 2. Double integrator: Response to large grid and the small grid.

This formula simplifies in important special cases. For single input systems and the standard saturation (that is, $U = [-1, 1]$), and when $R = 1$, $\nabla\rho$ is the standard saturation function. That is, $\nabla\rho(q) = \sigma(q) = -1$ if $q < -1$, q if $-1 \leq q \leq 1$, and 1 if $1 < q$. A similar formula holds whenever $R = r$ is a scalar and U is a closed interval $[u_-, u_+]$; then $\nabla\rho_{r,U}(q)$ equals $r^{-1}u_-$ if $q < u_-$, $r^{-1}q$ if $u_- \leq q \leq u_+$, $r^{-1}u_+$ if $u_+ < q$. For multiple input cases, when $R = \mathrm{diag}\{r_1, r_2, \ldots, r_k\}$ is diagonal and $U = U_1 \times U_2 \times \ldots \times U_k$ is a product of intervals, we have $\rho_{R,U}(q) = \rho_{r_1,U_1}(q_1) + \rho_{r_2,U_2}(q_2) + \ldots + \rho_{r_k,U_k}(q_k)$. Then, $\nabla\rho_{R,U}$ can be found coordinatewise.

Finally, as the optimal feedback for $\mathcal{CLQR}$ is continuous (and V is a smooth Lyapunov function), any sufficiently good approximation of the optimal feedback, that can be obtained via implementation of the procedure outlined above, is also stabilizing. This reflects the general principles on the robustness of stability; see [28, Ch. 9.1] and [29, Prop. 3.1].

V. Numerical Examples

Example 5.1: (Double Integrator): We consider $Q = 1$, $R = 0.1$, $u(t) \in [-1, 1]$, and

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad y(t) = [1 \quad 0]x(t)$$

as motivated by the computation in [30]. We calculate the feedback using the algorithm in Section IV. Solving the Riccati equation (7) yields $P = \begin{bmatrix} 0.795 & 0.316 \\ 0.316 & 0.256 \end{bmatrix}$. Initial points $x_i(0)$, $i = 1, 2, \ldots, 72$ are chosen on the boundary of the invariant ellipse $N = \{x \in I\!R^2 | x^T P x \leq 0.0398\}$. Solutions to (21), starting at $(x_i(0), -Px_i(0))$ are calculated on $[0, T]$ with $T = 10$ s. First,

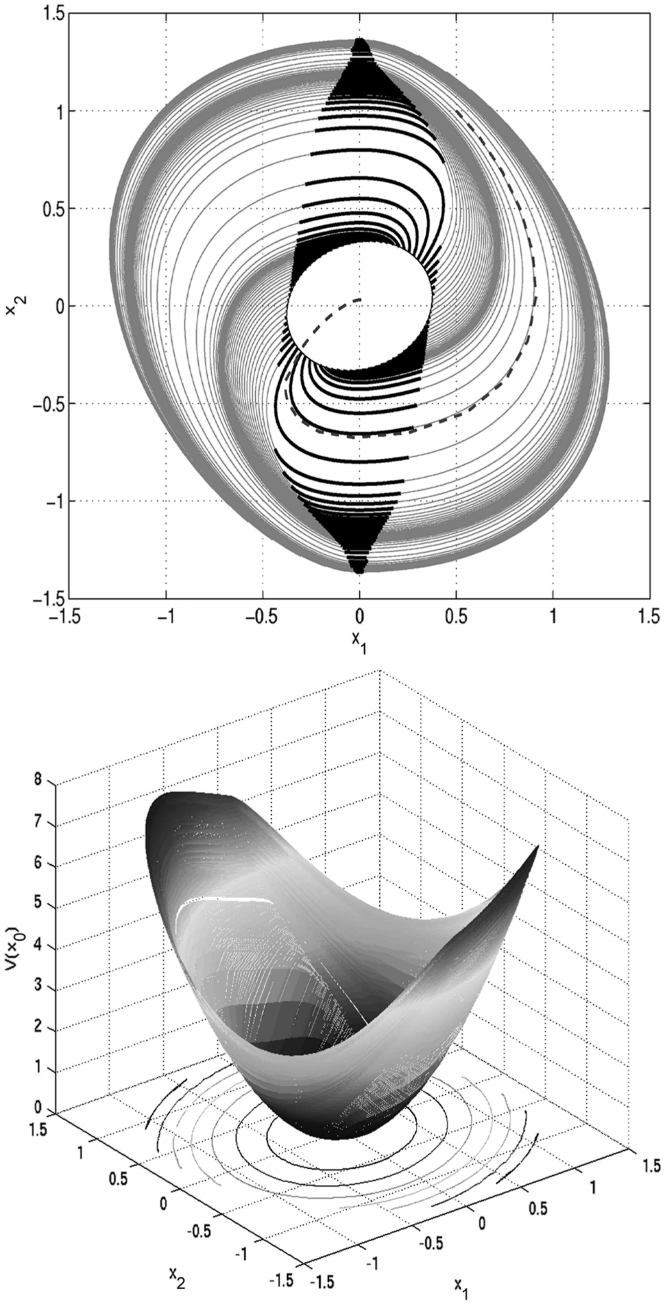

Fig. 3. Unstable system: Optimal trajectories and optimal value function.

we use $\Delta T = 0.005$ s as the sampling period, and store the points $(x_i(j\Delta T), p_i(j\Delta T))$ for $i = 1, 2, \ldots, 72$, $j = 0, 1, \ldots, 2000$. The corresponding trajectories $x_i(t)$ are shown in Fig. 1, first showing them on $[-3, 3] \times [-3, 3]$ (the darker shade indicates the "strip" in the plane where the control is not saturated), then showing the whole region the trajectories fill out. Fig. 1 also shows the trajectory starting at $x(0) = (1, -2.5)$ for the closed-loop system.

Given the stored grid and a state $x \notin N$, the control $u(x)$ is found as $\sigma(R^{-1}B^T p_i(j\Delta T))$, where $(x_i(j\Delta T), p_i(j\Delta T))$ is the grid point with $x_i(j\Delta T)$ closest to x. For $x \in N$, linear feedback $-R^{-1}B^T Px$ is used. The response of the system, from $x(0) = (1, -2.5)$, and the corresponding control sequence (sample time is $\tau = 0.25$ s) is in Fig. 2. The response is essentially the same as in [30]. To (significantly) reduce the number of stored points, we repeated the computation on the same interval [0, 10] s, but with $\Delta T = 0.5$ sec. and $j = 0, 1, \ldots, 20$. The sparser grid is shown on Fig. 2, together with the response for $x(0) = (1, -2.5)$. (The response, for this and other initial points, is very similar to that resulting from the denser grid.)

Example 5.2: (Unstable System): Consider the system

$$\dot{x}(t) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \quad y(t) = x(t) \tag{23}$$

with $Q = I$, $R = 1$, and $u(t) \in [-1, 1]$. As A is not semi-stable, $dom\ V$ is not the whole plane: there exist initial conditions that can not be driven to 0 with a constrained control. Thus, no matter how large T is chosen, the x-trajectories of (21) will not fill out arbitrarily chosen compact sets. In Fig. 3 we show the trajectories obtained with $T = 8$ s, $\Delta T = 0.005$ s, $i = 1, 2, \ldots, 72$ and $j = 0, 1, \ldots, 1600$. Fig. 3 also shows the closed-loop system trajectory starting at $x(0) = (0.5, 1)$. We also calculated the approximate values of V at the grid points. This is possible via the formula (20) for the time derivative of V along optimal trajectories: The third step of the algorithm is altered, by solving the following equation (which results from reversing time in (20), and substituting $\rho(B^T p(t))$ for the optimal control):

$$\frac{d}{dt}V(x(t)) = \frac{1}{2}x(t)C^T QCx(t) + \frac{1}{2}\rho\Big(B^T p(t)\Big)^T R\rho\Big(B^T p(t)\Big)$$

along with the backward Hamiltonian system (21) and storing the values of $V(x(t))$ along those of $(x(t), p(t))$. The initial points are taken to be $(x_i(0), -Px_i(0), (1/2)x_i(0)^T Px_i(0))$.

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A Lyapunov Proof of an Improved Maximum Allowable Transfer Interval for Networked Control Systems

Daniele Carnevale, Andrew R. Teel, and Dragan Nešić

***Abstract*—Simple Lyapunov proofs are given for an improved (relative to previous results that have appeared in the literature) bound on the maximum allowable transfer interval to guarantee global asymptotic or exponential stability in networked control systems and also for semiglobal practical asymptotic stability with respect to the length of the maximum allowable transfer interval.**

***Index Terms*—Lyapunov, networked control, nonlinear, stability.**

I. Introduction

A networked control system (NCS) is composed of multiple feedback control loops that share a serial communication channel. This architecture promotes ease of maintenance, greater flexibility, and low cost, weight and volume. On the other hand, if the communication is substantially delayed or infrequent, the architecture can degrade the overall system performance significantly. Results on the analysis of an NCS include [1]–[5]. In an NCS, the delay and frequency of communication between sensors and actuators in a given loop is determined by a combination of the channel's limitations and the transmission protocol used. Various protocols have been proposed in the literature, including the "round robin" (RR) and "try-once-discard" (TOD) protocols discussed in [1] and [2]. When the individual loops in an NCS are designed assuming perfect communication, the stability of the NCS is largely determined by the transmission protocol used and by the so-called "maximum allowable transfer interval" (MATI), i.e., the maximum allowable time between any two transmissions in the network. Following [1] and [2], we consider the problem of characterizing the length of the MATI for a given protocol to ensure uniform global asymptotic or exponential stability.

In [4], the authors were able to improve on the initial MATI bounds given in [1] and [2] by efficiently summarizing the properties of protocols through Lyapunov functions and characterizing the effect of transmission errors through $\mathcal{L}_p$ gains. They established uniform asymptotic or exponential stability and input–output stability when the $\mathrm{MATI} \in [0, \tau_{\mathrm{MATI}}]$ with

$$\tau_{\mathrm{MATI}} \leq \frac{1}{L} \ln \left(1 + \frac{1-\lambda}{\frac{\gamma}{L} + \lambda}\right) \tag{1}$$

where $\lambda \in [0, 1)$ characterized the contraction of the protocol's Lyapunov function at transmission times while $L > 0$ described its expansion between transmission times, and $\gamma > 0$ captured the effect

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