Solutions to hybrid inclusions via set and graphical convergence with stability theory applications

R. Goebel and A.R. Teel

Center for Control, Dynamical Systems and Computation
Department of Electrical and Computer Engineering
University of California, Santa Barbara 93106-9560

Abstract

Motivated by questions in stability theory for hybrid dynamical systems, we establish some fundamental properties of the set of solutions to such systems. Using the notion of a hybrid time domain and general results on set and graphical convergence, we establish under weak regularity and local boundedness assumptions that the set of solutions is sequentially compact and "upper semicontinuous" with respect to initial conditions and system perturbations. The general facts are then used to establish several results for the behavior of hybrid systems that have asymptotically stable compact sets. These results parallel what is already known for differential inclusions and difference inclusions. For example, the basin of attraction for a compact attractor is (relatively) open, the attractivity is uniform from compact subsets of the basin of attraction, and asymptotic stability is robust with respect to small perturbations.

Key words: Hybrid dynamical systems, differential and difference inclusions, robust stability, graphical convergence.

1 Introduction

The development of effective nonlinear control algorithms requires a clear understanding of stability and its robustness in nonlinear systems. For differential equations, this theory is well established and nicely summarized in the textbook Khalil (2002), for example. For discontinuous and/or switching systems, the theory is more recent and not yet complete.

Over the last decade, important pieces in the puzzle of stability theory for differential inclusions have been inserted. With the early work of Kurzweil (1956) for continuous differential equations as its precursor, Clarke, Ledyaev, and Stern (1998) established the existence of smooth Lyapunov functions for asymptotically stable differential inclusions. In the process, they showed that asymptotic stability for inclusions is a robust property. (This result was related to robust stabilization in Ledyaev and Sontag (1999). Additional results related to those in Clarke, Ledyaev, and Stern (1998) are in Bacciotti and Rosier (2001) and Teel and Praly (2000). For discrete-time "difference inclusions" see Kellett and Teel (2004a).) Around the same time, Ryan (1998) established a general invariance principle for differential inclusions, extending the seminal work of Krasovskii (1963) and LaSalle and Lefschetz (1961); LaSalle (1967). In addition, Artstein and Vigodner (1996); Artstein (1999) provided novel singular perturbation and averaging results. See also Teel, Moreau, and Nesic (2003).

The purpose of this paper is to provide some of the tools that will allow the mentioned results to be extended to hybrid systems: systems where the state flows according to a differential equation or inclusion and also jumps according to a difference equation or inclusion. (Control engineering motivation for considering such systems will be given in Section 2.) In differential inclusions theory, the main facts from which (robust) stability results follow are that sets of solutions are 1) sequentially compact under mild growth conditions (in particular, the limit of solutions is a solution), and 2) "upper semicontinuous" with respect to initial conditions and system perturbations (meaning that every perturbed solution is close, in an appropriate sense, to some unperturbed solution).

⋆ Current address for R. Goebel is 3518 NE 42 Street, Seattle, WA 98105.

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One of the first obstructions to address when considering such results for hybrid systems is the fact that the ordinary time domain may be insufficient to describe the evolution of the state of a hybrid system. There have been several characterizations of potentially suitable hybrid time domains given in the literature. We point to Tavernini (1987), Michel and Hu (1999), Lygeros, Johansson, Sastry, and Egerstedt (1999), and van der Schaft and Schumacher (2000) for particular examples. Most recently, the concurrent conference papers by Collins (2004) and the authors et al. Goebel, Hespanha, Teel, Cai, and Sanfelice (2004) have proposed treating the number of jumps as an independent variable and parameterizing the state of a hybrid system by (t, j) – that is, $x(t, j)$ is the state at time t and after j jumps. In Goebel, Hespanha, Teel, Cai, and Sanfelice (2004), the motivation for such parameterization was that it allows for the use of graphical convergence concepts to solutions of hybrid systems. Such convergence, and other tools of set-valued analysis, are well-developed; see Rockafellar and Wets (1998), or Aubin and Cellina (1984) for applications to differential inclusions theory. The need to use such nonclassical analysis tools is quite strong in hybrid systems, as, for example, the standard concepts like uniform convergence are not well-suited to handle discontinuous solutions.

Earlier results on the continuity of hybrid solutions with respect to initial conditions include those by Tavernini (1987), Broucke and Arapostathis (2002), and Lygeros, Johansson, Simić, Zhang, and Sastry (2003). These give, respectively: continuity near "regular states" under strong continuity properties of the data; existence of continuous selections from sets of solutions when Zeno behaviors are excluded; and continuity of solutions under a uniqueness assumption. In related work, Johansson, Egerstedt, Lygeros, and Sastry (1999) examine the limit of hybrid solutions as certain regularizing parameters converge to zero, while Chellaboina, Bhat, and Haddad (2003) give conditions for "quasi-continuous dependence" in state-dependent impulsive dynamical systems. See also Aubin, Lygeros, Quincampoix, Sastry, and Seube (2002). Additionally, the work of Collins (2004), which is restricted to hybrid systems with a compact state space, contains a statement about the upper semicontinuity of a map from initial conditions to jump values that are possible after a given number of jumps. In our work, we have no uniqueness assumptions, permit Zeno behaviors, and allow a noncompact state space. The regularity assumptions in our work, here and in Goebel, Hespanha, Teel, Cai, and Sanfelice (2004), extend those in Collins (2004) beyond compact state spaces, and appear to be the weakest possible for the results reported here. For example, from the differential inclusions describing the continuous evolution of the hybrid system we do not require more than what is needed for upper semicontinuity of solutions when no discrete behaviors are present; similarly for the difference inclusions describing the jumps.

In the results that follow, we establish that the solution set for hybrid inclusions satisfying basic conditions is sequentially compact (Theorem 4.4) and upper semicontinuous (Corollaries 4.8 and 5.5). As applications, we prove under a mild existence assumption that the basin of attraction for a compact attractor in a hybrid system is relatively open, that the convergence to the attractor is uniform over compact subsets of the basin of attraction, and that asymptotic stability is semiglobally practically robust with respect to perturbations. Further results, on a general LaSalle-like invariance principle and the construction of smooth Lyapunov functions for asymptotically stable hybrid systems, drawing upon the foundation established here, can be found in Sanfelice, Goebel, and Teel (2005) and Cai, Teel, and Goebel (2005).

The paper is organized as follows. Hybrid inclusions and the solution concept are described in Section 2. Also there, some control engineering justification for considering hybrid inclusions is presented. Prerequisites from set-valued analysis are in Section 3. In Section 4, graphical convergence is used to study semicontinuity and closeness properties of sets of solutions to hybrid inclusions. Section 5 extends some of these results to allow perturbations. The main results in these two sections are Theorem 4.4, which address sequential compactness of the sets of solutions to a nominal hybrid system, and Theorem 5.1, which describes the limits of solutions to a hybrid system with decreasing perturbations. Applications to stability theory of hybrid systems are in Section 6. Here, the main results are Theorems 6.5 and 6.6, giving a $\mathcal{KLL}$-bound on solutions to an asymptotically stable hybrid system, and describing how the bound, and so the stability, is affected by perturbations to the system.

2 Hybrid Inclusions

2.1 Solution description

We write $\mathbb{R}_{\geq 0}$ for $[0, +\infty)$ and $\mathbb{N}$ for $\{0, 1, 2, ...\}$. We call a subset $S \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ a *compact hybrid time domain* if

$$S = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_J$. We say S is a *hybrid time domain* if for all $(T, J) \in S$,

$$S \ \cap \ ([0, T] \times \{0, 1, ...J\})$$

is a compact hybrid domain; equivalently, if S is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the "last" interval possibly of the form $[t_j, T)$ with T finite or $T = +\infty$. Hybrid time domains were proposed in Collins (2004) and Goebel, Hespanha, Teel, Cai, and Sanfelice (2004). They are essentially equivalent to "hybrid time trajectories" of Lygeros, Johansson, Sastry, and Egerstedt (1999), Aubin, Lygeros, Quincampoix, Sastry, and Seube (2002), and Lygeros, Johansson, Simić, Zhang, and Sastry (2003), but give a more prominent role to the "discrete" variable j.

On each hybrid domain there is a natural ordering of points: $(t, j) \preceq (t', j')$ if $t + j \leq t' + j'$. Equivalently, $t \leq t'$ and $j \leq j'$, and furthermore, an obvious meaning can be given to $(t, j) \prec (t', j')$. Points from two different hybrid time domains need not be comparable.

A hybrid arc will be a function defined on a hybrid time domain. More specifically, by a *hybrid arc* we will understand a pair consisting of a hybrid time domain $\operatorname{dom} x$ and a function $x : \operatorname{dom} x \to \mathbb{R}^n$ such that $x(t, j)$ is locally absolutely continuous in t for a fixed j and $(t, j) \in \operatorname{dom} x$. We will not mention $\operatorname{dom} x$ explicitly, but always assume that given a hybrid arc x, the set $\operatorname{dom} x$ is exactly the set on which x is defined. Alternatively, one could think of a hybrid arc as a set-valued mapping from $\mathbb{R}_{\geq 0} \times \mathbb{N}$ (or $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$) to $\mathbb{R}^n$ whose domain is a hybrid time domain (for a set-valued mapping M, the *domain* $\operatorname{dom} M$ is the set of arguments for which the value is nonempty). Additionally, one would then assume that x is single-valued on $\operatorname{dom} x$, and absolutely continuous as before.

A sample solution of a hybrid system (corresponding to the height in the Bouncing Ball example, see for example Lygeros, Johansson, Simić, Zhang, and Sastry (2003)) in the hybrid coordinates is shown in Figure 1.

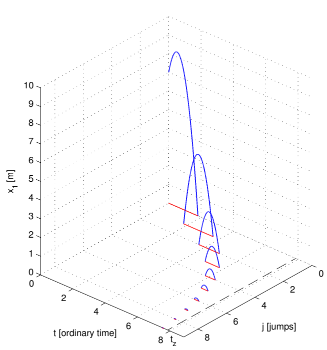

Fig. 1. *Solution of a hybrid system in hybrid coordinates*

The state of a hybrid system is often given by a "continuous" variable and "discrete" one. We will not explicitly distinguish between the two. The set of potential values of the discrete variable, often consisting of descriptive elements like "off" or "on", can be identified with a subset of integers. This leads to more compact notation. As long as the discrete variable has finite or countable range, no issues arise in understanding the continuity of mappings describing the behavior of the variable.

A hybrid system $\mathcal{H}$ will be given on a state space O by set-valued mappings F and G describing, respectively, the continuous and the discrete evolutions, and sets C and D where these evolutions may occur. A hybrid arc $x : \operatorname{dom} x \mapsto O$ is a *solution to the hybrid system* $\mathcal{H}$ if $x(0, 0) \in C \cup D$ and:

(S1) for all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \operatorname{dom} x$,

$$x(t, j) \in C, \quad \dot{x}(t, j) \in F(x(t, j)); \tag{1}$$

(S2) for all $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \quad x(t, j + 1) \in G(x(t, j)). \tag{2}$$

In (S1), the "almost all" refers to one-dimensional Lebesgue measure on $\operatorname{dom} x \cap ([0, \infty) \times \{j\})$. The fundamental conditions on $\mathcal{H}$ that will enable us to show, among other things, that an appropriately understood limit of solutions to $\mathcal{H}$ is itself a solution, are:

(A0) $O \subset \mathbb{R}^n$ *is an open set.*
(A1) C *and* D *are relatively closed sets in* O.
(A2) $F : O \rightrightarrows \mathbb{R}^n$ *is outer semicontinuous and locally bounded, and* $F(x)$ *is nonempty and convex for all* $x \in C$.
(A3) $G : O \rightrightarrows O$ *is outer semicontinuous and* $G(x)$ *is nonempty for all* $x \in D$.

The set C is relatively closed in O if $C = O \cap \overline{C}$, where $\overline{C}$ is the closure of C; similarly for D. The (set-valued) mapping F is *outer semicontinuous* if for all $x \in O$ and all sequences $x_i \to x$, $y_i \in F(x_i)$ such that $y_i \to y$, we have $y \in F(x)$. It is *locally bounded* if for any compact $K \subset O$ there exists $m > 0$ such that $F(K) \subset m\mathbb{B}$, where $\mathbb{B}$ denotes the closed unit ball. Alternative descriptions of outer semicontinuity are mentioned at the end of Section 3. Here, we note that in presence of local boundedness, it entails compactness of images of F and G.

A solution to $\mathcal{H}$ is called *maximal* if it cannot be extended (i.e. it is not a truncation of another solution), and *complete* if its domain is unbounded. Complete solutions are maximal and any solution can be extended to a maximal one. In our framework, a solution is *Zeno* if it is complete and $\operatorname{dom} x$ is bounded in the t-direction.

2.2 Control engineering justification

The control engineering motivation for set-valued right-hand sides, even in purely continuous-time or discrete-time systems, comes primarily from two sources: 1) using feedback control laws that are not continuous, and 2) accounting for uncertainty and/or multiple operating scenarios through an ensemble of solutions.

In the case of discontinuous feedback control, it has been argued starting at least from Hermes (1967) and continuing with Coron and Rosier (1994) that Filippov and/or

Krasovskii's notion of solution for discontinuous differential equations, which correspond to the solutions of a differential inclusion with right-hand side satisfying assumption (A2), are appropriate for capturing the possible closed-loop system behavior in the presence of arbitrarily small measurement noise. Similar observations have been made recently for discrete-time systems, especially in the context of model predictive control. See Kellett and Teel (2004a), Grimm, Messina, Tuna, and Teel (2004), Messina, Tuna, and Teel (2005). For asymptotically null controllable continuous-time systems, it is not possible in general to achieve asymptotic stabilization that is robust to measurement noise when using (discontinuous) continuous-time state feedback (see, for example, Ledyaev and Sontag (1999)). This is the case even though it is possible to achieve robustness with respect to additive disturbances (see Clarke, Ledyaev, Sontag, and Subbotin (1997) or Ancona and Bressan (1999).) In order to establish robustness to measurement noise, the approaches in Sontag (1999), Clarke, Ledyaev, Rifford, and Stern (2000), Kellett and Teel (2004b), and Prieur and Astolfi (2003) resort to hybrid feedback. Typically, the hybrid nature corresponded to sample and hold control. An exception is the approach taken in Prieur and Astolfi (2003) where hysteresis switching is used.

In the case of uncertain or varying systems, a typical class of systems to consider is linear differential (or difference) inclusions. These are often used to model time-varying linear systems with arbitrary variations within a class, linear systems in feedback with a time-varying nonlinearity described by a linear sector condition, etc.

In Goebel, Hespanha, Teel, Cai, and Sanfelice (2004) we used several examples to show that, from a concern for robustness, only relatively closed flow sets and jump sets should be used. This is in contrast to what is considered, for example, in Chellaboina, Bhat, and Haddad (2003) for state-dependent impulsive dynamical systems where these sets (being complements of one another) can not both be closed. Here, we discuss examples of hybrid inclusions that are related to control design.

Example 2.1 (Networked control systems, and sample and hold control)**.** The networked control systems considered in Walsh, Ye, and Bushnell (2002) (see also Nesic and Teel (2004)), with the time between consecutive updates in the interval $[\varepsilon, T]$ where $\varepsilon > 0$ to rule out Zeno solutions and the maximum allowable transfer interval $T \geq \varepsilon$, can be modeled as a hybrid system

$$\begin{aligned} &\dot{\xi} = f(\xi,u),\ \dot{u} = g(\xi,u),\ \dot{\tau} = 1 \\ &\qquad\qquad\qquad\qquad (\xi,u,\tau) \in \mathbb{R}^n \times \mathbb{R}^\ell \times [0,T]; \\ &\xi^+ = \xi,\ u^+ = h(\xi,u),\ \tau^+ = 0 \\ &\qquad\qquad\qquad\qquad (\xi,u,\tau) \in \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}_{\geq \varepsilon}\ . \end{aligned}$$

The rule $u^+ = h(\xi,u)$ is called the "protocol" of the networked control system. It often only depends on u and not on ξ. Sometimes it also depends on the jump number, which can be easily incorporated into the model with a jump counter $k^+ = k+1$. At points where h is discontinuous, it should be replaced by its set-valued regularization $H(\xi,u) = \bigcap_{\delta>0} \overline{h((\xi,u)+\delta\mathbb{B})}$. When H is multi-valued, this regularization introduces extra solutions that can be generated, to arbitrary precision, with arbitrarily small measurement noise e, i.e. by replacing $h(\xi,u)$ in the system above with $h(\xi+e_1, u+e_2)$. Even when the protocol is continuous and f and g are locally Lipschitz, when $T > \varepsilon$ the solutions will not be unique. Instead, the model captures all of the solutions that correspond to transmitting anywhere between ε and T seconds after the previous transmission. In the special case where $g(\xi,u) = 0$, h is independent of u, and $\varepsilon = T$, the model corresponds to applying the state feedback control $u = h(\xi)$ using a sample and hold mechanism with sampling period $T > 0$. Then, the solutions are unique (if f is locally Lipschitz and H is single valued) even though the flow and the jump sets overlap at points where $\tau = T$. Indeed, the rule $\dot{\tau} = 1$ and the flow constraint $\tau \in [0,T]$ make it impossible to flow when $\tau = T$.

Example 2.2 (Hysteresis switching control)**.** Let Q be a finite index set (a subset of the integers) and $f_q(\xi)$, $q \in Q$ be a family of continuous vector fields defined on the relatively closed sets $X_q \subset O \subset \mathbb{R}^n$. These may correspond to different feedbacks on different regions of the state space. To enable a hysteresis switching algorithm between control laws, one can assume that there exist relatively closed sets $K_q \subset \operatorname{int} X_q$ such that $\cup_{q\in Q} K_q = O$. Then, the set-valued map $\xi \mapsto G(\xi) := \{q \in Q \mid \xi \in K_q\}$ satisfies $(A3)$. (Outer semicontinuity follows from the graph of G, given by $\cup_{q\in Q} K_q \times \{q\}$, being relatively closed in $O \times \mathbb{R}$; see the comment after (3).) The system where the trajectories flow according to $\dot{\xi} = f_q(\xi)$ when $\xi \in X_q$ and q can be switched to satisfy $\xi \in K_q$ when the boundary of X_q is reached can be modelled by

$$\begin{aligned} \dot{\xi} = f_q(\xi),\ \dot{q} = 0 &\qquad (\xi,q) \in C; \\ \xi^+ = \xi,\ q^+ \in G(\xi) &\qquad (\xi,q) \in D. \end{aligned}$$

with $C = \cup_{q\in Q} X_q \times \{q\}$, $D = \cup_{q\in Q} (O \cap \overline{\mathbb{R}^n \setminus X_q}) \times \{q\}$. Solutions exist, but need not be unique (even if f_q's are locally Lipschitz) both due to $G(\xi)$ being not single-valued and to potential existence of points $(\xi,q) \in D$ from which flow $\dot{\xi} = f_q(\xi)$, $\xi \in X_q$ is possible. Bounded Zeno solutions are not possible, as $K_q \subset \operatorname{int} X_q$.

Example 2.3 (Dwell-time switching)**.** Let Q be a finite index set not containing zero and let $\xi \mapsto f_q(\xi)$, $q \in Q$ be a family of continuous vector fields. Consider modeling the set of trajectories for which each period of arbitrary switching lasts no longer than T before it is interrupted by a period of at least τ_D on which q is fixed. Such class of switching signals is motivated and considered in

Hespanha (2004). This set of trajectories is captured by

$$\begin{aligned} &\dot{\xi} \in F(\xi), \dot{\tau} = 1 && q = 0, \tau \leq T; \\ &\dot{\xi} = f_q(\xi), \dot{\tau} = 1 && q \neq 0; \\ &q^+ \in Q, \tau^+ = 0 && q = 0; \\ &q^+ \in Q \cup \{0\}, \tau^+ = 0 && q \neq 0, \tau \geq \tau_D; \end{aligned}$$

where $F(\xi)$ is the convex hull of $\bigcup_{q\in Q} f_q(\xi)$. Solutions to $\dot{\xi} \in F(\xi)$ include those generated by arbitrary switching between q's in $\dot{\xi} = f_q(\xi)$ and their limits.

2.3 Basic existence result

An elementary existence result and basic properties of maximal solutions of $\mathcal{H}$ are summarized below. Most were stated, for a similar setting, by Aubin, Lygeros, Quincampoix, Sastry, and Seube (2002); we include them for completeness of presentation. The set $T_C(x)$, the *tangent cone* to C at $x \in C$, is the set of all $v \in \mathbb{R}^n$ for which there exist real numbers $\alpha_i \searrow 0$ and vectors $v_i \to v$ such that for $i = 1, 2, ..., x + \alpha_i v_i \in C$; see Aubin (1991) or Chapter 6, Rockafellar and Wets (1998).

Proposition 2.4 *Assume (A0)-(A2). If $x^0 \in D$ or the following condition holds:*

(VC) $x^0 \in C$ and for some neighborhood U of x^0, for all $x' \in U \cap C$, $T_C(x') \cap F(x') \neq \emptyset$,

then there exists a solution x to $\mathcal{H}$ with $x(0,0) = x^0$ and $\operatorname{dom} x \neq (0,0)$. If (VC) holds for all $x^0 \in C \setminus D$, then for any maximal solution x at least one of the following statements is true:

- *(i) x is complete;*
- *(ii) x eventually leaves every compact subset of O: for any compact $K \subset O$, there exists $(T,J) \in \operatorname{dom} x$ such that for all $(t,j) \in \operatorname{dom} x$ with $(T,J) \prec (t,j)$, $x(t,j) \notin K$;*
- *(iii) for some $(T,J) \in \operatorname{dom} x$, $(T,J) \neq (0,0)$, we have $x(T,J) \notin C \cup D$.*

The case (iii) above does not occur if additionally

(VD) for all $x^0 \in D$, $G(x^0) \subset C \cup D$.

Proof. If $x^0 \in D$, then the arc $x(0,0) = x^0$, $x(0,1) = z$ with any $z \in G(x^0)$ provides the desired solution. Otherwise, there exists $\varepsilon > 0$ and an absolutely continuous $z : [0,\varepsilon] \mapsto U \cap C$ satisfying $\dot{x}(t) \in F(z(t))$ for almost all $t \in [0,\varepsilon]$, see Theorem 3.3.2 in Aubin (1991). Then the desired solution to $\mathcal{H}$ is provided by x given by $x(t,0) := z(t)$.

If x is maximal and $\operatorname{dom} x$ is bounded and closed (so that the supremum (T,J) of $\operatorname{dom} x$ is in $\operatorname{dom} x$, i.e. $x(T,J)$ is defined), then $x(T,J) \notin C \cup D$, as otherwise x could be extended. Consequently, a maximal solution x with $x(t,j) \in C \cup D$ for all $(t,j) \in \operatorname{dom} x$ that is not complete must satisfy $([T-\varepsilon, T), J) \in \operatorname{dom} x$ for some $\varepsilon > 0$, but $(T,J) \notin \operatorname{dom} x$. Then the truncation of x to $[T-\varepsilon, T) \times \{J\}$ can be identified with a maximal solution to the $\dot{x}(t) \in F(x(t))$, and as such, can not be contained in any compact subset of O, Theorem 2, § 7, Filippov (1988). The last property, by local boundedness of F, implies that x eventually leaves any compact subset of O. Otherwise, for some compact K and $t_i \to T$ we have $x(t_i) \in K$. Pick $\rho > 0$ such that $K + \rho\mathbb{B} \subset O$, and let $m = \max_{x\in(K+\rho\mathbb{B})\cap C} \|F(x)\|$. Then for each i and $t < t_i + \rho/m$, $x(t) \in K + \rho\mathbb{B}$. As $t_i \to T$, this implies that for all t close enough to T, $x(t) \in K + \rho\mathbb{B}$, and thus x is contained in a compact subset of O. □

The viability condition for the continuous evolution, (VC), is automatically satisfied at each point in the interior of C. Thus, if $C \cup D = O$, (VC) holds for all $x^0 \in C \setminus D$ (as $C \setminus D = O \setminus D$, and the latter set is open). Consequently, if $C \cup D = O$, for all $x^0 \in O$ there exists a nontrivial solution x with $x(0,0) = x^0$ and (VD) is automatically satisfied. Also note that condition (iii) above implies that (T,J) is the "last" element of $\operatorname{dom} x$, in other words, for all $(t,j) \in \operatorname{dom} x$, $(t,j) \preceq (T,J)$. The meaning of the last conclusion of Proposition 2.4 is that the only way a solution can leave $C \cup D$ is via a jump.

3 Preliminaries – Set Convergence

Consider a sequence $\{S_i\}_{i=1}^\infty$ of subsets of $\mathbb{R}^n$. The *outer limit* of the sequence, denoted $\limsup_{i\to\infty} S_i$, is the set of all $x \in \mathbb{R}^n$ for which there exists a subsequence $\{S_{i_k}\}_{k=1}^\infty$ and points $x_{i_k} \in S_{i_k}$, $k = 1, 2, ...$ such that $x_{i_k} \to x$. The *inner limit* of the sequence, denoted $\liminf_{i\to\infty} S_i$, is the set of all $x \in \mathbb{R}^n$ for which there exist points $x_i \in S_i$, $i = 1, 2, ...$ such that $x_i \to x$. The *limit* of the sequence exists if the outer and inner limits agree, and then $\lim_{i\to\infty} S_i = \limsup_{i\to\infty} S_i = \liminf_{i\to\infty} S_i$. The inner and outer limits of $\{S_i\}_{i=1}^\infty$ always exist, are closed (Proposition 4.4, Rockafellar and Wets (1998)) but may be empty. If the outer limit is empty (and then the inner limit and the limit are also empty), we say that the sequence *escapes to the horizon*; this can be equivalently described as: for all $\rho > 0$ there exists i_0 such that for all $i > i_0$, $S_i \cap \rho\mathbb{B} = \emptyset$.

Example 3.1 (converging intervals)**.** Let $S_i = [a_i, b_i] \subset \mathbb{R}$. The limit of S_i's exists if and only if the sequences of a_i's and b_i's converge (to finite or infinite limits). Then, $\lim_{i\to\infty} S_i = [\lim_{i\to\infty} a_i, \lim_{i\to\infty} b_i]$ if the latter two limits are finite (otherwise the infinite "endpoints" are not in the limit). In general, the inner limit is the interval with endpoints $a = \limsup_{i\to\infty} a_i$, $b = \liminf_{i\to\infty} b_i$ if $a \leq b$; otherwise it is empty. The outer limit need not be an interval; for example if $S_{2i-1} = [1,2]$, $S_{2i} = [3,4]$, then $\limsup_{i\to\infty} S_i = [1,2] \cup [3,4]$.

Example 3.2 (perturbations of a set). Consider a closed $S \subset \mathbb{R}^n$ and a locally bounded $\alpha : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$. Let

$$S_\alpha = \{x \in \mathbb{R}^n \mid x + \alpha(x)\mathbb{B} \cap S \neq \emptyset\}.$$

Such set perturbations appear in robustness analysis, see Section 5. The sets $S_{\delta\alpha}$ (defined with $\delta\alpha(x)$ in place of $\alpha(x)$) converge to S as $\delta \searrow 0$. Indeed, note that $\lim_{\delta\searrow 0} S_{\delta\alpha}$ exists since $S_{\delta\alpha}$'s are nonincreasing as δ decreases (Exercise 4.3, Rockafellar and Wets (1998)), and also $S \subset \lim_{\delta\searrow 0} S_{\delta\alpha}$. Now suppose $x \notin S$. By local boundedness of α, there exists a neighborhood U of x such that for all small enough δ, and all $x' \in U$, $x' \notin S_{\delta\alpha}$. Thus, $\limsup_{\delta\searrow 0} S_{\delta\alpha} \subset S$. Since the limit exists, it must equal S. We note that unless α is constant, S_α differs from the "set-inflation" $\bigcup_{x\in S} x+\alpha(x)\mathbb{B}$, in fact no inclusions between the two perturbations are valid in general.

Set convergence enjoys a certain uniformity property, which applies to unbounded sets if appropriate truncations are considered. This is formally stated as follows

Theorem 3.3 *(Rockafellar and Wets (1998), Theorem 4.10]) For any sequence of sets $\{S_i\}_{i=1}^\infty$ and a closed set S, $\lim_{i\to\infty} S_i = S$ if and only if for all $\varepsilon > 0$ and $\rho > 0$, there exists $i_0 \in \mathbb{N}$ such that $S \cap \rho\mathbb{B} \subset S_i + \varepsilon\mathbb{B}$ and $S_i \cap \rho\mathbb{B} \subset S + \varepsilon\mathbb{B}$ for all $i > i_0$.*

As an immediate consequence of this fact, one can show that arcs eventually get close to their omega limits.

Example 3.4 (Omega limit of a hybrid arc). Let $x : \operatorname{dom} x \mapsto \mathbb{R}^n$ be a complete hybrid arc. The omega limit of x, denoted $\Omega(x)$, is the set of all accumulation points of $x(t,j)$ as $t + j \to +\infty$. Equivalently,

$$\Omega(x) = \limsup_{i\to\infty} S_i \quad \text{where} \quad S_i = \{x(t,j) \mid t + j \geq i\}.$$

By a general property of set limits, $\Omega(x)$ is closed. When x is bounded, sets S_i do not escape to the horizon, and thus $\Omega(x) \neq \emptyset$. Also then, from Theorem 3.3, one obtains: for all $\varepsilon > 0$ there exists m_ε such that for all $(t,j) \in \operatorname{dom} x$ with $t + j \geq m_\varepsilon$, $x(t,j) \in \Omega(x) + \varepsilon\mathbb{B}$.

The bounds in Theorem 3.3 lead to distance-like quantities describing set convergence. That is, S_i's converge to S if and only if for each sufficiently large ρ, the infimum of all ε's satisfying the bounds tends to 0 as $i \to \infty$. Other set distance concepts, including an integrated set distance (describing set convergence via a single quantity), are discussed in Chapter 4 of Rockafellar and Wets (1998). When applied to graphs of hybrid arcs, such distances are related to the Skorokhod metric used in Broucke and Arapostathis (2002) and Collins (2004). [1]

We now show that the limit of a convergent (with respect to set convergence) sequence of hybrid time domains is itself a hybrid time domain.

Lemma 3.5 *Let $\{S_i\}_{i=1}^\infty$ be a convergent sequence of hybrid time domains. Then $S := \lim_{i\to\infty} S_i$ is a hybrid time domain. If also each S_i is unbounded, then so is S.*

Proof. Directly from the definition of set convergence, $\lim_{i\to\infty} S_i = S$ if and only if for all $J \in \mathbb{N}$, $S_i^J := S_i \cap (\mathbb{R}_{\geq 0} \times \{J\})$ converge to $S^J := S \cap (\mathbb{R}_{\geq 0} \times \{J\})$. Thus, each S^J is a closed interval (possibly empty, consisting of one point, or unbounded to the right). If S^{J+1} is nonempty, then so is S^J. Indeed, the right endpoints of S_i^J agree with left endpoints of S_i^{J+1}, and the latter converge to the left endpoint of S^{J+1}. This is enough to conclude that S is a hybrid time domain.

Now let $K^r = (\{r\} \times \{0,1,...,r\}) \cup ([0,r] \times \{r\})$ and note that a hybrid time domain S is unbounded if and only if for every $r \in \mathbb{N}$, $K^r \cap S$ is nonempty (similarly for S_i's). Suppose S is bounded, so that for some $r \in \mathbb{N}$, $S \subset [0,r] \times \{0,1,...,r\}$. If $\lim_{i\to\infty} S_i = S$, then from Theorem 3.3, for all sufficiently large i,

$$\begin{aligned} S_i \cap K^{2r} &\subset (S_i \cap [0,2r] \times \{0,1,...,2r\}) \cap K^{2r} \\ &\subset ([0, r + 1/2] \times \{0,1,...,r\}) \cap K^{2r} = \emptyset \end{aligned}$$

This contradicts the unboundedness of S_i's. □

For our purposes, another important property of set convergence is that, much like for real numbers, a sequence of sets either diverges or has a convergent subsequence.

Theorem 3.6 *(Rockafellar and Wets (1998), Theorem 4.18) Every sequence $\{S_i\}_{i=1}^\infty$ of nonempty subsets of $\mathbb{R}^n$ either escapes to the horizon or has a subsequence converging to a nonempty set S.*

Set convergence can be used to give sequential definitions of continuity of set-valued mappings. A mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *outer semicontinuous* at x if for all sequences $x_i \to x$, $y_i \in M(x_i)$ with $y_i \to y$, we have $y \in M(x)$; equivalently, if for all sequences $x_i \to x$ we have $\limsup_{i\to\infty} M(x_i) \subset M(x)$. Inner semicontinuity and continuity can also be defined, see Chapter 5 in Rockafellar and Wets (1998). The mapping M is outer semicontinuous on $\mathbb{R}^n$ if and only if the *graph* of M:

$$\operatorname{gph} M := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in M(x)\} \tag{3}$$

is closed (Theorem 5.7 in Rockafellar and Wets (1998)). Our assumptions can also be phrased in terms of graphs:

[1] The Skorokhod topology was originally designed to analyze convergence of stochastic processes, which can be often represented by right-continuous functions having left limits. See Pollard (1984) for a general discussion, or Kisynski (1990) for comparison of convergence in Skorokhod topology and graph convergence in Hausdorff distance.

$F : O \rightrightarrows \mathbb{R}^n$ is outer semicontinuous if $\operatorname{gph} F$, equal to $\{(x,y) \mid x \in O, y \in F(x)\}$, is relatively closed in $O \times \mathbb{R}^n$.

Outer semicontinuous mappings have closed values. If the mapping is also locally bounded, the values are compact. For locally bounded set-valued mappings with closed values, outer semicontinuity agrees with what is often referred to as *upper semicontinuity*: for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all x' with $\|x' - x\| < \delta$, $M(x') \subset M(x) + \varepsilon\mathbb{B}$. This need not be true in general.

Set convergence gives rise to a nonclassical concept of convergence of functions and set-valued mappings. A sequence $\{M_i\}_{i=1}^{\infty}$ of set-valued mappings *converges graphically* to a set-valued mapping M if $\lim_{i\to\infty} \operatorname{gph} M_i = \operatorname{gph} M$. See Chapter 5 in Rockafellar and Wets (1998) for more details, here we note that graphical convergence can treat mappings with different domains (in fact, domains need not even overlap).

Graphical convergence of solutions to differential inclusions, with different domains, is the topic of the preliminary result below. In the result, we do not explicitly mention sets constraining the flow of solutions. When such sets are present, it is sufficient to consider truncations of F and F_i's to C and C_i's. The proof is in the Appendix.

Lemma 3.7 *Assume that*

(i) $F : O \rightrightarrows \mathbb{R}^n$ *is a outer semicontinuous, locally bounded, and convex-valued mapping with* $\operatorname{dom} F$ *relatively closed in O;*
(ii) the sequence of mappings $\{F_i\}_{i=1}^{\infty}$ *is locally uniformly bounded on O and its graphical outer limit* F_0 *satisfies* $F_0(x) \subset F(x)$ *for all* $x \in 0$*;*
(iii) for $i = 1, 2, ...,$ $x_i : [a_i, b_i] \to O$ *is an absolutely continuous function satisfying* $\dot{x}_i(t) \in F_i(x_i(t))$ *for almost all* $t \in [a_i, b_i]$*;*
(iv) a_i*'s and* b_i*'s converge, respectively, to* a *and* b*, while* x_i*'s converge graphically to* x*.*

Then

(a) If x_i*'s are uniformly bounded with respect to O then* x *is an absolutely continuous function on* $[a, b]$ *satisfying* $\dot{x}(t) \in F(x(t))$ *for almost all* $t \in [a, b]$*.*
(b) If x_i*'s are not uniformly bounded with respect to O, but* $x_i(a_i) \to x_0 \in O$*, then* $a < b$ *and for some* $T \in (a, b]$ *we have that* x *is absolutely continuous on* $[a, T']$ *for all* $T' < T$*,* $\dot{x}(t) \in F(x(t))$ *for almost all* $t \in [a, T)$ *(which entails* $x(t) \in O$ *for all* $t \in [a, T)$*) and* $x(t)$ *leaves every compact subset of O as* $t \nearrow T$*.*

4 Graphical convergence of hybrid arcs

Part of our motivation to use graphical convergence is the difficulties encountered when treating hybrid systems with classical convergence notions.

Example 4.1 Consider a hybrid system on $O = \mathbb{R}$, with $C = \mathbb{R}$, $F(x) = 1$ for all $x \in C$, $D = \{1\}$, and $G(1) = 2$. For $\varepsilon \le 1$, a particular solution x_ε with $x_\varepsilon(0,0) = \varepsilon$ is: $\operatorname{dom} x_\varepsilon = [0, 1-\varepsilon] \times \{0\} \cup [1-\varepsilon, \infty) \times \{1\}$ and $x_\varepsilon(t, 0) = t+\varepsilon$ for $t \in [0, 1-\varepsilon]$, $x_\varepsilon(t, 1) = t+\varepsilon+1$ for $t \in [1-\varepsilon, \infty)$. As $\varepsilon \to 0$, x_ε converge graphically to x_0 (even though $\operatorname{dom} x_\varepsilon \ne \operatorname{dom} x_0$ for $\varepsilon \ne 0$). If solutions were considered in the more standard sense of left continuous functions of time, the objects corresponding to x_ε would be $y_\varepsilon : \mathbb{R}_{\ge 0} \to \mathbb{R}$ given by $y_\varepsilon(t) = t + \varepsilon$ for $t \in [0, 1-\varepsilon)$, $y_\varepsilon(t) = t+\varepsilon+1$ for $t \in [1-\varepsilon, \infty)$. As $\varepsilon \nearrow 0$, y_ε converge pointwise to y_0. As $\varepsilon \searrow 0$, while y_ε converge pointwise, the limit is not left, but right continuous. Also note that for $\varepsilon \ne \varepsilon'$ near 0, $\sup_{t\in[0,2]} |y_\varepsilon(t) - y_{\varepsilon'}(t)| = 1$, and thus y_ε do not converge uniformly as $\varepsilon \to 0$.

For various relationships between pointwise, uniform, and graphical convergences consult Chapter 5 of Rockafellar and Wets (1998).

We will usually be interested in graphical convergence of hybrid arcs subject to some boundedness assumptions. We will say that a sequence of hybrid arcs $x_i : \operatorname{dom} x_i \mapsto \mathbb{R}^n$ is *locally eventually bounded with respect to O* if

> for any $m > 0$, there exists $i_0 > 0$ and a compact set $K \subset O$ such that for all $i > i_0$, all $(t, j) \in \operatorname{dom} x_i$ with $t + j < m$, $x_i(t, j) \in K$.

If a locally eventually bounded with respect to O sequence converges graphically to an arc x, then in particular $x(t, j) \in O$ for all $(t, j) \in \operatorname{dom} x$.

In what follows, we will use a closeness concept related to graphical convergence, that does not require that jumps of "close" solutions occur at the same time (recall Example 4.1). One can say though that the times of j-th jumps of two close solutions do not differ by much, and, if there are no jumps for either solution near a particular time, solutions are close to each other there in the standard uniform sense. We will say that hybrid arcs $x : \operatorname{dom} x \to \mathbb{R}^n$, $y : \operatorname{dom} y \to \mathbb{R}^n$ are (T, K, ε)*-close* if:

(a) for all $(t, j) \in \operatorname{dom} x$ with $t \le T$, $j \le K$ there exists s such that $(s, j) \in \operatorname{dom} y$, $|t - s| < \varepsilon$, and
$$\|x(t, j) - y(s, j)\| < \varepsilon,$$
(b) for all $(t, j) \in \operatorname{dom} y$ with $t \le T$, $j \le K$ there exists s such that $(s, j) \in \operatorname{dom} x$, $|t - s| < \varepsilon$, and
$$\|y(t, j) - x(s, j)\| < \varepsilon.$$

Lemma 4.2 *Consider a sequence of hybrid arcs* $x_i : \operatorname{dom} x_i \mapsto \mathbb{R}^n$ *that is locally eventually bounded, and a hybrid arc* $x : \operatorname{dom} x \mapsto \mathbb{R}^n$*. The sequence* $\{x_i\}_{i=1}^{\infty}$ *converges graphically to* x *if and only if for all* $(T, K) \in \mathbb{R}_{\ge 0} \times \mathbb{N}$ *and* $\varepsilon > 0$*, there exists* $i_0 \in \mathbb{N}$ *such that, for all* $i > i_0$ *the hybrid arcs* x *and* x_i *are* (T, K, ε)*-close.*

This is a translation of the uniformity in set convergence, as stated in Theorem 3.3, to graphical convergence; see also Exercise 5.34 in Rockafellar and Wets (1998). Conditions (a) and (b) above do not restrict (t, j) to those for which, respectively, $x(t, j) \in \rho\mathbb{B}$ or $x_i(t, j) \in \rho\mathbb{B}$ (compare to Theorem 3.3). Thus, these conditions imply those required for graphical convergence (as stated in 5.34 in Rockafellar and Wets (1998)). On the other hand, local eventual boundedness of a graphically convergent sequence implies that the limit is locally bounded and the restrictions on $x(t, j)$ can be omitted.

The closedness, outer semicontinuity, and boundedness properties of the data of $\mathcal{H}$ guarantee that the graphical limit of solutions to $\mathcal{H}$, if it exists, is itself a solution.

Lemma 4.3 *Assume (A0)-(A3). Let* $x_i : \operatorname{dom} x_i \mapsto \mathbb{R}^n$, $i = 1, 2, ...$ *be solutions to* $\mathcal{H}$. *Suppose that the sequence* $\{x_i\}_{i=1}^{\infty}$ *is locally eventually bounded with respect to* O *and converges graphically to a set-valued mapping* $x : \mathbb{R}_{\geq 0} \times \mathbb{N} \mapsto \mathbb{R}^n$. *Then* $\operatorname{dom} x$ *is a hybrid time domain and* x *is a solution to* $\mathcal{H}$.

Proof. Let P be the projection of $\mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n$ onto $\mathbb{R}_{\geq 0} \times \mathbb{N}$, so that $\operatorname{dom} x_i = P(\operatorname{gph} x_i)$, $\operatorname{dom} x = P(\operatorname{gph} x)$. We first claim that $P(\lim_{i\to\infty} \operatorname{gph} x_i) = \lim_{i\to\infty} P(\operatorname{gph} x_i)$ (Theorem 4.26 in Rockafellar and Wets (1998) subsumes this, we give a direct argument). Directly from the definitions,

$$P(\lim_{i\to\infty} \operatorname{gph} x_i) = P(\liminf_{i\to\infty} \operatorname{gph} x_i) \subset \liminf_{i\to\infty} P(\operatorname{gph} x_i).$$

On the other hand, fix any $(t, j) \in \limsup_{i\to\infty} P(\operatorname{gph} x_i)$ and let $(t_{i_k}, j_{i_k}) \in \operatorname{dom} x_{i_k}$ converge to (t, j). As x_i's are locally eventually bounded, there exists a subsequence (which we do not relabel) such that $x_{i_k}(t_{i_k}, j_{i_k})$ converges. The limit is an element of $x(t, j)$, which, in particular, implies that $(t, j) \in P(\operatorname{gph} x)$. Consequently

$$\limsup_{i\to\infty} P(\operatorname{gph} x_i) \subset P(\operatorname{gph} x) = P(\lim_{i\to\infty} \operatorname{gph} x_i).$$

The two inclusions displayed above prove the claim. Lemma 3.5 implies that $\operatorname{dom} x$ is a hybrid time domain.

It is immediate, from local eventual boundedness of x_i's, that for all $(t, j) \in \operatorname{dom} x$, $x(t, j) \subset O$ (we have not shown yet that x is single-valued on $\operatorname{dom} x$). Furthermore, by relative closedness of C and D, $x(0, 0) \subset C \cup D$.

For any J such that $\operatorname{dom}^J x := \operatorname{dom} x \cap (\mathbb{R}_{\geq 0} \times \{J\}) \neq \emptyset$, let x^J be the truncation of x to $\operatorname{dom}^J x$ (similarly, let x_i^J be truncations of x_i to $\operatorname{dom}^J x_i := \operatorname{dom} x_i \cap (\mathbb{R}_{\geq 0} \times \{J\})$). From the definition of set convergence, x^J is the graphical limit of x_i^J. Lemma 3.7, used with $F_i = F$, shows that x^J is single-valued, absolutely continuous, and satisfies the second inclusion in (1) a.e. (when $\operatorname{dom}^J x$ equals $[t, t] \times \{J\}$, this just means that $x(t, J)$ is a singleton). For the first inclusion in (1), note that for $(t, J) \in \operatorname{dom}^J x$, $x(t, J)$ is a limit of $x_i(t_i, J)$ for some $t_i \to t$, while $x_i(t_i, J)$ are all in a compact subset of O. As C is relatively closed, $x(t, J) \in C$. Arguments just presented show that x is a hybrid arc, and that it satisfies (S1).

We now turn to condition (S2). Pick any $(t, j) \in \operatorname{dom} x$ so that $(t, j + 1) \in \operatorname{dom} x$. Then $(t, j, x(t, j)) = \lim_{i\to\infty}(t'_i, j, x_i(t'_i, j))$ for some sequence $(t'_i, j) \in \operatorname{dom} x_i$, while for another sequence $(t''_i, j + 1) \in \operatorname{dom} x_i$, $(t, j+1, x(t, j+1)) = \lim_{i\to\infty}(t''_i, j+1, x_i(t''_i, j+1))$. The properties of hybrid time domains imply that for some t_i with $t'_i \leq t_i \leq t''_i$ both (t_i, j) and $(t_i, j+1)$ are in $\operatorname{dom} x_i$, and thus $x_i(t_i, j) \in D$ and $x_i(t_i, j + 1) \in G(x_i(t_i, j))$. Also, as x_i's are locally eventually bounded and x is single valued (we showed this in the previous paragraph), we must have $\lim_{i\to\infty} x_i(t_i, j) = x(t, j)$, $\lim_{i\to\infty} x_i(t_i, j + 1)$, while $\lim_{i\to\infty} t_i = t$. This, outer semicontinuity of G, and relative closedness of D, leads immediately to (2). Thus x satisfies (S2). ∎

Theorem 3.6 applied to graphs shows that under mild growth conditions, from any sequence of set-valued mappings one can pick a graphically convergent subsequence (Theorem 5.36, Rockafellar and Wets (1998)). In particular, this can be applied to solutions of a hybrid system. The result of this, which we state below, can be viewed as the main result of this section. It will turn out to be the key tool in proving the results of Section 6, we add that it is also fundamental for studying invariance of certain sets; see Sanfelice, Goebel, and Teel (2005).

Theorem 4.4 (sequential compactness) *Assume (A0)-(A3). Let* $x_i : \operatorname{dom} x_i \mapsto \mathbb{R}^n$, $i = 1, 2, \ldots$, *be a locally eventually bounded with respect to* O *sequence of solutions to* $\mathcal{H}$. *Then there exists a subsequence of* x_i*'s graphically converging to a solution of* $\mathcal{H}$. *Such a limiting solution is complete if each* x_i *is complete, or more generally, if no subsequence of* x_i*'s has uniformly bounded domains (i.e. for any* $m > 0$, *there exists* $i_m \in \mathbb{N}$ *such that for all* $i > i_m$, *there exists* $(t, j) \in \operatorname{dom} x_i$ *with* $t + j > m$*).*

Proof. As $x_i(0, 0)$ are uniformly bounded, Theorem 3.6 implies that there is a subsequence of x_i's (which we do not relabel) for which the graphs converge (to a graph of some set-valued mapping x). Local boundedness of x_i's with respect to O and Lemma 4.3 imply that x is a solution of $\mathcal{H}$. If each x_i is complete, the completeness of x follows from the last statement of Lemma 3.5. If no subsequence of x_i is bounded, unboundedness of $\operatorname{dom} x$ (and so, completeness of x) follows from the arguments in the second part of the proof of Lemma 3.5. ∎

Completeness of the graphical limit of a sequence of solutions to $\mathcal{H}$ can be also guaranteed if these solutions are maximal, local existence of solutions can be guaranteed, and solutions to not leave $C \cup D$. The precise statement is given below, we will rely on it in Proposition 6.4. In particular, the result below says that if $C \cup D = O$, the limit of a graphically convergent and locally eventually bounded sequence of maximal solutions is complete.

Lemma 4.5 *Assume (A0)-(A3), that (VD) of Proposition 2.4 holds, and that (VC) of Proposition 2.4 holds for any $x^0 \in C \setminus D$. Then the graphical limit x of a graphically convergent and locally eventually bounded sequence of maximal solutions x_i to $\mathcal{H}$ is complete.*

Proof. Suppose that x is not complete. By the last conclusion of Theorem 4.4, some subsequence of x_i's has uniformly bounded domains. By local eventual boundedness, all but a finite number of elements of this subsequence are uniformly bounded with respect to O. But as (VD) holds, and (VC) holds for $x^0 \in C \setminus D$, each of the elements of the subsequence (except a finite number of them) must eventually leave any compact subset of O; see (ii) in Proposition 2.4. This contradicts the uniform boundedness. □

Recall that a solution to the hybrid system is complete if its domain is unbounded. The hybrid system $\mathcal{H}$ is called *forward complete* at x^0 if every maximal solution to $\mathcal{H}$ from x^0 is complete. In what follows, we will write $\mathcal{S}(x^0)$ to denote the set of all maximal solutions to $\mathcal{H}$ originating at x^0, and $\mathcal{S}(K)$ for all those originating in a set K.

Note that so far we have not required any growth conditions on G; local boundedness was only required of F. In what follows, we need a related condition on G:

(A4) $G : O \rightrightarrows O$ is locally bounded.

In light of outer semicontinuity of $G : O \rightrightarrows O$, local boundedness is in fact equivalent to local boundedness with respect to O: for any compact subset $K \subset O$, there exists a compact $K' \subset O$ such that $G(K) \subset K'$. Note that (A4) automatically holds if (VD) of Proposition 2.4 holds and both C and D are compact.

Theorem 4.6 (completeness and boundedness) *Assume (A0)-(A4). Suppose $\mathcal{H}$ is forward complete at each $x^0 \in K$ for some compact set $K \subset O$. Then for any $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ there exists $\delta > 0$ and a compact subset $K' \subset O$ such that any $x_\delta \in \mathcal{S}(K + \delta\mathbb{B})$ satisfies $x_\delta(t, j) \in K'$ for all $(t, j) \in \operatorname{dom} x_\delta$, $(t, j) \preceq (T, J)$.*

Proof. If the conclusion does not hold, then there exists a sequence of solutions $\{x_i\}_{i=1}^{\infty}$ with $x_i(0, 0)$ converging to $x^0 \in K$ and a point (T, J) such that truncations of x_i to $[0, T] \times \{0, 1, ...J\}$ are not uniformly bounded with respect to O. From now on, let x_i's be those truncations, and note that by passing to a subsequence, we can assume that x_i's converge graphically. We will argue that the graphical limit (or its further truncation) is a solution to $\mathcal{H}$ which is maximal, but not complete.

Let $\operatorname{dom}^j x_i := \operatorname{dom} x_i \cap (\mathbb{R}_{\geq 0} \times \{j\}) \subset [0, T] \times \{j\}$. There exists the smallest $j \leq J$ such that $x_i(\cdot, j)$ on $\operatorname{dom}^j x_i$ are not uniformly bounded with respect to O. Then x_i's truncated to $[0, T] \times \{0, 1, ...j-1\}$ are uniformly bounded, and by Lemma 4.3, x truncated to $[0, T] \times \{0, 1, ...j-1\}$ is a solution to $\mathcal{H}$. Let $t_i \to \bar{t}$ be the time coordinates of left endpoints of $\operatorname{dom}^j x_i$. Then $x_i(t_i, j)$'s are uniformly bounded with respect to O (and thus converge to $x(\bar{t}, j) \in O$). Indeed, if $j = 0$ then $t_i = 0$ and $x_i(t_i, j) \to x_0$. If $j > 0$, $x_i(t_i, j-1)$ are uniformly bounded with respect to O, and by (A4), so are $x_i(t_i, j) \in G(x_i(t_i, j-1))$. Passing to a subsequence of x_i's, we can assume that intervals $\operatorname{dom}^j x_i =: [a_i, b_i] \times \{j\}$ and points $x_i(t_i, j)$ converge to, respectively, $[a, b]$ and $x(\bar{t}, j)$. Applying Lemma 3.7 yields that for some $T \in (a, b]$, $x(\cdot, j)$ satisfies the differential inclusion on $[a, T)$ and leaves any compact subset of O as $t \nearrow T$. This leads to a maximal but not complete solution to $\mathcal{H}$ from x^0. □

Corollary 4.7 *Assume (A0)-(A4). Suppose $\mathcal{H}$ is forward complete at each $x^0 \in K$, for some compact set K. Then for all $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, the reachable set $\operatorname{reach}_{T,J}(K)$ is compact, where*

$$\operatorname{reach}_{T,J}(K) := \{x(t, j) \mid x \in \mathcal{S}(K), (t, j) \preceq (T, J)\}.$$

Proof. Any sequence of points in $\operatorname{reach}_{T,J}(K)$ can be written as $x_i(t_i, j_i)$ with $x_i \in \mathcal{S}(x_i^0)$, $x_i^0 \in K$, $(t_i, j_i) \preceq (T, J)$. Without changing the indices, we pass to a subsequence so that $x_i^0 \to x^0 \in K$. By Theorem 4.6, the sequence of x_i's truncated to $[0, t_i] \times \{0, 1, ..j_i\}$ is uniformly bounded, and by Theorem 4.4, we can pick from it a graphically convergent subsequence, with the limit denoted x. This subsequence can be picked so that (t_i, j_i) converge, say to (t, j). Now, $x(t, j) = \lim_{i \to \infty} x_i(t_i, j_i)$ and thus $x(t, j) \in \operatorname{reach}_{T,J}(K)$. □

The next result says that under some forward completeness assumptions, a solution starting close to a compact set K stays close, on compact subsets of its domain, to some solution starting in K.

Corollary 4.8 *Assume (A0)-(A4). Suppose that $\mathcal{H}$ is forward complete at every $x^0 \in K$ for some compact set K. For any $\varepsilon > 0$ and $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ there exists $\delta > 0$ with the following property: for any solution $x_\delta \in \mathcal{S}(K + \delta\mathbb{B})$ there exists a solution x to $\mathcal{H}$ with $x(0, 0) \in K$ such that x_δ and x are (T, J, ε)-close.*

Proof. If the conclusion is false, there exist $\varepsilon > 0$, $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, and a sequence of solutions x_i with $x_i(0, 0)$ converging to K such that, for each i, no solution $x \in S(K)$ is (T, K, ε)-close to x_i. The sequence of x_i's is locally eventually bounded with respect to O by Theorem 4.6 and has a graphically convergent subsequence, the limit $\bar{x}$ of which is a solution, see Theorem 4.4, and of course $\bar{x}(0,0) \in K$. Conclusions of Lemma 4.2 applied to this subsequence and $\bar{x}$ yield a contradiction. □

Note that if above one additionally assumes that (VD), and for all $x^0 \in C \setminus D$, (VC) of Proposition 2.4 hold (as in Lemma 4.5), then the solution x can be guaranteed to be complete (equivalently, maximal: $x \in \mathcal{S}(K)$).

To conclude this section, we mention a result by Collins (2004). Here, it can be derived from Theorem 4.4. See also Theorem 6.7. Following Collins (2004), we say that a given set of solutions to $\mathcal{H}$ is uniformly non-Zeno if there exist $T > 0$ and $J \in \mathbb{N}$ so that for any solution in that set, in any time period of length T, at most J jumps can occur. More specifically, if $(t, j), (t', j') \in \operatorname{dom} x$, then $|t - t'| \leq T$ implies $|j - j'| \leq J$. (Note though under such an assumption, multiple instantaneous jumps can still occur.) Forward invariance of K means that any $x \in \mathcal{S}(K)$ is such that $x(t, j) \in K$ for all $(t, j) \in \operatorname{dom} x$.

Corollary 4.9 *Assume (A0)-(A4). Let $K \subset O$ be a compact set that is forward invariant. Then either the set $\mathcal{S}(K)$ is uniformly non-Zeno or there exists an instantaneous Zeno solution (a complete solution x with* $\operatorname{dom} x = \{0\} \times \mathbb{N}$) *starting in K.*

5 Perturbations of hybrid systems

Below, we consider a sequence of hybrid systems $\mathcal{H}_i$ given by sets C_i, D_i and mappings $F_i : O \rightrightarrows \mathbb{R}^n$, $G_i : O \rightrightarrows O$ on the open set O. Since we do not require existence or outer semicontinuity from the solutions to $\mathcal{H}_i$ as studied for $\mathcal{H}$, in Sections 2, 3, we do not need the properties corresponding to (A1)-(A4) to hold for $\mathcal{H}_i$. We do assume the following:

(C1) Sequences of sets $\{C_i\}_{i=1}^\infty$, $\{D_i\}_{i=1}^\infty$ are such that

$$\left(\limsup_{i\to\infty} C_i\right) \cap O \subset C, \quad \left(\limsup_{i\to\infty} D_i\right) \cap O \subset D.$$

(C2) Sequences of set-valued mappings $\{F_i\}_{i=1}^\infty$, $\{G_i\}_{i=1}^\infty$ are such that

$$F_0(x) \subset F(x), \quad G_0(x) \subset G(x)$$

for all $x \in O$, where F_0, G_0 denote the outer graphical limits of F_i's, G_i's.

(C3) The sequence $\{F_i\}_{i=1}^\infty$ is locally eventually bounded: for any compact set $K \subset O$ there exists $m > 0$ such that for any $i = 1, 2, ...$ $F_i(K) \subset m\mathbb{B}$.

The outer graphical limit of $\{F_i\}_{i=1}^\infty$ is the mapping F_0 such that $\operatorname{gph} F_0 = \limsup_{i\to\infty} \operatorname{gph} F_i$. Assumption (C1) holds, in particular, when the sequences of C_i's and D_i's converge, and $C = O \cap \lim_{i\to\infty} C_i$, and similarly for D. An analogous statement can be made about (C2).

If we assume that each F_i is convex-valued (or more generally, connected-valued), then (C3) follows from (C2) and local boundedness of F as assumed in (A3); see Corollary 4.12 and Exercise 5.34 in Rockafellar and Wets (1998). Condition (C3) can be thought of as an extension of (A3) to sequences of hybrid systems, while (C4), which we formulate and rely on later, extends (A4).

Theorem 5.1 (solutions under perturbations) *Assume (A0)-(A4) and (C1)-(C3). Let $x_i : \operatorname{dom} x_i \mapsto \mathbb{R}^n$ be a solution to a hybrid system $\mathcal{H}_i$, $i = 1, 2, \ldots$. Suppose that the sequence $\{x_i\}_{i=1}^\infty$ is locally eventually bounded with respect to O and its graphical limit x exists. Then x is a solution to $\mathcal{H}$.*

Proof. The proof is essentially the same as that of Lemma 4.3. One just invokes Lemma 3.7 not with a constant sequence, but with the sequence $\{F_i\}_{i=1}^\infty$, and instead of relying on outer semicontinuity of G and closedness of C and D, one relies on the definitions of graphical and set convergence, and on (C1), (C2). □

(C4) The sequence $\{G_i\}_{i=1}^\infty$ is locally eventually bounded with respect to O: for any compact set $K \subset O$ there exists a compact $K' \subset O$ such that for any $i = 1, 2, ...$, $G_i(K) \subset K'$.

Below, $\mathcal{S}_i(x^0)$ is the set of maximal solutions to $\mathcal{H}_i$ from x^0; later in this section $\mathcal{S}_\delta(x^0)$ has a similar meaning.

Corollary 5.2 *Assume (A0)-(A4) and (C1)-(C4). Suppose $\mathcal{H}$ is forward complete at x^0. Then, for any $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ there exists $\delta > 0$, $i_0 > 0$, and a compact subset $K \subset O$ such that for all $i > i_0$, any $x \in \mathcal{S}_i(x^0 + \delta\mathbb{B})$ satisfies $x(t, j) \in K$ for all $(t, j) \in \operatorname{dom} x$, $(t, j) \preceq (T, J)$.*

Proof. If the conclusion is false, there exists (T, J) and $x_i \in \mathcal{S}_i(x_i^0)$ with $x_i^0 \to x_0$, truncations of which to $[0, T] \times \{0, 1, ...J\}$ are not uniformly bounded with respect to O. Now one proceeds as in the proof of Theorem 4.6, to construct, via Theorem 5.1, a solution $x \in \mathcal{S}(x^0)$ which is not complete (a contradiction). The extra assumption (C3) replaces the one on local boundedness of G with respect to E, and is used to show that if $x_i(t_i, j-1)$ are uniformly bounded then so are $x_i(t_i, j) \in G_i(x(t_i, j-1))$. Also, Lemma 3.7 is invoked not with a constant sequence, but with the sequence $\{F_i\}_{i=1}^\infty$. □

Robustness analysis of $\mathcal{H}$ calls for consideration of perturbations $\mathcal{H}_\delta$. These can be often understood as given on O by sets C_δ, D_δ, and mappings F_δ, G_δ, for a continuously varying parameter $\delta > 0$. We will say that perturbations $\mathcal{H}_\delta$ have the *convergence property* if

(CP) for any sequence $1 > \delta_1 > \delta_2 > ... > 0$ converging to 0, sequences $\{C_i\}_{i=1}^{\infty}$, $\{D_i\}_{i=1}^{\infty}$ and $\{F_i\}_{i=1}^{\infty}$, $\{G_i\}_{i=1}^{\infty}$ satisfy assumptions (C1), (C2), (C3), and (C4), where for each $i = 1, 2, ..., C_i = C_{\delta_i}$, similarly for D_i, F_i, G_i.

A particular kind of perturbations appears in converse Lyapunov theory for differential and difference inclusion, see Clarke, Ledyaev, and Stern (1998), Teel and Praly (2000), Kellett and Teel (2004a).

Example 5.3 ("outer perturbations")**.** Let $\alpha : O \mapsto \mathbb{R}_{\geq 0}$ be a continuous function such that, for all $x \in O$ we have $x + \alpha(x)\mathbb{B} \subset O$. Then, one considers systems $\mathcal{H}_\delta$ on O, given for $\delta \in (0, 1)$ by the sets C_δ, D_δ (we discussed similar perturbations in Example 3.2), and the mappings $F_\delta : O \rightrightarrows \mathbb{R}^n$, $G_\delta : O \rightrightarrows O$ given by

$$C_\delta = \{x \in O \mid x + \delta\alpha(x)\mathbb{B} \cap C \neq \emptyset\},$$
$$D_\delta = \{x \in O \mid x + \delta\alpha(x)\mathbb{B} \cap D \neq \emptyset\},$$
$$F_\delta(x) = \operatorname{con} F\,(x + \delta\alpha(x)\mathbb{B}) + \delta\alpha(x)\mathbb{B},$$
$$G_\delta(x) = \{y \mid y \in \eta + \delta\alpha(\eta)\mathbb{B},\ \eta \in G\,(x + \delta\alpha(x)\mathbb{B})\}\,.$$

Above, $\operatorname{con} S$ denotes the convex hull of a set S. Obviously, F_δ is convex-valued. It can be verified that F_δ is nonempty-valued on C_δ, similarly for G_δ on D_δ.

Theorem 5.4 *Assume (A1)-(A4). The perturbations of Example 5.3 have the convergence property (CP).*

Proof. Pick any decreasing sequence of δ_i's, converging to 0, and construct sequences of sets and mappings as needed in the convergence property. Convergence of C_i's, D_i's follows from Example 3.2. As graphs of F_i's, G_i's are decreasing, the graphical limits exist, and moreover $\operatorname{gph} F \subset \lim_{i\to\infty} \operatorname{gph} F_i$, $\operatorname{gph} G \subset \lim_{i\to\infty} \operatorname{gph} G_i$.

Suppose $(x, y) \in \lim_{i\to\infty} \operatorname{gph} F_i$ and $x \in O$. There exist x_i, y_i such that $x_i \to x$, $y_i \to y$, and $y_i \in \operatorname{con} F(x_i + \delta_i\alpha(x_i)\mathbb{B}) + \delta_i\alpha(x_i)\mathbb{B}$. Thus, $y_i = \sum_{k=0}^{n} \lambda_i^k y_i^k + \delta_i\alpha(x_i)\mathbb{B}$, where $\lambda_i^k \geq 0$, $\sum_{k=0}^{n} \lambda_i^k = 1$, and $y_i^k \in F(x_i^k)$ where $x_i^k \in x_i + \delta_i\alpha(x_i)\mathbb{B}$. For each $k = 0, 1, ..n$, x_i^k's converge to x, and by local boundedness of F, y_i^k's are bounded. Passing to subsequences, so that for y_i^k's converge to some $y^k \in F(x)$, while λ_i^k's converge to some λ^k's in the simplex, we get $y = \sum_{k=0}^{n} \lambda^k y^k \in F(x)$, as $F(x)$ is convex. Thus $(x, y) \in \operatorname{gph} F$.

Now say $(x, y) \in \lim_{i\to\infty} \operatorname{gph} G_\delta$ with $x \in O$. There exist x_i, y_i such that $x_i \to x$, $y_i \to y$, and $y_i \in \eta_i + \delta_i\alpha(\eta_i)\mathbb{B}$ while $\eta_i \in G(x_i + \delta_i\alpha(x_i)\mathbb{B})$. By local boundedness of G, we can pick a subsequence of η_i's converging to some η, and $\eta \in G(x)$. But the limit of y_i's must be the same, so $y = \eta$. This shows $(x, y) \in \operatorname{gph} G$.

These arguments show (C1) and (C2). The discussion following the statement of (C3) implies that condition is satisfied. We now argue that (C4) holds. If it did not, there would exist $x \in O$ and sequences $x_i \to x$, $y_i \in G_i(x_i)$ such that the sequence $\{y_i\}_{i=1}^{\infty}$ was not uniformly bounded with respect to O. Now $y_i \in G_i(x_i)$ means that $y_i \in \eta_i + \delta_i\alpha(\eta_i)\mathbb{B}$ while $\eta_i \in G(x_i + \delta_i\alpha(x_i)\mathbb{B})$. Since $x_i + \delta_i\alpha(x_i)$ are eventually contained in an arbitrarily small neighborhood of x, and G is locally bounded with respect to O, η_i's are bounded with respect to O. Using local boundedness of α again we obtain that y_i's are bounded with respect to O. This is a contradiction. □

Continuity of α is not essential for the above arguments, local boundedness is sufficient. Continuity implies that the outer perturbations actually satisfy (A1)-(A4).

Corollary 5.5 *Assume (A0)-(A4) (for the system $\mathcal{H}$) and (C1)-(C4). Suppose that $\mathcal{H}$ is forward complete at every $x^0 \subset K$ for some compact set K. Assume that perturbations $\mathcal{H}_\delta$ have the convergence property (CP). Then, for any $\varepsilon > 0$ and $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ there exists $\delta^* > 0$ with the following property: for any $\delta \in (0, \delta^*]$ and any $x_\delta \in \mathcal{S}_\delta(K + \delta\mathbb{B})$ there exists a solution x to $\mathcal{H}$ with $x(0, 0) \in K$ such that x_δ and x are (T, J, ε)-close.*

Proof. If the conclusion is false, then there exist $\varepsilon > 0$; $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, a sequence of systems $\mathcal{H}_i$ given by C_i, D_i, F_i, and G_i as in Theorem 5.4, and a sequence $x_i \in \mathcal{S}_i(x_i^0)$ with $x_i^0 \to x^0$ for some $x^0 \in K$ such that, for each i, no solution x to $\mathcal{H}$ with $x(0, 0) = x^0$ is (T, K, ε)-close to x_i. Without loss of generality, we can suppose that for each i, $\operatorname{dom} x_i \subset [0, T] \times \{0, 1, ...J\}$. By Lemma 5.2, all sufficiently large x_i's are uniformly bounded on $[0, T] \times \{0, 1, ...J\}$, and a subsequence of them converges to some $x \in \mathcal{S}(x^0)$, by Theorem 5.1. Lemma 4.2 applied to this subsequence and x yields a contradiction. □

Example 5.6 (Temporal regularization)**.** Given a hybrid system $\mathcal{H}$ on O, with F nonempty and convex valued on O (not just on C), consider an augmented system $\widetilde{\mathcal{H}}_\delta$ with state space $\widetilde{O} = O \times \mathbb{R}$ and variable $\tilde{x} = (x, \tau)$:

$$\widetilde{C}_\delta = (C \times \mathbb{R}_{\geq 0}) \cup (O \times [0, \delta]), \quad \widetilde{D}_\delta = D \times \mathbb{R}_{\geq \delta},$$
$$\widetilde{F}(\tilde{x}) = F(x) \times \{1\}, \quad \widetilde{G}(\tilde{x}) = G(x) \times \{0\},$$

Similar augmented systems with $\delta > 0$ were considered in Johansson, Egerstedt, Lygeros, and Sastry (1999) to eliminate Zeno behavior. Indeed, when $\delta > 0$ jumps are separated by at least δ amount of time. When $\delta = 0$, the behavior of the x component of the solution is exactly that of $\mathcal{H}$. Such a temporal regularization has the convergence property (CP). Thus, the conclusions of Theorem 5.5 are valid. In turn, the x component of each solution to the regularized system is close to some solution of $\mathcal{H}$ on compact hybrid time domains.

6 Applications to stability: compact attractors

For a hybrid system $\mathcal{H} = (F, G, C, D)$ with state space O, a compact set $\mathcal{A} \subset O$ is called stable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x^0 \in (\mathcal{A} + \delta\mathbb{B}) \cap (C \cup D)$, each solution $x \in \mathcal{S}(x^0)$ is complete and satisfies $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \operatorname{dom} x$ (here and in what follows, $|x|_{\mathcal{A}}$ denotes the distance from $\mathcal{A}$ to x). It is called attractive if there exists $\mu > 0$ such that, for each $x^0 \in (\mathcal{A} + \mu\mathbb{B}) \cap (C \cup D)$, each $x \in \mathcal{S}(x^0)$ is complete and satisfies $\lim_{t+j\to\infty} |x(t,j)|_{\mathcal{A}} = 0$. [2] The set of points from which solutions are complete and converge to $\mathcal{A}$ is called the basin of attraction for $\mathcal{A}$ and is denoted $\mathcal{B}_{\mathcal{A}}$. The set $\mathcal{A}$ is called locally asymptotically stable if it is both stable and attractive, and uniformly attractive from the compact set $\mathcal{K} \subset O$ if for each $x^0 \in \mathcal{K}$ each $x \in \mathcal{S}(x^0)$ is complete and for each $\varepsilon > 0$ there exists m such that $x \in \mathcal{S}(\mathcal{K})$ and $t + j \geq m$ imply $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$.

From now on, we assume (A0) through (A4). Other assumptions will be made when needed. Propositions 6.1 – 6.4 when specialized to differential or difference inclusions, have appeared recently in Clarke, Ledyaev, and Stern (1998), Teel and Praly (2000), Kellett and Teel (2004a), and Kellett (2002).

Proposition 6.1 *If a compact set $\mathcal{A}$ is forward invariant and uniformly attractive from a compact set containing a neighborhood of $\mathcal{A}$ in $C \cup D$ then it is stable, and hence locally asymptotically stable.*

Proof. Let $\varepsilon > 0$ be given. Using the uniform attractivity, let $\mu > 0$ and m be such that for each $x^0 \in (\mathcal{A} + \mu\mathbb{B}) \cap (C \cup D)$, each $x \in \mathcal{S}(x^0)$ is complete and $t + j \geq m$ implies $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$. Next, using forward invariance and Corollary 4.8, let $\delta \in (0, \mu)$ be such that $x^0 \in (\mathcal{A} + \delta\mathbb{B}) \cap (C \cup D)$ implies that, for each $x \in \mathcal{S}(x^0)$, $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $t + j \leq m$. It now follows that for each $x^0 \in (\mathcal{A} + \delta\mathbb{B}) \cap (C \cup D)$ and each $x \in \mathcal{S}(x^0)$, $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \operatorname{dom} x$. Thus, $\mathcal{A}$ is stable. ◻

Proposition 6.2 *A locally asymptotically stable compact set $\mathcal{A} \subset O$ is uniformly attractive from each compact set $K \subset \mathcal{B}_{\mathcal{A}}$.*

Proof. Assume otherwise. Then for some $\varepsilon > 0$, $m > 0$, and a sequence of complete $x_i \in \mathcal{S}(K)$, some $(t_i, j_i) \in \operatorname{dom} x_i$ with $t_i + j_i > m$ satisfies $|x_i(t_i, j_i)|_{\mathcal{A}} > \varepsilon$. By Theorem 4.6 and forward completeness of $\mathcal{H}$ on K, the sequence of x_i's is locally eventually bounded. By Theorem 4.4, we can assume it converges graphically to a complete $x \in \mathcal{S}(K)$. By stability of A, there exists $\delta > 0$ such that $|x_i(t,j)|_{\mathcal{A}} > \delta$ for all $(t,j) \in \operatorname{dom} x_i$ with $(t,j) \preceq (t_i, j_i)$, $i = 1, 2, \ldots$. This implies that $\|x(t,j)\|_{\mathcal{A}} \geq \delta$ for all $(t,j) \in \operatorname{dom} x$, which contradicts $x(0,0) \in \mathcal{B}_{\mathcal{A}}$. ◻

[2] In the definitions of stability and attractivity, the condition that $x^0 \in C \cup D$ can be omitted, as by definition, solutions of the hybrid system have initial points in $C \cup D$.

Proposition 6.3 *Let $\mathcal{A} \subset O$ be compact and locally asymptotically stable, and let $K \subset \mathcal{B}_{\mathcal{A}}$ be compact. Then*

$$\mathcal{A} \cup \bigcup_{(T,J)\in\mathbb{R}_{\geq 0}\times\mathbb{N}} \operatorname{reach}_{T,J}(K) \tag{4}$$

is a compact subset of $\mathcal{B}_{\mathcal{A}}$.

Proof. Any sequence of points in (4) either has a subsequence that approaches $\mathcal{A}$ (from which a convergent sequence with the limit in $\mathcal{A}$ can be picked by compactness of $\mathcal{A}$) or is contained in $\operatorname{reach}_{T,J}(K)$ for some T, J, by Proposition 6.2. But $\operatorname{reach}_{T,J}$ is compact by Corollary 4.7, and this leads to compactness of (4). ◻

Based on the compactness of the reachable set, and Theorem 4.4, we observe that given a compact subset K of the basin of attraction, either times between jumps are uniformly bounded below by $\tau > 0$ over all trajectories originating in K, or for some trajectory there exist multiple instantaneous jumps. Consequently, in our setting, existence of a uniform bound over all trajectories, and existence of a bound dependent on trajectory, are equivalent. This may not be the case in other frameworks (see for example Ye, Michel, and Hou (1998) for different converse Lyapunov theorems for the two cases).

Proposition 6.4 *Assume that (VD) of Proposition 2.4 holds, and that (VC) of Proposition 2.4 holds for any $x^0 \in C \setminus D$. Then, for any attractive compact set $\mathcal{A} \subset O$, its basin of attraction $\mathcal{B}_{\mathcal{A}}$ is open relative to $C \cup D$.*

Proof. Assume otherwise, that for some $x^0 \in \mathcal{B}_{\mathcal{A}}$ there exist maximal solutions x_i to $\mathcal{H}$, each of which is either not complete or does not converge to $\mathcal{A}$, and such that $x_i(0,0) \to x^0$. If x_i is not complete, then by Proposition 2.4 it eventually leaves any compact subset of O, and thus it does not converge to $\mathcal{A}$. Hence, we can assume that each of x_i's does not converge to A. By Theorem 4.6 and forward completeness of $\mathcal{H}$ at x^0, the sequence of x_i's is locally eventually bounded. By Theorem 4.4, we can assume it converges graphically to a solution x. By Lemma 4.5, x is complete. Lastly, by attractivity of A, there exists $\mu > 0$ such that $|x_i(t,j)|_{\mathcal{A}} > \mu$ for all $(t,j) \in \operatorname{dom} x_i$, all $i = 1, 2, \ldots$. This implies that $|x(t,j)|_{\mathcal{A}} \geq \mu$ for all (t,j), which contradicts $x(0,0) = x^0 \in \mathcal{B}_{\mathcal{A}}$. ◻

A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KLL}$ if it is continuous, $\beta(\cdot, t, j)$ is 0 at 0 and nondecreasing, $\beta(s, \cdot, j)$ and $\beta(s, t, \cdot)$ are nonincreasing and converge to 0 as arguments increase to ∞. A function $\omega : U \to \mathbb{R}_{\geq 0}$ is a proper indicator of a compact $\mathcal{A} \subset U$ with respect to an open U if it is continuous, positive definite with respect to $\mathcal{A}$, and such that $\omega(x) \to \infty$ as $x \to \partial U$ (boundary of U) or $\|x\| \to \infty$.

The proof of Theorem 6.5 follows that of Proposition 3 in Teel and Praly (2000), while the proof of Theorem 6.6 uses some ideas of the proof of Theorem 1 in Teel, Moreau, and Nesic (2003).

Theorem 6.5 ($\mathcal{KLL}$-bound) *Suppose that the basin of attraction $\mathcal{B}_\mathcal{A}$ of a compact set $\mathcal{A} \subset O$ is open relative to $C \cup D$. Let $U \subset O$ be any open set such that $\mathcal{B}_\mathcal{A} = (C \cup D) \cap U$. For each proper indicator $\omega : U \to \mathbb{R}_{\geq 0}$ of $\mathcal{A}$ with respect to U there exists $\beta \in \mathcal{KLL}$ such that, for all solutions starting in $\mathcal{B}_\mathcal{A}$,*

$$\omega(x(t,j)) \leq \beta(\omega(x(0,0)), t, j) \qquad \forall (t,j) \in \operatorname{dom} x\ . \quad (5)$$

Proof. Given ω, define $\alpha : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$\alpha(r,s) := \sup\{\omega(x(t,j)) \mid \omega(x(0,0)) \leq r, t+j \geq s\},$$

with the supremum taken over all solutions x to $\mathcal{H}$ and $(t,j) \in \operatorname{dom} x$. By Proposition 6.3, $\alpha(r,s)$ is finite for all (r,s). By definition of α, we have

$$\omega(x(t,j)) \leq \alpha(\omega(0,0), t+j)$$

for all solutions x with $x(0,0) \in \mathcal{B}_\mathcal{A}$, all $(t,j) \in \operatorname{dom} x$. Also, $\alpha(r,s)$ is nondecreasing in r and nonincreasing in s. By stability of $\mathcal{A}$, $\lim_{r \searrow 0} \alpha(r,s) = 0$ for all s, while by continuity and growth properties of ω (in particular, by compactness of $\{z \in \mathcal{B}_\mathcal{A} \mid \omega(z) \leq r\}$) and uniform convergence as stated in Proposition 6.2, $\lim_{s \to \infty} \alpha(r,s) = 0$ for all r. Thus, α has the properties normally required of $\mathcal{KL}$ functions except continuity. There does exist a function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that has these properties and is continuous, and moreover $\alpha(r,s) \leq \gamma(r,s)$ for all $r, s \in \mathbb{R}_{\geq 0}$; see for example Remark 3 in Teel and Praly (2000). Now, let

$$\beta(r,t,j) := \gamma(r, t+j).$$

This function has the required properties. ☐

Theorem 6.6 ($\mathcal{KLL}$-bound under perturbations) *Suppose that the basin of attraction $\mathcal{B}_\mathcal{A}$ of a compact set $\mathcal{A} \subset O$ is open relative to $C \cup D$, $U \subset O$ is any open set such that $\mathcal{B}_\mathcal{A} = (C \cup D) \cap U$, and $\omega : U \to \mathbb{R}_{\geq 0}$ is a proper indicator of $\mathcal{A}$ with respect to U, and $\beta \in \mathcal{KLL}$ is such that, for all solutions starting in $\mathcal{B}_\mathcal{A}$, (5) holds. Assume that the family of perturbed systems $\mathcal{H}_\delta$, $\delta \in (0,1)$, has the convergence property (CP) of the previous section. Then, for each compact set $\mathcal{K} \subset \mathcal{B}_\mathcal{A}$ and each $\varepsilon > 0$ there exists $\delta^* > 0$ such that for each $\delta \in (0, \delta^*]$, the solutions x_δ of $\mathcal{H}_\delta$ from $\mathcal{K}$ satisfy, for all $(t,j) \in \operatorname{dom} x_\delta$,*

$$\omega(x_\delta(t,j)) \leq \beta(\omega(x_\delta(0,0)), t, j) + \varepsilon\ . \quad (6)$$

Proof. As ω is continuous, for each compact $\mathcal{K}$ there exists $m > 0$ such that $\mathcal{K} \subset \{x \in \mathcal{B}_\mathcal{A} \mid \omega(x) \leq m\}$. The latter set is also compact by the growth properties of ω. Fix $\varepsilon > 0$ and $m > \varepsilon$. Pick $T > 0$, $J > 0$ large enough so that $\beta(m,t,j) \leq \varepsilon/2$ when either $t \geq T$ or $j \geq J$. We claim that there exists $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$, all solutions x_δ to $\mathcal{H}_\delta$ with $\omega(x_\delta(0,0)) \leq m$ satisfy

$$\omega(x_\delta(t,j)) \leq \beta(\omega(x_\delta(0,0)), t, j) + \varepsilon/2 \quad (7)$$

for all $(t,j) \in \operatorname{dom} x_\delta$ with $t \leq 2T$, $j \leq 2J$. This implies $\omega(x_\delta(t,j)) \leq \varepsilon$ for all $(t,j) \in \operatorname{dom} x$ with $t \leq 2T$, $j \leq 2J$ but either $t \geq T$ or $j \geq J$. Using this fact recursively and relying on $m > \varepsilon$ shows that $\omega(x_\delta(t,j)) \leq \varepsilon$ when $t \geq T$ or $j \geq J$. This, and (7), shows (6).

To see the claim, suppose otherwise: that there exists a sequence x_i of solutions to $\mathcal{H}_{\delta_i}$, $\delta_i \searrow 0$ with $\omega(x_i(0,0)) \leq m$ and points $(t_i, j_i) \in \operatorname{dom} x_i$ with $t_i \leq 2T$, $j_i \leq 2J$ so that (7) does not hold, i.e.

$$\omega(x_i(t_i, j_i)) > \beta(\omega(x_i(0,0)), t_i, j_i) + \varepsilon/2.$$

Since $\omega(x_i(0,0)) \leq m$ implies that points $x_i(0,0) \in C_i \cup D_i$ all lay in some compact subset of U, one can assume that they converge to some point in $U \cap (C \cup D) = \mathcal{B}_\mathcal{A}$. At this point, $\mathcal{H}$ is forward complete. Relying on Theorem 5.1 and Corollary 5.2, one can extract a graphically convergent, to a solution x of $\mathcal{H}$, subsequence of x_i's. Now, extracting a convergent, to some (t,j), subsequence of (t_i, j_i)'s, and using continuity of β and ω shows that (5) is violated by x at (t,j). This is a contradiction. ☐

Theorem 6.6 does not address the existence of solutions to perturbed systems $\mathcal{H}_\delta$. Under our assumptions, not much can be said about this, as we do not assume any regularity of C_δ, D_δ, F_δ, and G_δ. When $\mathcal{H}_\delta$ is the "outer perturbation" of $\mathcal{H}$ as in Example 5.3, then the data for $\mathcal{H}_\delta$ satisfies (A1), (A2), (A3), and if furthermore $C \cup D = O$, and consequently $C_\delta \cup D_\delta = O$, then solutions to $\mathcal{H}_\delta$ do exist for any initial point in O. However, even then the existence of solutions to a system with exogenous inputs $\dot{x} \in F(x+e)$ if $x + e \in C$ and $x^+ \in G(x+e)$ if $x + e \in D$, for example to $\mathcal{H}$ under measurement error, remains problematic. We do not pursue this topic here.

The following result is inspired by and relies on the result of Collins (2004) (see Corollary 4.9).

Theorem 6.7 *Under the assumptions of Theorem 6.6, if the hybrid system $\mathcal{H}$ has no instantaneous Zeno solutions in $\mathcal{B}_\mathcal{A}$, then for each compact set $\mathcal{K} \subset \mathcal{B}_\mathcal{A}$, there exists $\delta^* > 0$ such that all solutions of $\mathcal{H}_\delta$ starting in $\mathcal{K}$ are uniformly non-Zeno for all $\delta \in (0, \delta^*]$.*

Proof. Suppose otherwise, that there exist solutions $x_i \in \mathcal{S}_{1/i}(\mathcal{K})$ and points $(t_i, j_i), (t_i', j_i') \in \operatorname{dom} x_i$ with $(t_i, j_i) \preceq (t_i', j_i')$ and $t_i' - t_i \leq 1/i$ and $j_i' - j_i \geq i$. For each

i define a solution x_i' to $\mathcal{H}_{1/i}$ by $x_i'(t,j) = x_i(t+t_i, j+j_i)$. As $x_i'(0,0) = x_i(t_i, j_i)$, we can find ω and β as in Theorem 6.6 so that $\omega(x_i'(0,0)) \leq \beta(\omega(x_i(0,0)),0,0) + \varepsilon$ for all large enough i. As $x_i(0,0) \in \mathcal{K}$, $\omega(x_i'(0,0)) \leq m$ for some $m > 0$ and large enough i's. As in the last part of the proof of Theorem 6.6, we can assume that x_i's are locally eventually bounded and converge graphically to a solution x to $\mathcal{H}$. As $(t_i' - t_i, j_i' - j_i) \in \operatorname{dom} x_i'$, x is complete and instantaneous Zeno. This is a contradiction. □

Example 6.8 (Temporal regularization for a stabilizing controller that produces Zeno solutions)**.** Consider controlling the state x_1 of an integrator using an actuator, with state x_2, that can decrease at a constant rate (normalized to minus one) and can be reset in the positive direction instantaneously. The flow equation is $\dot{x}_1 = x_2$, $\dot{x}_2 = -1$. With $k_r > k_c \geq 2$, $\eta \in [0,1)$, we consider the flow set $C := \left\{x : x_1 \geq 0 \text{ or } x_2 \geq \sqrt{-k_c x_1}\right\}$, the jump set $D := \overline{\mathbb{R}^2 \backslash C}$ and the reset rule

$$x_2^+ = \max\{0, -\eta x_2\} + \sqrt{-k_r x_1}.$$

One can verify that $\{x : x_1 \geq 0\}$ is forward invariant, that trajectories reach this set in finite time, and within this set the solutions are the same as those of the bouncing ball example (see, for example, Lygeros, Johansson, Simić, Zhang, and Sastry (2003)) with $g = 1$. In particular, the origin is forward invariant (the only solution from the origin is the constant instantaneous Zeno solution) and the origin is uniformly attractive. Thus, by Proposition 6.1, the origin is globally asymptotically stable. To eliminate the said instantaneous Zeno solution, we use a temporal regularization with the flow set $\widetilde{C}_\delta$ and $\widetilde{D}_\delta$ as in Example 5.6 for $\delta \geq 0$, and additional dynamics $\dot{\tau} = 1 - \tau$, $\tau^+ = 0$. We note that the set $\mathbb{R}_{\geq 0}$ is forward invariant for τ, independently of $\delta \geq 0$. When $\delta = 0$, the x component of the solution is exactly the solution without the temporal regularization and τ converges uniformly to the interval $[0,1]$. Thus, the compact set $\mathcal{A} := \{(x,\tau) : \|x\| = 0\ ,\ \tau \in [0,1]\}$ is asymptotically stable when $\delta = 0$ with basin of attraction $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. By Theorem 6.5, with $z = (x,\tau)$ and $\omega(z)$ given by $\|x\| + \max\{0, \delta - 1\}$, there exists $\beta \in \mathcal{KLL}$ such that,

$$\omega(z(t,j)) \leq \beta(\omega(z(0,0)), t, j)$$

for each solution to the system. According to Theorem 6.6 and the remark that follows it, for each $\varepsilon > 0$ and compact set $K \subset \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ there exists $\delta^* > 0$ such that the solutions of the hybrid system $\widetilde{H}_\delta$, $\delta \in (0, \delta^*]$, satisfy

$$\omega(z_\delta(t,j)) \leq \beta(\omega(z_\delta(0,0)), t, j) + \varepsilon.$$

In particular, for $\tau(0,0) \in [0,1]$,

$$\|x_\delta(t,j)\| \leq \beta(\|x_\delta(0,0)\|, t, j) + \varepsilon.$$

7 Conclusions

We have proposed a novel framework for the modeling and analysis of hybrid systems, and established elementary properties – in particular, sequential compactness – of the sets of solutions to such systems, also in the presence of perturbations. We have applied these results to the study of basic properties of asymptotically stable hybrid systems, and to the question of how the said stability behaves under perturbations. These developments confirm that in a right framework, and with an appropriate notions of a solution to a hybrid system and of convergence of solutions, many results that have proved fundamental in the study of continuous time (or discrete time) systems can be extended to a hybrid setting.

We view the results of this paper not as a complete body of work, but rather, as a foundation upon which many further investigations can be built. Already, some of the results have been applied to very important problems in control theory. More specifically:

- Working with hybrid systems in the framework subsuming that of the current paper, Sanfelice, Goebel, and Teel (2005) established several invariance principles and related detectability and invariance to asymptotic stability. For such results, Lemma 4.3 and Theorem 4.4 were the key tool.
- In Cai, Teel, and Goebel (2005), the existence of smooth Lyapunov functions for hybrid systems was tied to the robustness of asymptotic stability, and sufficient conditions for the latter were given. To a large extent, it was Theorem 5.1 and Corollary 5.5 that made such results possible.
- The existence of a hybrid feedback that stabilizes any given asymptotically controllable nonlinear system, robustly to measurement noise, actuator error, and external disturbances, was shown in Prieur, Goebel, and Teel (2005). This required versions of Theorems 6.5 and 6.6, and Lemma 4.3.

Many other issues related to hybrid controllers and hybrid systems await solutions. We do believe that the framework and the results of this paper will make some of the solutions possible.

8 Appendix: proof of Lemma 3.7

We first show (a). If $a < b$, we can assume that $[a_i, b_i] = [a, b]$ for all i. (Otherwise, one considers $\widetilde{x}_i : [a,b] \to \mathbb{R}^n$ given by $\widetilde{x}_i(t) = x_i\left(\frac{b-t}{b-a}a_i + \frac{t-a}{b-a}b_i\right)$ and $\widetilde{F}_i = \frac{b_i - a_i}{b-a}F_i$.) By boundedness assumptions on x_i's and F_i's, arcs x_i are Lipschitz continuous (with the same constant). This in particular implies that x is single-valued, x_i's converge pointwise to x, and also, that from every sequence of $\dot{x}_i$'s, we can pick a weakly convergent (in $L^1[a,b]$) subsequence (Dunford-Pettis Theorem). Boundedness and

Lipschitz continuity of x_i's implies that the first inclusion below holds for some $\rho > 0$, while the second holds for all $\varepsilon > 0$, all $i > i_\varepsilon$ by Theorem 3.3: for a.a. $t \in [a, b]$,

$$(x_i(t), \dot{x}_i(t)) \in \operatorname{gph} F_i \cap \rho\mathbb{B} \subset \operatorname{gph} F + \varepsilon\mathbb{B}.$$

Now, the Convergence Theorem of Aubin and Cellina (1984) concludes that $\dot{x}(t) \in F(x(t))$ for almost all $t \in [a, b]$. If $a = b$, uniform Lipschitz continuity of x_i's easily leads to $x(a)$ being a singleton.

To see (b), let T be the minimum of all τ's for which there exist $t_i \in [a_i, b_i]$ with $t_i \to \tau$ and $\lim_{i\to\infty} x_i(t_i) \notin O$ (at least one such $\tau \leq b$ exists, as x_i's are not uniformly bounded with respect to O). Pick $\varepsilon > 0$, $L > 0$ so that $1+\|F_i(x)\| \leq L$ for all i, all x with $x \in x_0+2\varepsilon\mathbb{B}$. As for all large enough i, $x_i(a_i) \in x_0+\varepsilon\mathbb{B}$, we have $x_i(t) \in x_0+2\varepsilon\mathbb{B}$ for all $t \in [a_i, a_i + \varepsilon/L]$. Thus $a < T$. By (a), for any $a < T' < T$ the arc x is absolutely continuous on $[a, T']$, and $\dot{x}(t) \in F(x(t))$ a.e. on $[a, T)$. If x does not eventually leave a given compact $K \subset O$, then by arguments as in the last paragraph of the proof of Proposition 2.4, show that $x(t)$ is contained in some compact $K' \subset O$, for all $t \in [a, T)$. This contradicts the definition of T.

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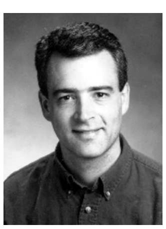
Rafal Goebel received his M.Sc. degree in mathematics in 1994 from the University of Maria Curie Sklodowska in Lublin, Poland, and his Ph.D. degree in mathematics in 2000 from University of Washington, Seattle. He held postdoctoral positions at the Departments of Mathematics at the University of British Columbia and Simon Fraser University in Vancouver, Canada, 2000 – 2002, and at the Electrical and Computer Engineering Department at the University of California, Santa Barbara, 2002 – 2004. Since then he has held part-time research and teaching positions at Electrical and Computer Engineering Department at the University of California, Santa Barbara, and the Department of Mathematics at the University of Washington. His interests include convex, nonsmooth, and set-valued analysis, control, hybrid systems, and optimization.



Andrew R. Teel received his A.B. degree in Engineering Sciences from Dartmouth College in Hanover, New Hampshire, in 1987, and his M.S. and Ph.D. degrees in Electrical Engineering from the University of California, Berkeley, in 1989 and 1992, respectively. After receiving his Ph.D., Dr. Teel was a postdoctoral fellow at the Ecole des Mines de Paris in Fontainebleau, France. In September of 1992 he joined the faculty of the Electrical Engineering Department at the University of Minnesota where he was an assistant professor until September of 1997. In 1997, Dr. Teel joined the faculty of the Electrical and Computer Engineering Department at the University of California, Santa Barbara, where he is currently a professor.