PRE-ASYMPTOTIC STABILITY AND HOMOGENEOUS APPROXIMATIONS OF HYBRID DYNAMICAL SYSTEMS *

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Abstract. Hybrid dynamical systems are systems that combine features of continuous-time dynamical systems and discrete-time dynamical systems, and can be modeled by a combination of differential equations or inclusions, difference equations or inclusions, and constraints. Pre-asymptotic stability is a concept that results from separating the conditions that asymptotic stability places on the behavior of solutions from issues related to existence of solutions. In this paper, techniques for approximating hybrid dynamical systems that generalize classical linearization techniques are proposed. The approximation techniques involve linearization, tangent cones, homogeneous approximations of functions and set-valued mappings, and tangent homogeneous cones, where homogeneity is considered with respect to general dilations. The main results deduce pre-asymptotic stability of an equilibrium point for a hybrid dynamical system from pre-asymptotic stability of the equilibrium point for an approximate system. Further results relate the degree of homogeneity of a hybrid system to the Zeno phenomenon that can appear in the solutions of the system.

Key words. hybrid dynamical system, asymptotic stability, homogeneous approximation, tangent cone, linearization

AMS subject classifications. 34A38, 93D20, 41A29, 34A60, 49J53

1. Introduction. A fundamental result in the theory of differential equations deduces asymptotic stability of an equilibrium point from asymptotic stability for the linear approximation to the differential equation. The main goal of this paper is to give parallel and far more general results in the setting of hybrid dynamical systems.

Hybrid dynamical systems combine behaviors typical of continuous-time dynamical systems with behaviors typical of discrete-time dynamical systems and are of increasing interest in control engineering, computer science, and systems biology. Examples include circuits combining analog and digital components, mechanical devices controlled by computers, biological systems exhibiting impulsive behavior, or systems where various behaviors occur on dramatically different time scales, like in mechanical systems with impacts. This paper models hybrid systems by hybrid inclusions, which generalize differential equations as well as difference equations. A hybrid inclusion combines a differential inclusion, a difference inclusion, and constraints on the motions resulting from the differential and difference inclusions.

The framework of hybrid inclusions builds on the settings used in [5], [6], [21], and [13]. The work [13], and the concurrent [10], outlines some elementary structural properties of the solution space to a hybrid inclusion. Consequences of those properties for asymptotic stability in hybrid systems, some of which we rely on in this paper, appear in [14]. There are mathematical frameworks that also target continuous-time dynamical systems with occasional discontinuous behaviors, like measure driven differential equations or inclusions [24], [11], [29], or that unify differential and difference equations, like dynamical systems on time scales [7]. In general, these frameworks do not overlap with the hybrid systems considered here.

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Homogeneity is often used in stability analysis for differential or difference equations or inclusions. Results deducing asymptotic stability of a system from asymptotic stability of its homogeneous approximation date back to [20], where linearization was used, [22] and [23], where standard homogeneity is considered, and in [17], [26], where non-standard dilations are used. Control design results relying on homogeneity appear in [19], [16], and [31]. Generalized homogeneity for hybrid systems is studied in [32], and a result related to linearization for hybrid systems is given in [21].

Tangent cones to constraint sets play an important role in viability theory for differential equations or inclusions [3]. Conical approximations of differential inclusions and control systems were used in [12] and [30] in the study of controllability and stabilizability properties. The most general result of the current paper blends homogeneous approximation and conical approximation concepts. In particular, it involves homogeneous approximations of set-valued mappings, similar to the idea of graphical derivatives [25, Chapter 8, Section G] but relying on generalized homogeneity. Homogeneous, with respect to non-standard dilations, approximations of sets were used in [2] in the study of optimal control problems. The tangent homogeneous cones used in this paper agree in some cases with the set of homogeneous tangent vectors of [2].

The main results of this paper show how pre-asymptotic stability of an equilibrium point can be deduced for a hybrid system from pre-asymptotic stability of the equilibrium point for a homogeneous approximation of the hybrid system. Pre-asymptotic stability is closely related to asymptotic stability, but allows for solutions that are not complete. Rigorous statements of the results require introducing several definitions. However, the ideas can be illustrated now.

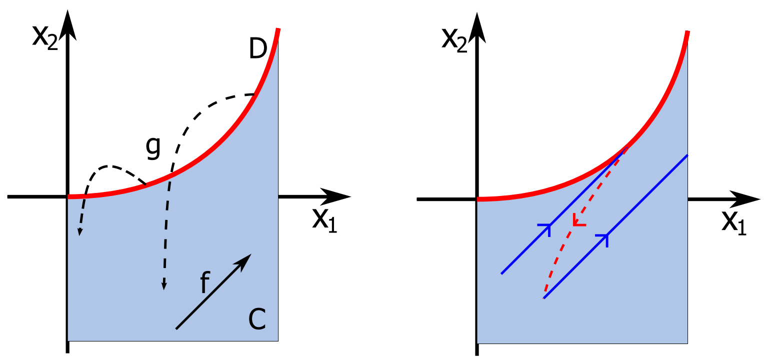

Fig. 1.1. *The hybrid system from Example 1.1. Solutions may flow in the set* $C = \{x \in \mathbb{R}^2 \,|\, x_1 \geq 0, x_2 \leq x_1^2\}$ *and may jump from the set* $D = \{x \in \mathbb{R}^2 \,|\, x_1 \geq 0, x_2 = x_1^2\}$. *The solid arrow indicates the direction of flow, given by* $f(x) = (1,1)$, *and the dashed arrows indicate jumps, given by* $g(x) = (x_1^2, -x_1/2)$. *The left graph shows a sample divergent solution.*

Example 1.1. *Consider a hybrid system in* $\mathbb{R}^2$ *for which solutions may flow according to a differential equation*

$$\dot{x} = f(x) := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{in the set} \quad C := \left\{x \in \mathbb{R}^2 \,|\, x_1 \geq 0, x_2 \leq x_1^2\right\},$$

and they may jump according to a difference equation

$$x^+ = g(x) := \begin{pmatrix} x_1^2 \\ -x_1/2 \end{pmatrix} \qquad \text{from the set} \quad D := \left\{x \in \mathbb{R}^2 \,|\, x_1 \geq 0, x_2 = x_1^2\right\}.$$

For example, a solution from $(1/4, 0)$ may flow with velocity $(1, 1)$ while remaining in the set C for $1/4$ units of time, at time $1/4$ reach the set D at the point $z = (1/2, 1/4)$, jump to a point $g(z) = (1/4, -1/4)$, and then flow with velocity $(1, 1)$ forever. See Figure 1.1. Careful analysis shows that solutions from initial points in C sufficiently close to the origin remain close to the origin and converge to it by flowing, reaching D and jumping, flowing again, reaching D and jumping again, and so on. One can ask whether there is a simpler way to predict that solutions from near the origin remain close to the origin and converge to it.

This paper suggests approximating the hybrid system with a simpler one. For example, one can consider an approximation for which the flow is determined by

$$\dot{x} = f(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{in the set} \quad T_C(0) = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\leq 0},$$

with $T_C(0)$ being the tangent cone to C at 0, and the jumps are determined by

$$x^+ = g'(0)x = \begin{pmatrix} 0 \\ -x_1/2 \end{pmatrix} \qquad \text{from the set} \quad T_D(0) = \mathbb{R}_{\geq 0} \times \{0\},$$

with $T_D(0)$ being the tangent cone to D at 0. See Figure 1.2. For this approximation, solutions from nonzero initial conditions experience infinitely many periods of flow with jumps in between those periods and converge to the origin, and solutions from near the origin remain near it. This can be shown using a Lyapunov function $V(x) = 2x_1 - 3x_2$, which is positive definite on $T_C(0) \cup T_D(0)$ and satisfies $\nabla V(x) \cdot f(0) = -1$ for all $x \in T_C(0)$ and $V(g'(0)x) = 3/4 \cdot V(x)$ for all $x \in T_D(0)$. Theorem 3.3 concludes, from the properties of the approximation, that solutions to the original system, from initial points close to the origin, remain close to the origin and converge to it.

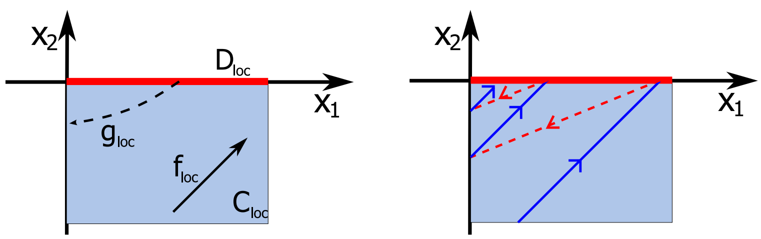

Fig. 1.2. *Approximation of the hybrid system from Example 1.1. Solutions may flow in the $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\leq 0}$, which is the tangent cone to C, and they may jump from the set $\mathbb{R}_{\geq 0} \times \{0\}$, which is the tangent cone to D. The solid arrow indicates the direction of flow, which is determined by $f(0) = (1, 1)$, and the dotted arrows indicate the jumps, which are determined by $g'(0)x = (0, -x_1/2)$. The graph on the right shows a sample solution.*

Another question to ask is "how much time does it take for such solutions to converge to the origin?". Clearly, solutions from near the origin experience infinitely many jumps, but it is less clear that those jumps happen in a finite amount of time.

Another use of homogeneity, in the framework of differential equations, is its implication for convergence rates. Non-exponential convergence is associated with positive degree of homogeneity for a system, exponential convergence is associated with degree zero, and finite-time convergence is associated with negative degree. In this paper, we connect homogeneity to the amount of time — but not to the number of jumps — it takes for solutions to converge to the origin in a pre-asymptotically stable hybrid system. This issue is closely related to the Zeno phenomenon, which is the occurrence of infinitely many jumps of a solution in a finite amount of ordinary time. The Zeno phenomenon poses challenges for simulation and may have an adverse effect on performance of hybrid control algorithms. A vast literature in control engineering and computer science tackles these issues, for example [18], [8], and [33]. Conditions guaranteeing or ruling out Zeno phenomenon have also been studied, via approximation of solutions [28], or approximation of systems [1]. An often used example of a system with Zeno phenomenon is the bouncing ball.

Example 1.2. *Consider a hybrid system in $\mathbb{R}^2$ for which solutions may flow according to a differential equation*

$$\dot{x} = f(x) := \begin{pmatrix} x_2 \\ \gamma \end{pmatrix} \qquad \text{in the set} \quad C := \left\{ x \in \mathbb{R}^2 \,|\, x_1 \geq 0 \right\},$$

and they may jump according to a difference equation

$$x^+ = g(x) := \begin{pmatrix} x_1 \\ -\rho x_2 \end{pmatrix} \qquad \text{from the set} \quad D := \left\{ x \in \mathbb{R}^2 \,|\, x_1 = 0, x_2 < 0 \right\}.$$

This is a simple model of a ball bouncing on a surface, with x_1 representing the ball's height, x_2 the velocity, $\gamma < 0$ the acceleration due to gravity, and $\rho \in (0,1)$ a dissipation of energy factor. The flow describes the ball evolving according to Newton's Law when above the surface, the jumps represent the instantaneous reversal of velocity when the ball bounces, with some energy dissipation. A calculus exercise shows that, for a ball dropped from a positive height, the times of consecutive bounces form a convergent sequence. In short, infinitely many bounces occur in finite time.

One of the results in this paper, Theorem 6.2, lets one conclude that infinitely many jumps occur in finite time without explicitly solving for the bounce times. Instead, it is sufficient to note that this hybrid system is homogeneous with a negative degree with respect to an appropriate dilation. Corollary 6.3 makes similar conclusions from a homogeneous approximation of a system. These two results also exclude nondegenerate Zeno behavior for systems with a nonnegative degree of homogeneity.

2. Preliminaries.

2.1. Hybrid systems.

The data of a hybrid system with state $x \in \mathbb{R}^n$ consists of four elements: a *flow set* C, a *flow map* F, a *jump set* D, and a *jump map* G. The following is assumed about these elements:

- C and D are sets in $\mathbb{R}^n$;
- $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping with $F(x) \neq \emptyset$ when $x \in C$;
- $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping with $G(x) \neq \emptyset$ when $x \in D$.

Roughly, solutions to a hybrid system are allowed to flow in the flow set according to a differential inclusion given by the flow map, and are also allowed to jump from the jump set according to a difference inclusion given by the jump map. This behavior can be summarized in the following representation:

$$\begin{cases} x \in C & \dot{x} \;\in\; F(x) \\ x \in D & x^+ \in G(x). \end{cases} \tag{2.1}$$

The notation $\dot{x}$ denotes the time derivative, while the notation x^+ represents the state after a jump. A special case of (2.1) is provided by systems where the flow and jump maps are functions, so that we have

$$\begin{cases} x \in C & \dot{x} \ = \ f\left(x\right) \\ x \in D & x^{+} = g\left(x\right). \end{cases} \tag{2.2}$$

Each solution of a hybrid system is parameterized by elements of a *hybrid time domain*, an ordered subset of $\mathbb{R}^2_{\geq 0}$. Hybrid time domains include $\mathbb{R}_{\geq 0} \times \{0\}$, which corresponds to a solution that never jumps; $\{0\} \times \mathbb{N}$, corresponding to a solution that never flows; and sets that are, in a sense, in between these two possibilities. A distinguishing feature of the framework adopted here is that different solutions to the same hybrid system may have different domains.

Definition 2.1. *A set $E \subset \mathbb{R}^2$ is a* compact hybrid time domain *if $E = \bigcup_{j=0}^{J} I_j \times \{j\}$, where $J \in \mathbb{N}$ and $I_j = [t_j, t_{j+1}]$, $j = 0, 1, \ldots, J$, for some $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{J+1}$. A set E is a* hybrid time domain *if, for each $(T, J) \in E$, the set $\{(t, j) \in E \,|\, t \leq T, j \leq J\}$ is a compact hybrid time domain.*

Equivalently, a hybrid time domain is a union of finitely or infinitely many intervals $[t_j, t_{j+1}] \times \{j\}$, where $0 = t_1 \leq t_2 \leq \ldots$, with the last interval — if it exists — possibly of the form $[t_j, t_{j+1})$ or $[t_j, \infty)$.

Definition 2.2. *A function $\phi : E \to \mathbb{R}^n$ is a* solution to the hybrid system (2.1) *if E is a hybrid time domain, $\phi(0, 0) \in \overline{C} \cup D$, and*

- *if $I_j := \{t \,|\, (t, j) \in E\}$ has nonempty interior, then $t \mapsto \phi(t, j)$ is locally absolutely continuous on I_j and*

$$\phi(t,j) \in C \ \text{ for all } t \in \operatorname{int} I_j \ \text{ and } \ \frac{d}{dt}\phi(t,j) \in F(\phi(t,j)) \ \text{ for almost all } t \in I_j;$$

- *if $(t, j) \in E$ and $(t, j + 1) \in E$ then*

$$\phi(t,j) \in D \ \text{ and } \ \phi(t,j+1) \in G(\phi(t,j)).$$

A solution $\phi : E \to \mathbb{R}^n$ is maximal *if it cannot be extended, and* complete *if E is unbounded.*

To save on notation, the domain of a solution ϕ to a hybrid system will be denoted $\operatorname{dom} \phi$. Caratheodory solutions to a differential equation $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$, correspond to solutions to a hybrid system with the flow map f, flow set $\mathbb{R}^n$, empty jump set, and arbitrary jump map. More precisely, a solution to $\dot{x} = f(x)$ on $[0, T]$ corresponds to a solution to the hybrid system with domain $[0, T] \times \{0\}$. Similarly, solutions to a difference equation or inclusion correspond to solutions of an appropriate hybrid system, whose domains are subsets of $\{0\} \times \mathbb{N}$.

Example 2.3. *Consider the hybrid system, proposed as an approximation of another system in Example 1.1, with the following data:*

$$C = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\leq 0}, \qquad f(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$D = \mathbb{R}_{\geq 0} \times \{0\}, \qquad g(x) = \begin{pmatrix} 0 \\ -x_1/2 \end{pmatrix}.$$

The maximal solution with initial condition $(0, -1)$, to the system (2.2) with the data as above, is given by

$$\operatorname{dom} \phi = \bigcup_{j=0}^{\infty} \left[\sum_{i=1}^{j} 2^{-i}, 2^{j} + \sum_{i=1}^{j} 2^{-i} \right] \times \{j\}, \qquad \phi(t,j) = \begin{pmatrix} t - \displaystyle\sum_{i=1}^{j} 2^{-i} \\ t - 2^{-j} \end{pmatrix}.$$

The maximal solution with initial condition 0 *is* $\operatorname{dom}\phi = \{0\} \times \mathbb{N}$, $\phi(0,j) = 0$. *Both of these solutions are complete. For the former solution, it is easy to check that only for* $t \in [0,2)$ *there exists* j *such that* $(t,j) \in \operatorname{dom}\phi$.

2.2. Pre-asymptotic stability. Usually, definitions of asymptotic stability are posed in a setting where existence of solutions is guaranteed. Alternatively, explicit requirements that maximal solutions be complete are sometimes included. For example, the classical result that asymptotic stability of the equilibrium point at the origin for $\dot{x} = f(x)$ follows from asymptotic stability of the origin for $\dot{x} = f'(0)x$ requires not only differentiability of f at the origin but also properties of f away from origin to ensure that, for every initial condition close to the origin, a solution to $\dot{x} = f(x)$ exists. Continuous differentiability is often assumed for f in order to guarantee the latter property. The existence of solutions for the linear approximation $\dot{x} = f'(0)x$, on its own, does not have any bearing on existence of solutions for the original equation. This observation suggests separating the issues of existence of solutions from the behavior of the solutions that do exist. It is worth noting that existence of solutions is not related to Lyapunov inequalities. In fact, even in the setting of hybrid systems, it is pre-asymptotic stability, as defined below, that turns out to be equivalent to the existence of smooth Lyapunov functions [9].

Since hybrid systems blend differential equations or inclusions, in which asymptotic properties describe what happens as time approaches infinity, and difference equations or inclusions, in which asymptotic properties describe what happens as the number of jumps approaches infinity, the definition below considers behavior of solutions to hybrid systems as either time or the number of jumps approaches infinity. Note that this is connected to the concept of completeness of a solution, which requires that the domain of a solution be unbounded, but does not require that it be unbounded in both "the t direction" and "the j direction".

Definition 2.4. *For the hybrid system* (2.1)*, the origin is*

- stable *if for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that every solution* ϕ *to* (2.1) *with* $|\phi(0,0)| < \delta$ *satisfies* $|\phi(t,j)| < \varepsilon$ *for all* $(t,j) \in \operatorname{dom}\phi$;
- pre-attractive *if there exists* $\delta > 0$ *such that every solution* ϕ *to* (2.1) *with* $|\phi(0,0)| < \delta$ *is bounded and if it is complete, then* $|\phi(t,j)| \to 0$ *as* $(t,j) \in \operatorname{dom}\phi$, $t + j \to \infty$.
- pre-asymptotically stable *if it is stable and pre-attractive.*
- attractive *if there exists* $\delta > 0$ *such that every maximal solution* ϕ *to* (2.1) *with* $|\phi(0,0)| < \delta$ *is complete and* $|\phi(t,j)| \to 0$ *as* $(t,j) \in \operatorname{dom}\phi$, $t + j \to \infty$.
- asymptotically stable *if it is stable and attractive.*

Pre-asymptotic stability of the origin agrees with asymptotic stability of the origin when maximal and bounded solutions to a hybrid system, from initial conditions near the origin, are complete. This is always the case when, for each initial condition near the origin, existence of nontrivial solutions is guaranteed and stability is present. Sufficient conditions for existence of nontrivial solutions, and their implications for maximal solutions, are given in Proposition 2.4 in [14].

Pre-asymptotic stability, used in the setting of differential equations, yields the following version of a classical result: if $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable at the origin, then the origin is pre-asymptotically stable for $\dot{x} = f(x)$ if it is pre-asymptotically stable for $\dot{x} = f'(0)x$. Such a result is a special case of Theorem 3.3. Of course, for $\dot{x} = f'(0)x$, pre-asymptotic stability is equivalent to asymptotic stability.

A different justification for the term "pre-asymptotic" is that every hybrid system with pre-asymptotically stable origin can be augmented to yield asymptotic stability.

Indeed, consider (2.1) and suppose that the origin is pre-asymptotically stable. Replace the map G by $x \mapsto G(x) \cup \{0\}$ if $x \in D$, $x \mapsto \{0\}$ if $x \notin D$ and then replace D by $\mathbb{R}^n$. The resulting system has the origin asymptotically stable, and every solution to the original system is also a solution to the augmented system.

Example 2.5. *A hybrid system given in $\mathbb{R}^2$ by (2.2) with*

$$C := \mathbb{R}_{\geq 0}{}^2, \qquad f(x) := \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \ \ \forall x \in \mathbb{R}^2,$$

empty D, and arbitrary g has the origin pre-asymptotically stable. Let $|x|$ denote the Euclidean norm of x. Then, for every solution ϕ, $|\phi(0,0)| < \varepsilon$ implies $|\phi(t,j)| = |\phi(0,0)| < \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$, so the origin is stable. The unique complete solution is $\phi(t,0) = 0$ for all $t \in \mathbb{R}_{\geq 0}$, while solutions with nonzero initial conditions are not complete, with $(t,0) \in \operatorname{dom}\phi$ implying $t \leq \pi/2$, and bounded. Hence the origin is pre-attractive, and pre-asymptotically stable.

Now consider C and f as above but let $D = \mathbb{R}^2$, $g(x) = 0$ for all $x \in \mathbb{R}^2$. This system has the origin asymptotically stable.

3. Main results. The classical result about linear approximation and asymptotic stability deduces asymptotic stability of the origin for a differential equation $\dot{x} = f(x)$, with a sufficiently regular right-hand side and $f(0) = 0$, from asymptotic stability for the linear approximation of the differential equation at the origin: $\dot{x} = f'(0)x$. Here, $f'(0)$ is the matrix representing the derivative of f at 0, i.e.,

$$\lim_{|h| \to 0} \frac{f(h) - f(0) - f'(0)h}{|h|} = 0.$$

A similar result holds for difference equations.

Local approximations of sets, often used in optimization in optimality conditions, and in viability theory in conditions for existence of solutions to constrained differential equations or inclusions, are given by tangent cones.

Definition 3.1. *The* tangent cone *to a set $S \subset \mathbb{R}^n$ at a point $x \in \overline{S}$, denoted $T_S(x)$, is the set*

$$T_S(x) = \{ v \in \mathbb{R}^n \mid \textit{there exist } \lambda_i \searrow 0, x_i \to x \textit{ such that } (x_i - x)/\lambda_i \to v \textit{ as } i \to \infty \}.$$

When S is a $\mathcal{C}^1$ manifold, the tangent cone to it agrees with the tangent space. The example below discusses the case of a hybrid system with the flow set described by inequalities and the jump set being a part of the boundary of the flow set.

Example 3.2. *Consider*

$$C = \{ x \in \mathbb{R}^n \mid h_i(x) \leq 0, \ i = 1, 2, \ldots, m \}, \ D = C \cap \bigcup_{i=1}^{l} \{ x \in \mathbb{R}^n \mid h_i(x) = 0 \} \quad (3.1)$$

where $h_i : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable and $l \leq m$. Without loss of generality, suppose $h_i(0) = 0$ for $i = 1, 2, \ldots, m$. If $\nabla h_i(0)$, $i = 1, 2, \ldots, m$, are linearly independent, then

$$T_C(0) = \{ v \in \mathbb{R}^n \mid \nabla h_i(0) \cdot v \leq 0, \ i = 1, 2, \ldots, m \}, \quad (3.2)$$

$$T_D(0) = T_C(0) \cap \bigcup_{i=1}^{l} \{ v \in \mathbb{R}^n \mid \nabla h_i(0) \cdot v = 0 \}. \quad (3.3)$$

See [25, Theorem 6.31]. An alternative condition for (3.2) *to hold, independently of linear independence of* $\nabla h_i(0)$*, is that the set on the right of* (3.2) *have interior; see the example following [4, Definition 4.1.1].*

Combining the two ideas on approximation, and accounting for the possibility that a hybrid system can be pre-asymptotically stable even when the flow map is not 0 at 0, leads to a result on pre-asymptotic stability of hybrid systems.

THEOREM 3.3. *Suppose that a function* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is continuous at* 0*, and if* $f(0) = 0$ *then it is differentiable at* 0*, a function* $g : \mathbb{R}^n \to \mathbb{R}^n$ *is differentiable at* 0 *and* $g(0) = 0$*, and* C*,* D *are sets in* $\mathbb{R}^n$*. If the origin is pre-asymptotically stable for the hybrid system*

$$\begin{cases} x \in T_C(0) & \dot{x} \;= \begin{cases} f(0) & \text{if} \quad f(0) \neq 0 \\ f'(0)x & \text{if} \quad f(0) = 0 \end{cases} \\ x \in T_D(0) & x^+ = \; g'(0)x \end{cases} \tag{3.4}$$

then the origin is pre-asymptotically stable for the hybrid system (2.2)*.*

This theorem justifies the statements in Example 1.1. In that example, concluding pre-asymptotic stability of 0 for the system (3.4) is straightforward, while doing it for the original system is not as immediate if done directly.

Of course, there are pre-asymptotically stable, or even asymptotically stable systems for which the approximation (3.4) is not pre-asymptotically stable. This is the case for the bouncing ball model of Example 1.2. There, $T_C(0) = C = \{x \in \mathbb{R}^2 \,|\, x_1 \geq 0\}$ and $f(0) = (0, \gamma)$, and hence (3.4) has flowing solutions from arbitrarily close to 0, even from 0, that diverge.

Note that (3.4) can turn out to be a "trivial" hybrid system, i.e., a system where solutions only flow or only jump. In some cases, this makes concluding pre-asymptotic stability quite simple.

EXAMPLE 3.4. *Consider the hybrid system* (2.2) *with the data*

$$\begin{aligned} C &= \{x \in \mathbb{R}^2 \,|\, x_1 \geq 0, 0 \leq x_2 \leq x_2^2\}, & f(x) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ D &= \{x \in \mathbb{R}^2 \,|\, x_1 \geq 0, x_1 = x_2^2\}, & g(x) &= \begin{pmatrix} x_1/2 \\ 0 \end{pmatrix}. \end{aligned}$$

Then $T_C(0) = T_D(0) = \mathbb{R}_{\geq 0} \times \{0\}$*, and there are no solutions to* $\dot{x} = f(0) = (1, 1)$ *that start and remain in* $T_C(0)$ *for a positive amount of time. Hence, the solutions to the approximation* (3.4) *correspond to solutions to* $x \in T_D(0)$*,* $x^+ = (x_1/2, 0)$*. Obviously,* (3.4) *has* 0 *pre-asymptotically stable, and in fact asymptotically stable. Theorem 3.3 implies that the origin is pre-asymptotically stable for the system displayed above.*

EXAMPLE 3.5. *Consider the hybrid system* (2.2) *with the flow set and jump set given by* (3.1)*, the flow map* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is continuously differentiable and* $f(0) \neq 0$*, while* $g : \mathbb{R}^n \to \mathbb{R}^n$ *is differentiable at* 0*,* $g(0) = 0$ *and* $g(D) \subset C \cup D$*. Theorem 3.3 concludes that if the origin is pre-asymptotically stable for*

$$\begin{cases} x \in T_C(0) & \dot{x} \;= f(0) \\ x \in T_D(0) & x^+ = \; g'(0)x, \end{cases}$$

then the origin is pre-asymptotically stable for (2.2)*. Example 3.2 provided simple descriptions* (3.2)*,* (3.3) *of* $T_C(0)$*,* $T_D(0)$*. It is also easy to give a sufficient condition* *that ensures that pre-asymptotic stability for* (2.2) *is in fact asymptotic stability. The condition is* $\nabla f(0)\cdot\nabla h_i(0) < 0$ *for* $i = l+1, l+2, \dots, m$. *Indeed, then* $\nabla f(x)\cdot\nabla h_i(x) < 0$ *for all* $x \in C \setminus D$ *close enough to* 0*, and consequently,* $f(x) \in T_C(x)$ *for all such* x, $i = l+1, l+2, \dots, m$. *This ensures that for all* $x \in C \cup D$ *close enough to the origin there exists a nontrivial solution to* (2.2)*. In the presence of pre-asymptotic stability, this ensures that maximal solutions from initial points close enough to* 0 *are complete. For further details, see [14, Section 2.3].*

When the approximation (3.4) of (2.2) does not yield useful conclusions, other "tighter" approximations can be considered. Even in the setting of differential equations, the linear approximation may vanish and hence asymptotic stability for it may be absent even if it is present for the differential equation. Higher order approximations, homogeneous with respect to the standard dilation with order greater than 1, can be then used. Alternatively, one can consider approximations that are homogeneous with respect to more general dilations. For an exposition, see [17]. These approximations motivate much of what follows. First, to familiarize the reader with the concept, we give a shortened version of Example 3.2 from [17].

Example 3.6. *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be given by*

$$f(x) = \begin{pmatrix} x_2^2 - x_1^3 \\ -x_2^{5/3} \end{pmatrix}.$$

The function f *and its derivative vanish at the origin, and the origin is not asymptotically stable for the linearization. However,* $f = f_2 + f_5$*, where*

$$f_2(x) = \begin{pmatrix} -x_1^3 \\ -x_2^{5/3} \end{pmatrix}, \qquad f_5(x) = \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix},$$

and, for the differential equation $\dot{x} = f_2(x)$*, the origin is obviously asymptotically stable. Now, Theorem 3.3 of [17] concludes that, for the differential equation* $\dot{x} = f(x)$*, the origin is asymptotically stable, because both* f_2 *and* f_5 *are homogeneous with respect to a particular dilation, and the degree of homogeneity for* f_5 *is higher than the order for* f_2*. Indeed, for each* $\lambda \in \mathbb{R}_{\geq 0}$*, let* $M(\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^3 \end{bmatrix}$*. Then, for each* $\lambda \in \mathbb{R}_{\geq 0}$ *and each* $x \in \mathbb{R}^2$*,* $f_2(M(\lambda)x) = \lambda^2 M(\lambda) f_2(x)$ *and* $f_5(M(\lambda)x) = \lambda^5 M(\lambda) f_5(x)$*.*

Definition 3.7. A dilation *of* $\mathbb{R}^n$*, with parameters* $r_1, r_2, \dots, r_n \geq 0$*, is the family of mappings* $x \mapsto M(\lambda)x$*,* $\lambda \in \mathbb{R}_{\geq 0}$*, where* $M(\lambda)$ *is the diagonal matrix*

$$M(\lambda) = \operatorname{diag}\{\lambda^{r_1}, \lambda^{r_2}, \dots, \lambda^{r_n}\} \tag{3.5}$$

A dilation is proper *if* $r_1, r_2, \dots, r_n > 0$*.*

The standard dilation, $x \mapsto \lambda x$, corresponds to $r_i = 1$, $i = 1, 2, \dots, n$. With some abuse of terminology, we will often speak of the dilation $M(\lambda)$ or just the dilation M, when referring to the object defined in Definition 3.7. Every tangent cone is a cone, i.e., a set that is homogeneous with respect to the standard dilation. Homogeneity of sets and mappings with respect to general dilations can be considered.

Definition 3.8. *A set* $S \subset \mathbb{R}^n$ *is* homogeneous with respect to a dilation M *if, for each* $\lambda \in \mathbb{R}_{>0}$*,* $M(\lambda)S = S$*. A set-valued mapping* $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *is* homogeneous with respect to a dilation M *and the* degree of homogeneity *is* d *if, for each* $\lambda \in \mathbb{R}_{>0}$*, each* $x \in \mathbb{R}^n$*,* $\Phi(M(\lambda)x) = \lambda^d M(\lambda)\Phi(x)$*.*

The sets $\{x \in \mathbb{R}^2 \,|\, x_1 \geq 0, x_2 \leq x_1^2\}$ and $\{x \in \mathbb{R}^2 \,|\, x_1 \geq 0, 0 \leq x_2 \leq x_1^2\}$ are homogeneous with respect to $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^2 \end{bmatrix}$. A constant set-valued mapping is homogeneous with respect to a standard dilation with degree -1. Some examples of functions homogeneous with respect to a nonstandard dilation were given in Example 3.6.

To extend the idea of homogeneous approximations from differential equations to hybrid systems, a concept of a homogeneous approximation of a set, and later of a set-valued mapping, is needed. A concept of a tangent M-cone, generalizing that of a tangent cone, is now proposed to address the first need.

Definition 3.9. *Given a set $S \subset \mathbb{R}^n$ and a dilation M, the* tangent homogeneous with respect to M cone *to S at x (the* tangent M-cone *for short) is the set*

$$T_S^M(x) = \{v \in \mathbb{R}^n \mid \text{there exist } \lambda_i \nearrow \infty, x_i \to x \text{ such that } M(\lambda_i)(x_i - x) \to v\}.$$

In [2], homogeneous tangent vectors to a reachable set of a control system were defined. Only dilations with integer $r_i \geq 1$ were considered. Mimicking that definition to define homogeneous tangent vectors to a set leads to: $v \in \mathbb{R}^n$ is a homogeneous tangent vector to $S \subset \mathbb{R}^n$ at $x \in S$ if there exists a continuous $y : [0, \overline{\varepsilon}] \to S$ with $y(0) = x$ such that $v = \lim_{\varepsilon \searrow 0} M\left(\varepsilon^{-1}\right)(y(\varepsilon) - x)$. For the case of the standard dilation, such a definition resembles that of a derivable tangent vector in [25, Definition 6.1], where no continuity of y is required. Not all tangent vectors are derivable. An illustration of this and further details are in [25, Chapter VI] and [4, Chapter 4]. A key property of homogeneous tangent cones to a set is that, locally, their homogeneous neighborhoods contain the set. Details of this fact are given in Lemma 5.1.

Lemma 3.10. *For any set $S \subset \mathbb{R}^n$, any dilation M, and any $x \in \mathbb{R}^n$, the tangent M-cone to S at x is closed and homogeneous with respect to M.*

Proof. The set $T_S^M(x)$ is the outer limit, as $i \to \infty$, of the sequence of sets given by $M(i)(S - x)$. Hence it is closed [25, Proposition 4.4]. Regarding homogeneity, pick any $v \in T_S^M(x)$ and any $\lambda > 0$. Let $\lambda_i \nearrow \infty$ and $x_i \in S$ be such that $M(\lambda_i)(x_i - x) \to v$. Then $M(\lambda)M(\lambda_i)(x_i - x) = M(\lambda\lambda_i)(x_i - x) \to M(\lambda)v$ and thus $M(\lambda)v \in T_S^M(x)$. Consequently, $M(\lambda)T_S^M(x) \subset T_S^M(x)$ for all $\lambda > 0$. Then $M(\lambda^{-1})T_S^M(x) \subset T_S^M(x)$, and $T_S^M(x) \subset M(\lambda)T_S^M(x)$, for all $\lambda > 0$. Thus $T_S^M(x)$ is homogeneous. $\square$

Theorem 3.11. *Consider the hybrid system* (2.2) *and suppose that there exists a proper dilation M and $d \in \mathbb{R}$ such that, for all $x \in \mathbb{R}^n$,*

$$f(x) = f_d(x) + \sum_{k=1}^{K} f_{\delta_k}(x), \qquad g(x) = g_0(x) + \sum_{l=1}^{L} g_{\delta_l}(x),$$

where $f_d : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and homogeneous with respect to M with order d; $g_0 : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and homogeneous with respect to M with order 0; $f_{\delta_k} : \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded and homogeneous with respect to M with order $\delta_k > d$, $k = 1, 2, \ldots, K$; and $g_{\delta_l} : \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded and homogeneous with respect to M with order $\delta_l > 0$, $l = 1, 2 \ldots, L$. Then, if the origin is pre-asymptotically stable for the hybrid system

$$\left\{ \begin{array}{ll} x \in T_C^M(0) & \dot{x} \ = f_d\,(x) \\ x \in T_D^M(0) & x^+ = g_0\,(x) \end{array} \right. \tag{3.6}$$

then the origin is pre-asymptotically stable for the hybrid system (2.2).

Example 3.12. *In $\mathbb{R}^2$, consider $\dot{x} = f(x)$ constrained by $x \in C$, where*

$$f(x) = \begin{pmatrix} x_1^3 + x_2 \\ x_1^2 x_2 \end{pmatrix}, \qquad C = \left\{x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq x_1^2\right\}.$$

One can ask whether the origin is pre-asymptotically stable for this system. (It is not asymptotically stable, because for all solutions, $\dot{x}_1 \geq 0, \dot{x}_2 \geq 0$.) Theorem 3.3 suggests looking at $\dot{x} = f'(0)x = (x_2, 0)$ constrained to $x \in T_C(0) = {\mathbb{R}_{\geq 0}}^2$. For this system, the origin is not pre-asymptotically stable, hence Theorem 3.3 is not helpful.

Consider $M(\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^2 \end{bmatrix}$ *and note that* $T_C^M(0) = C$ *while*

$$f(x) = f_1(x) + f_2(x), \qquad \text{where } \; f_1(x) = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, \; f_2(x) = \begin{pmatrix} x_1^3 \\ x_1^2 x_2 \end{pmatrix},$$

and f_i are homogeneous with respect to M with degree i, $i = 1, 2$. For $\dot{x} = f_1(x)$, $x \in C$, maximal solutions are not complete, except the constant solution $x(t) = 0$, and satisfy $|x(t)| \leq \sqrt{|x(0)|(|x(0)|+1)}$ on their intevals of existence. Hence, this constrained differential equation is pre-asymptotically stable. Theorem 3.11 concludes pre-asymptotic stability for $\dot{x} = f(x)$, $x \in C$.

Note that a set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is homogeneous with respect to $M(\lambda)$ with degree d if and only if its graph is homogeneous, as a set, with respect to

$$N(\lambda) = \begin{bmatrix} M(\lambda) & 0 \\ 0 & \lambda^d M(\lambda) \end{bmatrix}. \tag{3.7}$$

That is, $\Phi(M(\lambda)x) = \lambda^d M(\lambda)\Phi(x)$ for all x if and only if $\operatorname{gph}\Phi = N(\lambda)\operatorname{gph}\Phi$.

Definition 3.13. *Given a set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, a dilation M, and N is given by* (3.7)*, the* homogeneous with respect to M approximation *of Φ at* 0 with degree d *is the set-valued mapping $\Phi^{M,d} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by*

$$\operatorname{gph}\Phi^{M,d} = T^N_{\operatorname{gph}\Phi}(0).$$

For the case of a standard dilation and $d = 0$, $\Phi^{M,d}$ reduces to the graphical derivative of Φ at 0 [25, Definition 8.33]. Lemma 3.10 implies that for any Φ, any M, and any d, $\Phi^{M,d}$ has closed graph, and thus is outer semicontinuous, and furthermore, it is homogeneous with respect to M with degree d.

Example 3.14. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous at* 0 *and M be the standard dilation. Then*

$$f^{M,-1}(x) = f(0) \qquad \text{for all } x \in \mathbb{R}^n,$$

in other words, the constant function $x \mapsto f(0)$ is the homogeneous with respect to M approximation of f at 0 with degree -1. Indeed, let $y \in f^{M,-1}(x)$. Then, by definition, there exist $\lambda_i \nearrow \infty$, $x_i \in \mathbb{R}^n$, such that $M(\lambda_i)x_i = \lambda_i x_i \to x$, $\lambda_i^{-1}M(\lambda_i)f(x_i) = f(x_i) \to y$. Then $x_i \to 0$, and by continuity of f at 0, $y = f(0)$. More generally, if $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous at 0, then $F^{M,-1}(x) = F(0)$ for all $x \in \mathbb{R}^n$. If f is differentiable at 0 and $f(0) = 0$ then

$$f^{M,0}(x) = f'(0)x \qquad \text{for all } x \in \mathbb{R}^n.$$

Definition 3.15. *Given a proper dilation M, a* homogeneous with respect to M quasinorm *on $\mathbb{R}^n$ is a continuous function $\omega : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that satisfies $\omega(x) = 0$ if and only if $x = 0$, $\lim_{|x|\to\infty}\omega(x) = \infty$, and $\omega(M(\lambda)x) = \lambda\omega(x)$ for all $\lambda > 0$, $x \in \mathbb{R}^n$.*

For the standard dilation, any norm is a homogeneous quasinorm. For a general proper dilation, an example of a quasinorm is $\omega(x) = \sqrt{|x_1|^{2/\lambda_1} + \cdots + |x_n|^{2/\lambda_n}}$.

Theorem 3.16. *Consider the hybrid system* (2.1)*. Suppose that there exists a proper dilation M, a homogeneous with respect to M quasinorm ω, and $d \in \mathbb{R}$ such that the set-valued mappings*

$$x \mapsto \omega^{-d}(x) M\left(\omega^{-1}(x)\right) F(x), \qquad x \mapsto M\left(\omega^{-1}(x)\right) G(x)$$

are locally bounded at $x = 0$. If 0 is pre-asymptotically stable for the hybrid system

$$\begin{cases} x \in T_C^M(0) & \dot{x} \in \operatorname{con} F^{M,d}(x) \\ x \in T_D^M(0) & x^+ \in G^{M,0}(x) \end{cases} \tag{3.8}$$

then 0 is pre-asymptotically stable for the hybrid system (2.1).

In (3.8), $\operatorname{con} F^{M,d}(x)$ stands for the closed convex hull of $F^{M,d}(x)$. The role of the boundedness assumptions in Theorem 3.16 is to ensure that $\operatorname{con} F^{M,d}$ and $G^{M,0}$ be locally bounded at all $x \neq 0$ and that they approximate F and G, respectively, in an appropriate local sense. Details are in Lemma 5.1. Here, note that the boundedness assumption on $\omega^{-d}(x) M\left(\omega^{-1}(x)\right) F(x)$, for the standard dilation and $d = -1$ amounts to F being locally bounded at 0, while for $d = 0$ it amounts to $F(x)/|x|$ being locally bounded at 0. This suggests that Theorem 3.16 implies Theorem 3.3.

Proof. (of Theorem 3.3.) By Example 3.14, (3.4) is obtained from (2.2) just as (3.8) is obtained from (2.1), by considering the standard dilation, $d = -1$ for the case of $f(0) \neq 0$, and $d = 0$ for the case of $f(0) = 0$. In Theorem 3.16, consider the standard dilation M and $\omega(x) = |x|$. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous at 0 function, then $|x|^1 M\left(|x|^{-1}\right) f(x) = f(x)$ is bounded at $x = 0$. If $f(0) = 0$ and f is differentiable at 0, then $|x|^0 M\left(|x|^{-1}\right) f(x) = f(x)/|x|$ is bounded at $x = 0$; similarly for g. $\square$

Theorems 3.11 and 3.16 are proved in Section 5. Theorem 3.11 is shown as a consequence of Theorem 3.16. The idea behind Theorem 3.16 is that the data of the system (3.8) is regular enough to ensure that pre-asymptotic stability of 0 for (3.8) is robust, as a consequence of generic robustness results for hybrid systems. Homogeneity of (3.8) yields that this robustness is not just to some sufficiently small perturbation, but to a homogeneous perturbation. The data of a homogeneous perturbation of (3.8) contains, locally around 0, the data of (2.1). Thus, locally around 0, solutions to (2.1) are solutions to the homogeneous perturbation of (3.8) and pre-asymptotic stability for the homogeneous perturbation of (3.8) implies that for (2.1).

4. Pre-asymptotic stability and robustness for homogeneous hybrid systems. This section develops material on homogeneous hybrid systems, needed to prove the main results. Homogeneous hybrid systems are defined, homogeneous perturbations of such systems are presented, and robustness of pre-asymptotic stability to homogeneous perturbations is shown.

4.1. Homogeneous hybrid systems.

Definition 4.1. *Let M be a dilation and $d \in \mathbb{R}$. The hybrid system* (2.1) *is* homogeneous with respect to a dilation M *and the* degree of homogeneity is d *if*

- *the sets C and D are homogeneous with respect to M,*
- *the mapping F is homogeneous with respect to M with degree d,*
- *the mapping G is homogeneous with respect to M with degree 0.*

For example, the Bouncing Ball system of Example 1.2 is homogeneous with respect to $\begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{bmatrix}$, with degree $d = -1$. The linear/tangent cone approximation (3.4) is homogeneous with respect to the standard dilation and $d = 0$ when $f(0) = 0$, $d = -1$ when $f(0) \neq 0$. Homogeneity of G with nonzero degree is not considered, since it does not translate nicely to properties of solutions.

Given a hybrid system (2.1), $\lambda > 0$, and $d \in \mathbb{R}$, consider a system with the rescaled flow map:

$$\begin{cases} x \in C & \dot{x} \in \lambda^{-d} F(x) \\ x \in D & x^+ \in G(x) \end{cases} . \tag{4.1}$$

Lemma 4.2. *Suppose that the hybrid system* (2.1) *is homogeneous with respect to a proper dilation M with order d. Then, a function $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ is a solution to* (2.1) *if and only if the function $\psi : \operatorname{dom}\phi \to \mathbb{R}^n$ given by $\psi(t,j) = M(\lambda)\phi(t,j)$ is a solution to* (4.1).

The proof is straightforward. In other words, the lemma says that if ϕ solves (2.1), then $M(\lambda)\phi$ solves system like (2.1), but in which the flow occurs faster (if $\lambda^{-d} > 1$) or slower (if $\lambda^{-d} < 1$).

The proposition below shows that for homogeneous hybrid systems, pre-asymptotic stability can be concluded from the behavior of solutions with initial points on a "sphere" determined by a quasinorm.

Proposition 4.3. *Consider a hybrid system* (2.1)*, a proper dilation M, and $d \in \mathbb{R}$ such that the system* (2.1) *is homogeneous with respect to dilation M with degree d and there exist $R > r > 0$, $m > 0$, and a homogeneous with respect to M quasinorm ω such that for any solution ϕ to* (2.1) *with $\omega(\phi(0,0)) = r$ either*

(i) $\operatorname{dom}\phi$ is compact, with $t + j \le m$ for all $(t,j) \in \operatorname{dom}\phi$ and for all such (t,j), $\omega(\phi(t,j)) \le R$, or

(ii) there exists $(T,J) \in \operatorname{dom}\phi$ with $T + J \le m$, $\omega(\phi(T,J)) \le r/2$, and $\omega(\phi(t,j)) \le R$ for all $(t,j) \in \operatorname{dom}\phi$, $t \le T$, $j \le J$.

Then, 0 *is pre-asymptotically stable for* (2.1).

Proof. Let ψ be a solution to (2.1) with $2^{i-1} r \le \omega(\psi(0,0)) \le 2^i r$ for some $i \in \mathbb{Z}$. Pick $2^{-i} \le \lambda \le 2^{-i+1}$ such that $\lambda\omega(\psi(0,0)) = r$ and consider ϕ given by $\phi(t,j) = M(\lambda)\psi(t,j)$. Lemma 4.2 implies that ϕ is a solution to the system (4.1), with $\omega(\phi(0,0)) = \omega(M(\lambda)\psi(0,0)) = \lambda\omega(\psi(0,0)) = r$. Assumptions (i) and (ii) apply to ϕ, with m replaced by $m' = \lambda^d m$ if $\lambda^{-d} < 1$ (because then, solutions to (4.1) flow slower than those to (2.1)) and $m' = m$ if $\lambda^{-d} \ge 1$ (because then, solutions to (4.1) flow faster than those to (2.1)). Translating this to ψ yields

(i') $\operatorname{dom}\psi$ is compact, with $t + j \le m'$ for all $(t,j) \in \operatorname{dom}\psi$, and for all such (t,j), $\omega(\psi(t,j)) \le R/\lambda \le 2^i R$, or

(ii') there exists $(T,J) \in \operatorname{dom}\psi$ with $T + J \le m'$, $\omega(\psi(T,J)) \le r/(2\lambda) \le 2^{i-1} r$, and such that $\omega(\psi(t,j)) \le R/\lambda \le 2^i R$ for all $(t,j) \in \operatorname{dom}\psi$, $t \le T$, $j \le J$.

This is enough to conclude that 0 is pre-asymptotically stable for (2.1). ∎

4.2. Homogeneous perturbations of hybrid systems.

Definition 4.4. *Given a hybrid system* (2.1)*, a proper dilation M, and a homogeneous with respect to M quasinorm ω, a* homogeneous perturbation of (2.1) *of* *size $\rho > 0$ is the hybrid system:*

$$\begin{cases} x \in C_\rho & \dot{x} \in F_\rho(x) \\ x \in D_\rho & x^+ \in G_\rho(x) \end{cases} \tag{4.2}$$

with the data given by

$$\begin{aligned} C_\rho &= \{x \mid (x + \rho M(\omega(x))\mathcal{B}) \cap C \neq \emptyset\}, \\ F_\rho(x) &= \mathrm{con} F\left((x + \rho M(\omega(x))\mathcal{B}) \cap C\right) + \rho\omega^d(x) M(\omega(x))\mathcal{B}, \\ D_\rho &= \{x \mid (x + \rho M(\omega(x))\mathcal{B}) \cap D \neq \emptyset\}, \\ G_\rho(x) &= G\left((x + \rho M(\omega(x))\mathcal{B}) \cap D\right) + \rho M(\omega(x))\mathcal{B}, \end{aligned}$$

where $\mathcal{B} = \{x \in \mathbb{R}^n \mid \omega(x) \leq 1\}$.

Using the term “homogeneous” in the name of the perturbation in Definition 4.4 above is justified by the following result.

PROPOSITION 4.5. *Let M be a proper dilation, $d \in \mathbb{R}$, and ω a homogeneous with respect to M quasinorm. If the hybrid system* (2.1) *is homogeneous with respect to M with degree d then, for each $\rho > 0$, the hybrid system* (4.2) *is homogeneous with respect to M with degree d.*

Proof. Recall that $\omega(M(\lambda)x) = \lambda\omega(x)$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$ and note that $M(\omega(M(\lambda)x)) = M(\lambda)M(\omega(x))$. Then $F_\rho(M(\lambda)x)$ turns into

$$\mathrm{con} F\left[M(\lambda)x + \rho M(\lambda)M(\omega(x))\mathcal{B}\right] + \rho\lambda^d\omega^d(x)M(\lambda)M(\omega(x))\mathcal{B}$$

which is exactly $\lambda^d M(\lambda)F_\rho(x)$. The case of G is similar; consider $d = 0$ above. Now, $M(\lambda)C_\rho$ is $\{M(\lambda)x \mid (x + \rho M(\omega(x))\mathcal{B}) \cap C \neq \emptyset\}$ which, by taking $y = M(\lambda)x$ and so $x = M(\lambda^{-1})y$, turns into $\left\{y \mid M(\lambda^{-1})(y + \rho M(\omega(y))\mathcal{B}) \cap C \neq \emptyset\right\}$ and, since $M(\lambda)C = C$, to $\{y \mid (y + \rho M(\omega(y))\mathcal{B}) \cap C \neq \emptyset\} = C_\rho$. The case of D_ρ is parallel. ◻

4.3. Robustness of pre-asymptotic stability for homogeneous systems. A set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* at $x \in \mathbb{R}^n$ if, for each $x_i \to x$ and each convergent sequence $y_i \in \Phi(x_i)$, $\lim_{i\to\infty} y_i \in \Phi(x)$. Φ is locally bounded at x if there exists a neighborhood U of x such that $\Phi(U)$ is bounded.

ASSUMPTION 4.6 (basic assumptions). *The system* (2.1) *is said to satisfy* the basic assumptions *on a set $S \subset \mathbb{R}^n$ if*

(a) C and D are relatively closed in S;
(b) F is locally bounded and outer semicontinuous at x and $F(x)$ is nonempty and convex for all $x \in C \cap S$;
(c) G is locally bounded and outer semicontinuous at x and $G(x)$ is nonempty for all $x \in D \cap S$.

DEFINITION 4.7. *A continuous function $\beta : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $r \mapsto \beta(r, s)$ is nondecreasing and $\beta(0, s) = 0$ for all s while $s \mapsto \beta(r, s)$ is nonincreasing and* $\lim_{s\to\infty} \beta(r, s) = 0$ *for all r is called a $\mathcal{KL}$ function.*

LEMMA 4.8. *Let M be a proper dilation and ω be a homogeneous with respect to M quasinorm. Suppose that the hybrid system* (2.1) *satisfies the basic assumptions on $\mathbb{R}^n$ and 0 is pre-asymptotically stable for* (2.1). *Then there exists a $\mathcal{KL}$ function β and for each $\varepsilon > 0$, each $K > 0$, there exists $\delta > 0$ such that*

$$\omega(\phi(t, j)) \leq \beta\left(\omega(\phi(0, 0)), t + j\right) + \varepsilon \quad \textit{for all } (t, j) \in \mathrm{dom}\,\phi \tag{4.3}$$

for all solutions to the hybrid system

$$\left\{\begin{array}{ll} x \in C + \delta\mathbb{B} & \dot{x} \in F\left(x + \delta\mathbb{B}\right) + \delta\mathbb{B} \\ x \in D + \delta\mathbb{B} & x^{+} \in G\left(x + \delta\mathbb{B}\right) + \delta\mathbb{B} \end{array}\right. \tag{4.4}$$

with $\omega(\phi(0,0)) \leq K$.

Proof. The system (2.1) can be augmented, as was illustrated at the end of Section 2.2, to yield a system whose solutions subsume those of (2.1) and for which 0 is asymptotically stable. Then, Proposition 6.4 and Theorems 6.5, 6.6 in [14] yield a $\mathcal{KLL}$ function γ with properties paralleling those of the needed β. Considering $\beta(r,s) = \sup_{t+j=s} \gamma(r,t,j)$ finishes the proof. □

The lemma above stated that pre-asymptotic stability in a hybrid system meeting the basic assumptions is robust, to constant perturbations, in a semiglobal practical sense. A result on global robustness to nonconstant, and vanishing near 0, perturbations is in [9]. The theorem below shows that for homogeneous hybrid systems, pre-asymptotic stability is robust to homogeneous perturbations.

THEOREM 4.9. *Consider a hybrid system* (2.1)*, a proper dilation M, $d \in \mathbb{R}$, and a homogeneous with respect to M quasinorm ω. Suppose that the hybrid system* (2.1) *is homogeneous with respect to M with degree d, satisfies the basic assumptions on* $\mathbb{R}^n \setminus \{0\}$*, and has* 0 *pre-asymptotically stable. Then, there exists $\rho > 0$ such that* 0 *is pre-asymptotically stable for* (4.2).

Proof. Consider the hybrid system

$$\left\{\begin{array}{ll} x \in C \setminus \operatorname{int}\mathcal{B} & \dot{x} \in F\left(x\right) \\ x \in D \setminus \operatorname{int}\mathcal{B} & x^{+} \in G\left(x\right) \end{array}\right. . \tag{4.5}$$

Since, for (2.1), 0 is pre-asymptotically stable, 0 is also pre-asymptotically stable for (4.5). Furthermore, (4.5) meets the regularity assumptions in Lemma 4.8. Hence, there exists a $\mathcal{KL}$ function β, and $\delta > 0$ such that (4.3) holds with $\varepsilon = 1$ for each solution ϕ to

$$\left\{\begin{array}{ll} x \in (C \setminus \operatorname{int}\mathcal{B}) + \delta\mathbb{B} & \dot{x} \in F\left(x + \delta\mathbb{B}\right) + \delta\mathbb{B} \\ x \in (D \setminus \operatorname{int}\mathcal{B}) + \delta\mathbb{B} & x^{+} \in G\left(x + \delta\mathbb{B}\right) + \delta\mathbb{B} \end{array}\right. \tag{4.6}$$

with $\omega(\phi(0,0)) \leq 4$. Pick $m > 0$ such that $\beta(4,m) \leq 1$. Then, pick $\rho > 0$ such that $\rho M(\omega(x))\mathcal{B} \subset \delta\mathbb{B}$ for all $x \in \mathbb{R}^n$ with $\omega(x) \leq R := \beta(4,0)+1$ and $\rho\omega^d(x)M(\omega(x))\mathcal{B} \subset \delta\mathbb{B}$ for all $x \in \mathbb{R}^n$ with $1 \leq \omega(x) \leq R$. With such ρ, consider the system (4.2) and recall that by Proposition 4.5, it is homogeneous with respect to M with degree d.

Let ϕ be a solution to (4.2) with $\omega(\phi(0,0)) = 4$. Note that ϕ is also a solution to (4.6) as long as it remains in $\{x \,|\, 1 \leq \omega(x) \leq R\}$. Let $(T,J) \in \operatorname{dom}\phi$ be the "first" element in $\operatorname{dom}\phi$ such that $\omega(\phi(T,J)) \leq 2$. If such (T,J) does not exist, it means that $\operatorname{dom}\phi$ is compact, with $t + j \leq m$ for all $(t,j) \in \operatorname{dom}\phi$, and for all such (t,j), $\omega(\phi(t,j)) \leq R$. For such a (T,J), thanks to (4.3), $T + J \leq m$ and $\omega(\phi(t,j)) \leq R$ for all $(t,j) \in \operatorname{dom}\phi$, $t \leq T$, $j \leq J$. Proposition 4.3 finishes the proof. □

5. Proofs of the main pre-asymptotic stability results.

LEMMA 5.1. *Let M be a proper dilation, $d \in \mathbb{R}$, and $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping such that* *(a) for some homogeneous with respect to M quasinorm ω, the mapping $x \mapsto \omega^{-d}(x)M\left(\omega^{-1}(x)\right)\Phi(x)$ is locally bounded at $x=0$.*

Then $\Phi^{M,d}$ is locally bounded at each $x \neq 0$. Furthermore, if $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping such that

(b) $\operatorname{gph}\Phi^{M,d} \setminus \{0\} \subset \operatorname{int}\operatorname{gph}\Psi$*;*

(c) Ψ is homogeneous with respect to M with order d;

then, for all x sufficiently close to 0, $\Phi(x) \subset \Psi(x)$.

Proof. To prove the first conclusion, suppose that, on the contrary, there exists a convergent sequence x_i, $i = 1, 2, \dots$, with $x_i \to x \neq 0$, and a sequence $y_i \in \Phi^{M,d}(x_i)$ with $|y_i| \to \infty$. The definition of $\Phi^{M,d}$ yields $\lambda_i \nearrow \infty$, a sequence x_i' with $M(\lambda_i)x_i' \to x$, and a sequence y_i' with $y_i' \in \Phi(x_i')$ and $|\lambda_i^d M(\lambda_i)y_i'| \to \infty$. Note that since M is proper, $x_i' \to 0$ as $i \to \infty$. Let ω be as in assumption (a), and note that $\omega(M(\lambda_i)x_i') = \lambda_i\omega(x_i') =: a_i \to a$ where $a = \omega(x) \neq 0$. Then $\lambda_i^d M(\lambda_i)y_i' = a_i^d M(a_i)\omega^{-d}(x_i')M\left(\omega^{-1}(x_i')\right)y_i'$ and this remains bounded as $i \to \infty$, by assumption (a) and since $a_i \to a \neq 0$. This contradicts $|\lambda_i^d M(\lambda_i)y_i'| \to \infty$.

To prove the second conclusion, suppose on the contrary, that there exists a sequence x_i, $i = 1, 2, \dots$, such that $x_i \neq 0$, $x_i \to 0$ and a sequence y_i, $i = 1, 2, \dots$, such that $y_i \in \Phi(x_i)$ but $y_i \notin \Psi(x_i)$. With the quasinorm ω from assumption (a), pick $\lambda_i = \omega^{-1}(x_i)$. Let $x_i' = M(\lambda_i)x_i$, so that $\omega(x_i') = 1$, and $y_i' = \lambda_i^d M(\lambda_i)y_i \in \lambda_i^d M(\lambda_i)\Phi(x_i)$. By assumption (a), y_i''s are uniformly bounded. Without relabeling, pass to a subsequence so that x_i''s and y_i''s converge to, respectively, $\overline{x}$ and $\overline{y}$. By Definition 3.9, $(\overline{x}, \overline{y}) \in \operatorname{gph}\Phi^{M,d}$. Hence, by (b) and since $\omega(\overline{x}) = 1$, $(\overline{x}, \overline{y}) \in \operatorname{gph}\Psi$. Then, for all large enough i, $(x_i', y_i') \in \operatorname{gph}\Psi$, and thus $\lambda_i^d M(\lambda_i)y_i \in \Psi(M(\lambda_i)x_i)$. Homogeneity of Ψ, as in (c), implies that $y_i \in \Psi(x_i)$, which is a contradiction. ◻

Corollary 5.2. *Let M be a proper dilation and $S, S' \subset \mathbb{R}^n$ be sets such that $T_S^M(0) \setminus \{0\} \subset \operatorname{int} S'$ and S' is homogeneous with respect to M. Then for all x sufficiently close to 0, $x \in S$ implies $x \in S'$.*

Proof. In Lemma 5.1, consider $\Phi(x) = 0$ if $x \in S$, $\Phi(x) = \emptyset$ otherwise, and $\Psi(x) = \mathbb{R}^n$ if $x \in S'$, $\Psi(x) = \emptyset$ otherwise. ◻

Proof. (of Theorem 3.16.) Lemma 3.10 showed that $T_C^M(0)$, $T_D^M(0)$ are closed. It also implies that graphs of $F^{M,d}$ and $G^{M,0}$ are closed, hence the mappings are outer semicontinuous. Lemma 5.1 implies that $F^{M,d}$ and $G^{M,0}$ are locally bounded at each $x \neq 0$. Pointwise convexification of $F^{M,d}$ does not change the local boundedness, which is clear, and outer semicontinuity at $x \neq 0$, as follows from [25, Proposition 4.30]. Now Theorem 4.9 can be invoked to yield $\rho > 0$ such that the homogeneous perturbation of (3.8) with size ρ, as in Definition 4.4, is pre-asymptotically stable. Let $\left(T_C^M(0)\right)_\rho$, $\left(\operatorname{con}F^{M,d}\right)_\rho$, $\left(T_D^M(0)\right)_\rho$, $\left(G^{M,0}\right)_\rho$ be the data of this perturbation. Then

$$T_C^M(0)\setminus\{0\} \subset \operatorname{int}\left(T_C^M(0)\right)_\rho, \ \left\{(x,y) \,|\, x \in T_C^M(0),\ y \in F^{M,d}(x)\right\} \subset \operatorname{int}\operatorname{gph}\left(\operatorname{con}F^{M,d}\right)_\rho,$$

and a similar containment holds for the objects associated with D and G, with $d = 0$ for the latter. Lemma 5.1 and Corollary 5.2 imply that the data of (2.1) is, locally around 0, contained in the data of the homogeneous perturbation of (3.8) with size ρ. More precisely, we have, for some $\varepsilon > 0$:

$$\begin{aligned} C \cap \varepsilon\mathbb{B} &\subset \left(T_C^M(0)\right)_\rho, \quad F(x) \subset \left(\operatorname{con}F^{M,d}\right)_\rho(x) \quad \forall x \in C \cap \varepsilon\mathbb{B}, \\ D \cap \varepsilon\mathbb{B} &\subset \left(T_D^M(0)\right)_\rho, \quad G(x) \subset \left(G^{M,0}\right)_\rho(x) \qquad \forall x \in D \cap \varepsilon\mathbb{B}. \end{aligned}$$

This is sufficient to conclude pre-asymptotic stability of 0 for (2.1) from that for the homogeneous perturbation of (3.8) with size ρ. ◻

LEMMA 5.3. *Given a proper dilation M and $d \in \mathbb{R}$, suppose that a set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ can be decomposed as*

$$\Phi(x) = \Phi_d(x) + \sum_{k=1}^{K} \Phi_{\delta_k}(x),$$

where Φ_d is outer semicontinuous, locally bounded, and homogeneous with respect to M with degree d and Φ_{δ_k}, $k = 1, 2, \ldots, K$ are locally bounded and homogeneous with respect to M with degree δ_k, and $\delta_k > d$, $k = 1, 2, \ldots, K$. Then, for any homogeneous with respect to M quasinorm ω, the set-valued mapping $x \mapsto \omega^{-d}(x)M\left(\omega^{-1}(x)\right)\Phi(x)$ is locally bounded at $x = 0$ and $\Phi^{M,d} \subset \Phi_d$. If, furthermore, $\operatorname{dom}\Phi_d = \operatorname{dom}\Phi_{\delta_k}$ *for $k = 1, 2, \ldots, K$, then $\Phi^{M,d} = \Phi_d$.*

Proof. First, note that

$$\begin{aligned}\omega^{-d}(x)M\left(\omega^{-1}(x)\right)\Phi(x) &= \omega^{-d}(x)M\left(\omega^{-1}(x)\right)\Phi_d(x) + \sum_{k=1}^{K}\omega^{-d}(x)M\left(\omega^{-1}(x)\right)\Phi_{\delta_k}(x)\\ &= \Phi_d\left(M\left(\omega^{-1}(x)\right)x\right) + \sum_{k=1}^{K}\omega^{\delta_k-d}(x)\Phi_{\delta_k}\left(M\left(\omega^{-1}(x)\right)x\right)\end{aligned}$$

and $\omega\left(M\left(\omega^{-1}(x)\right)x\right) = \omega^{-1}(x)\omega(x) = 1$. This, local boundedness of Φ_d, Φ_{δ_k}'s, and $\delta_k - d > 0$ shows that $x \mapsto \omega^{-d}(x)M\left(\omega^{-1}(x)\right)\Phi(x)$ is locally bounded at $x = 0$,

By definition, $y \in \Phi^{M,d}(x)$ means that there exist $\lambda_i \nearrow \infty$, $x_i, y_i \in \mathbb{R}^n$ such that $y_i \in \Phi(x_i)$, $M(\lambda_i)x_i \to x$, $\lambda^d M(\lambda_i)y_i \to y$. The inclusion $y_i \in \Phi(x_i)$ is equivalent to

$$\begin{aligned}\lambda_i^d M(\lambda_i)y_i \in \lambda_i^d M(\lambda_i)\Phi_d(x_i) &+ \sum_{k=1}^{K}\lambda_i^{d-\delta_k}\lambda_i^k M(\lambda_i)\Phi_{\delta_k}(x_i)\\ = \Phi_d(M(\lambda_i)x_i) &+ \sum_{k=1}^{K}\lambda_i^{d-\delta_k}\Phi_{\delta_k}(M(\lambda_i)x_i)\end{aligned}$$

Local boundedness of Φ_{δ_k}'s and the fact that $\lambda_i^{d-\delta_k} \to 0$ implies that y is an element of $\limsup_{i\to\infty}\Phi_d(M(\lambda_i)x_i) \subset \Phi_d(x)$, where the last inclusion comes from outer semicontinuity of Φ_d. Hence $\Phi^{M,d} \subset \Phi_d$. The reverse inclusion is shown by considering any $\lambda_i \nearrow \infty$, any x, any $y \in \Phi_d(x)$, $x_i = M(\lambda_i^{-1})x$, and $y_i \in \Phi(x_i)$ given by $y_i = \lambda_i^{-d}M(\lambda_i^{-1})y + \sum_{k=1}^{K}\lambda_i^{-\delta_k}M(\lambda_i^{-1})y_i^k$, where $y_i^k \in \Phi_{\delta_k}(x)$ are arbitrary. ∎

Proof. (of Theorem 3.11.) Combine Theorem 3.16 and Lemma 5.3. ∎

6. Uniform small ordinary time property and Zeno behavior. Homogeneity of hybrid systems sheds light on the Zeno phenomenon — the occurrence of infinitely many jumps in a finite amount of time — in asymptotically stable hybrid systems. In particular, homogeneity turns out to be closely related to the amount of time it takes solutions to such hybrid systems to converge to the origin.

Given a set $S \subset \mathbb{R}^n$ and a solution ϕ to (2.1), let

$$\mathcal{T}_S(\phi) = \sup\left\{t \in \mathbb{R}_{\geq 0} \mid \exists j \in \mathbb{N} \text{ such that } (t,j) \in \operatorname{dom}\phi,\ \phi(t,j) \in S\right\}.$$

In particular, $\mathcal{T}_{\mathbb{R}^n}(\phi) = \sup\{t \in \mathbb{R}_{\geq 0} \mid \exists j \in \mathbb{N} \text{ such that } (t,j) \in \operatorname{dom}\phi\}$.

DEFINITION 6.1. *The origin is* uniformly small ordinary time pre-asymptotically stable *for the hybrid system* (2.1) *if it is pre-asymptotically stable and for each $\varepsilon >$*

0 *there exists* $\delta > 0$ *such that, for each solution* ϕ *to* (2.1) *with* $|\phi(0,0)| < \delta$, $\mathcal{T}_{\mathbb{R}^n\setminus\{0\}}(\phi) < \varepsilon$.

The result below ties the issue of $\mathcal{T}_{\mathbb{R}^n\setminus\{0\}}(\phi)$ being small or infinite to the degree of homogeneity of a hybrid system.

THEOREM 6.2. *Consider a hybrid system* (2.1)*, a proper dilation* M*,* $d \in \mathbb{R}$*, and a homogeneous with respect to* M *quasinorm* ω*. Suppose that* 0 *is pre-asymptotically stable for* (2.1)*. Then*

- *(a) if* $d < 0$ *and* (2.1) *satisfies the basic assumptions on* $\mathbb{R}^n$ *except possibly the local boundedness of* F *at* 0*, then* 0 *is uniformly small ordinary time pre-asymptotically stable for* (2.1)*;*
- *(b) if* $d \geq 0$*,* (2.1) *satisfies the basic assumptions on* $\mathbb{R}^n$*, and* $\mathcal{T}_{\mathbb{R}^n\setminus\{0\}}(\phi) > 0$ *for every complete solution to* (2.1) *with* $\omega(\phi(0,0)) = 1$ *then* $\mathcal{T}_{\mathbb{R}^n\setminus\{0\}}(\phi) = \infty$ *for all complete solutions to* (2.1) *with* $\omega(\phi(0,0)) > 0$*.*

Proof. Showing (a) relies on the proof of Theorem 4.9. That proof obtained $m > 0$ such that, for any solution ϕ to (4.5) (since solutions to (4.5) are solutions to (4.6)) with $\omega(\phi(0,0)) = 4$, either $t + j \leq m$ for all $(t,j) \in \operatorname{dom}\phi$ or there exists $(T,J) \in \operatorname{dom}\phi$ with $T + J \leq m$ and $\omega(\phi(T,J)) \leq 2$. Since solutions ϕ to (2.1) are also solutions to (4.5) when restricted to $(t,j) \in \operatorname{dom}\phi$ with $t \leq T$, $j \leq J$, one can conclude the following: for any solution ϕ to (2.1) with $\omega(\phi(0,0)) = 4$, either $t \leq m$ for all $(t,j) \in \operatorname{dom}\phi$ or there exists $(T,J) \in \operatorname{dom}\phi$ with $T \leq m$ and $\omega(\phi(T,J)) \leq 2$.

Now take a solution ψ to (2.1) with $2\,2^r \leq \omega(\psi(0,0)) \leq 4\,2^r$ for some $r \in \mathbb{Z}$. Let $\lambda = 4/\omega(\psi(0,0))$, so that $2^{-r} \leq \lambda \leq 2^{-r+1}$, and let $\phi(t,j) = M(\lambda)\psi(t,j)$. Then $\omega(\phi(0,0)) = 4$ and ψ is a solution to (4.1). Then either $t \leq \lambda^d m \leq 2^{d(-r+1)}m$ for all $(t,j) \in \operatorname{dom}\phi = \operatorname{dom}\psi$, or there exists $(T,J) \in \operatorname{dom}\phi = \operatorname{dom}\psi$ with $T \leq \lambda^d m \leq 2^{d(-r+1)}m$ and $\omega(\phi(T,J)) \leq 2$, and thus $\omega(\psi(T,J)) \leq 2/\lambda \leq 2\,2^r = 4\,2^{r-1}$. Now (a) follows, since $d < 0$, from the recursive bound

$$\mathcal{T}_{\mathbb{R}^n\setminus\{0\}}(\psi) \leq 2^{d(-r+1)}m + 2^{d(-(r-1)+1)}m + \dots = 2^{d(-r+1)}m\left(1 + 2^d + 2^{2d} + \dots\right) < \infty.$$

To see (b), note that there exist $m > 0$, $r > 0$ such that, for each complete solution ϕ to (2.1) with $\omega(\phi(0,0)) = 1$ there exists $(t,j) \in \operatorname{dom}\phi$, $t > m$, such that $\omega(\phi(t,j)) > r$. Indeed, otherwise there exist complete solutions ϕ_i to (2.1) with $\omega(\phi_i(0,0)) = 1$ such that $\omega(\phi_i(t,j)) < 1/i$ for all $(t,j) \in \operatorname{dom}\phi_i$ with $t > 1/i$. By Lemma 4.8, this sequence is uniformly bounded. It also has a graphically convergent subsequence, see Theorem 4.18 in [25], the graphical limit ϕ of which is a complete solution to (2.1), with $\mathcal{T}_{\mathbb{R}^n\setminus\{0\}} = 0$. This is a contradiction.

Now, for any complete solution ϕ and (t_1,j_1) with $t_1 > m$, $\omega(\phi(t_1,j_1)) > r$ let $\lambda = 1/\omega(\phi(t_1,j_1))$ and consider $M(\lambda)\phi$. Then $\omega(M(\lambda)\phi(t_1,j_1)) = 1$, $M(\lambda)\phi$ is a solution to (4.1), and there exists $(t_2,j_2) \in \operatorname{dom}\phi$ with $t_2 - t_1 > \lambda^d m$, and thus $t_2 > m\left(1 + \omega^{-d}(\phi(t_1,j_1))\right)$ such that $\omega(M(\lambda)\phi(t_2,j_2)) > r$, and thus $\omega(\phi(t_2,j_2)) > r^2$. Repeating this argument shows the existence of $(t_k,j_k) \in \operatorname{dom}\phi$ with $t_k > m\left(1 + \sum_{i=1}^{k-1}\omega^{-d}(\phi(t_i,j_i))\right)$ Since (2.1) is pre-asymptotically stable, ϕ is bounded, and hence $\omega(\phi(t_i,j_i))$ are bounded above. Since $d \geq 0$, this implies that $t_k \to \infty$ as $k \to \infty$, and thus $\mathcal{T}_{\mathbb{R}^n\setminus\{0\}}(\phi) = \infty$. ◻

Theorem 6.2 verifies the claims made in Example 1.2, about the convergence of solutions to 0 in a finite amount of time. Indeed, the Bouncing Ball system is homogeneous with respect to $\begin{bmatrix}\lambda^2 & 0\\ 0 & \lambda\end{bmatrix}$, with $d = -1$. In general, Theorem 6.2 excludes Zeno behavior of solutions converging to 0 when the degree of homogeneity is nonnegative and jumps to 0 do not occur, and it suggests Zeno behavior when the degree of homogeneity is negative and solutions converging to 0 do jump infinitely many times.

COROLLARY 6.3. *Under the assumptions of Theorem 3.16,*

(a) if 0 *is pre-asymptotically stable for* (3.8) *and* $d < 0$*, then* 0 *is uniformly small ordinary time pre-asymptotically stable for* (2.1)*;*

(b) if 0 *is pre-asymptotically stable for* (3.8), $d \geq 0$*, and* $\mathcal{T}_{\mathbb{R}^n\setminus\{0\}}(\phi) > 0$ *for every complete solution to* (3.8) *with* $\omega(\phi(0,0) = 1$*, then* 0 *is pre-asymptotically stable for* (2.1) *and* $\mathcal{T}_{\mathbb{R}^n\setminus\{0\}}(\phi) = \infty$ *for every complete solution to* (2.1).

Proof. Combine the proof of Theorem 3.16 with Theorem 6.2. □

DEFINITION 6.4. *The origin is* uniformly Zeno asymptotically stable *for the system* (2.1) *if it is uniformly small ordinary time pre-asymptotically stable and there exists* $\delta > 0$ *such that each maximal solution* ϕ *to* (2.1) *with* $0 < |\phi(0,0)| < \delta$ *is* Zeno, *i.e.,* ϕ *is complete,* $\mathcal{T}_{\mathbb{R}^n}(\phi) < \infty$*, and there is no* $j \in \mathbb{N}$ *with* $(\mathcal{T}_{\mathbb{R}^n}(\phi), j) \in \operatorname{dom}\phi$.

To see the difference between uniform Zeno asymptotic stability and "Zeno asymptotic stability", a property that combines asymptotic stability with the existence of a neighborhood of 0 from which all solutions are Zeno, consider the next example.

EXAMPLE 6.5. *In* $\mathbb{R}^2$*, consider a hybrid system given by*

$$C = \bigcup_{i=0}^{\infty} \left\{x \in \mathbb{R}^2 \,|\, x_1 \geq 0, x_2 \geq 0, 2^{-2i-1} \leq |x| \leq 2^{-2i}\right\}, \qquad f(x) = -\frac{x}{|x|},$$

$$D = \bigcup_{i=0}^{\infty} \left\{x \in \mathbb{R}^2 \,|\, x_1 \geq 0, x_2 \geq 0, |x| = 2^{-2i-1}\right\}, \qquad g(x) = \frac{x}{2}.$$

This system is pre-asymptotically stable. Furthermore, $|\phi(0,0)| \leq 2^{-2i}$ *implies that* $\mathcal{T}_{\mathbb{R}^n\setminus\{0\}}(\phi) \leq 2^{-2i+1}/3$, $i = 0, 1, \ldots$*, and every solution is Zeno. Thus* 0 *is uniformly Zeno asymptotically stable. Now consider a system with the same* D, g*, but with*

$$C = \bigcup_{i=0}^{\infty} \left\{x \in \mathbb{R}^2 \,|\, x_1 \geq 0, x_2 \geq x_1^2, 2^{-2i-1} \leq |x| \leq 2^{-2i}\right\}, \qquad f(x) = -\frac{x}{|x|}\frac{x_2}{x_1}.$$

Then 0 *is asymptotically stable, all solutions are Zeno, but the uniform small ordinary time property is missing: there exist solutions* ϕ *from initial points* x *with* $|x| = 2^{-2i}$, $x_2 = x_1^2$ *and arbitrarily close to* 0*, such that* $\mathcal{T}_{\mathbb{R}^n\setminus\{0\}}(\phi) \geq 2/3$.

PROPOSITION 6.6. *Suppose the hybrid system* (2.1) *satisfies the basic assumptions on* $\mathbb{R}^n$, 0 *is uniformly small ordinary time pre-asymptotically stable for* (2.1)*, and:*

(a) there exists $\varepsilon > 0$ *such that each maximal solution* ϕ *to* (2.1) *with* $0 < |\phi(0,0)| < \varepsilon$ *satisfies* $T_{\mathbb{R}^n\setminus\{0\}}(\phi) > 0$*;*

(b) there does not exist an absolutely continuous $\psi : [0,\varepsilon] \to \mathbb{R}^n$, $\varepsilon > 0$*, such that* $\dot{\psi}(t) \in -F(\psi(t))$ *for almost all* $t \in [0,\varepsilon]$, $\psi(t) \in C$ *all* $t \in (0,\varepsilon)$, $\psi(0) = 0$, $\psi(\varepsilon) \neq 0$.

Then 0 *is uniformly Zeno pre-asymptotically stable for* (2.1).

Proof. We need to find $\delta > 0$ such that each maximal solution ϕ to (2.1) with $|\phi(0,0)| < \delta$ is Zeno. Using pre-asymptotic stability pick $\delta > 0$ so that each maximal solution ϕ to (2.1) with $|\phi(0,0)| < \delta$ is bounded, satisfies $|\phi(t,j)| < \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$ where $\varepsilon \geq \delta$ is such that each maximal solution ψ to (2.1) with $|\psi(0,0)| < \varepsilon$ satisfies $0 < T_{\mathbb{R}^n\setminus\{0\}}(\psi) < \infty$. Existence of such a ε comes from (a) and the uniform small ordinary time property. Let ϕ be a solution to (2.1) with $0 < |\phi(0,0)| < \delta$. Suppose ϕ is not complete. If $\operatorname{dom}\phi$ was not closed, ϕ could be extended to $\overline{\operatorname{dom}\phi}$. Hence, maximality of ϕ implies that $\operatorname{dom}\phi$ is closed. Let (T, J) be the "last" element in $\operatorname{dom}\phi$. Since $|\phi(T,J)|<\varepsilon$, maximal solutions ψ from $\phi(T,J)$ satisfy $T_{\mathbb{R}^n\setminus\{0\}}(\psi)>0$. Concatenating such a ψ with ϕ contradicts maximality of ϕ. Thus ϕ is complete and $T_{\mathbb{R}^n\setminus\{0\}}(\phi)<\infty$. To see that $T_{\mathbb{R}^n}(\phi)<\infty$ it is enough to note that there does not exist $(t,j)\in\operatorname{dom}\phi$ with $\phi(t,j)=0$. This follows from assumptions (a) and (b): (a) excludes the possibility of $\phi(t,j)=0$ with $(t,j-1)\in\operatorname{dom}\phi$ and $\phi(t,j-1)\neq 0$, (b) excludes the possibility of $\phi(t,j)=0$ with $(t',j)\in\operatorname{dom}\phi$, $t'<t$, and $\phi(t',j)\neq 0$. Thus $T_{\mathbb{R}^n}(\phi)<\infty$. Finally, if there exists $J\in\mathbb{N}$ with $(\mathcal{T}_{\mathbb{R}^n}(\phi),j)\in\operatorname{dom}\phi$, then $\psi(t,j):=\phi\left(t+\mathcal{T}_{\mathbb{R}^n}(\phi),j+J\right)$ violates the property that $0<T_{\mathbb{R}^n\setminus\{0\}}(\psi)$ for all maximal solution ψ to (2.1) with $|\psi(0,0)|<\varepsilon$. Thus ϕ is Zeno. □

7. Hybrid systems with logical modes.

7.1. Definitions and the main result. The data of a hybrid system with logical modes consists of a set $Q=\{1,2,\ldots,q_{max}\}$, and for each $q\in Q$, a flow set C_q, a flow map F_q, a jump set D_q, and a jump map G_q. The following is assumed, for each $q\in Q$: C_q and D_q are sets in $\mathbb{R}^n$; $F_q:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ is a set-valued mapping with $F_q(x)\neq\emptyset$ when $x\in C_q$; and $G_q:\mathbb{R}^n\rightrightarrows Q\times\mathbb{R}^n$ is a set-valued mapping with $G_q(x)\neq\emptyset$ when $x\in D_q$. Such systems can be represented in the following form:

$$\left\{\begin{array}{lcl} x\in C_q & \dot{x} & \in F_q\,(x)\\ x\in D_q & \begin{pmatrix}q^+\\ x^+\end{pmatrix} & \in G_q\,(x)\end{array}\right. . \tag{7.1}$$

A *solution* to (7.1) consists of functions $q:E\to Q$, $\phi:E\to\mathbb{R}^n$, where E is a hybrid time domain, $\phi(0,0)\in\overline{C_{q(0,0)}}\cup D_{q(0,0)}$, and, if $I_j:=\{t\,|\,(t,j)\in E\}$ has nonempty interior, then $t\mapsto q(t,j)$ is constant, $t\mapsto\phi(t,j)$ is absolutely continuous on I_j, and

$$\phi(t,j)\in C_{q(t,j)}\ \text{ for all } t\in\operatorname{int}I_j\ \text{ and }\ \frac{d}{dt}\phi(t,j)\in F(\phi(t,j))\ \text{ for almost all } t\in I_j;$$

if $(t,j)\in E$ and $(t,j+1)\in E$ then

$$\phi(t,j)\in D_{q(t,j)}\ \text{ and }\ \begin{pmatrix}q(t,j+1)\\ \phi(t,j+1)\end{pmatrix}\in G_{q(t,j)}(\phi(t,j)).$$

Pre-asymptotic stability for (7.1) considers the behavior of $\phi(t,j)$ and not of $q(t,j)$. For (7.1), the origin is *stable* if for each $\varepsilon>0$ there exists $\delta>0$ such that every solution (q,ϕ) to (7.1) with $|\phi(0,0)|<\delta$ satisfies $|\phi(t,j)|<\varepsilon$ for all $(t,j)\in\operatorname{dom}(q,\phi)$; *pre-attractive* if there exists $\delta>0$ such that every solution (q,ϕ) to (7.1) with $|\phi(0,0)|<\delta$ is bounded and if it is complete, then $|\phi(t,j)|\to 0$ as $(t,j)\in\operatorname{dom}(q,\phi)$, $t+j\to\infty$; and *pre-asymptotically stable* if it is stable and pre-attractive.

THEOREM 7.1. *Consider the hybrid system* (7.1). *Suppose that there exists a proper dilation M of $\mathbb{R}^n$, a homogeneous with respect to M quasinorm ω, and $d\in\mathbb{R}$ such that, for each $q\in Q$, the set-valued mappings*

$$x\mapsto\omega^{-d}(x)M\left(\omega^{-1}(x)\right)F_q(x),\qquad x\mapsto M\left(\omega^{-1}(x)\right)G_q(x)$$

are locally bounded at $x=0$. If 0 is pre-asymptotically stable for the hybrid system

$$\left\{\begin{array}{lcl} x\in T^M_{C_q}(0) & \dot{x} & \in \operatorname{con}F_q^{M,d}\,(x)\\ x\in T^M_{D_q}(0) & \begin{pmatrix}q^+\\ x^+\end{pmatrix} & \in\ \ G_q^{M,0}\,(x)\end{array}\right. \tag{7.2}$$

then 0 is pre-asymptotically stable for the hybrid system (7.1).

Hybrid systems with logical modes (7.1) can be formulated as (2.1). To this end, one constructs a hybrid system in $\mathbb{R}^{n+1}$ with the variable (q,x) as follows:

$$C=\bigcup_{q\in Q}\{q\}\times C_q,\quad F(q,x)=\begin{pmatrix}0\\ F_q(x)\end{pmatrix},\quad D=\bigcup_{q\in Q}\{q\}\times D_q,\quad G(q,x)=G_q(x). \tag{7.3}$$

Details can be seen, for example, in [27]. However, Theorem 7.1 is not a special case of Theorem 3.16. Pre-asymptotic stability of the origin for (7.1) corresponds to pre-asymptotic stability of $Q\times\{0\}$ for (7.3). Furthermore, in Theorem 7.1, the "tangent approximation" is carried out in each mode q separately, reformulation of (7.2) along the lines of (7.3) is homogeneous but with respect to a dilation that is not proper, etc. Still, the key ideas of the proof are similar, and many elements of the proof of Theorem 7.1 just need to be repeated in each mode q. We only outline the argument.

7.2. Outline of the proof of Theorem 7.1. The system (7.2) is *homogeneous* in the following sense: for each $q\in Q$, C_q, D_q are homogeneous with respect to M, F_q is homogeneous with respect to M with order d, and for each $x\in\mathbb{R}^n$, $\lambda>0$, $G_q(M(\lambda)x)=\begin{bmatrix}1 & 0\\ 0 & M(\lambda)\end{bmatrix}G_q(x)$. For (7.2), and for each system homogeneous in the same sense, a result parallel to Lemma 4.2 is straightforward: (q,ϕ) is a solution to (7.1) if and only if $(q,M(\lambda)\phi)$ is a solution to a system like (7.1) but where F_q is replaced by $\lambda^{-d}F_q$. Then, a result like Proposition 4.3 follows. Furthermore, for each $q\in Q$, the flow and the jump sets for (7.2) are closed, while the flow and the jump maps are outer semicontinuous and locally bounded at each point except $x=0$. In other words, (7.2) reformulated as (2.1) along the lines of (7.3) satisfies the basic assumptions on $\mathbb{R}^{n+1}\setminus(Q\times\{0\})$.

If 0 for (7.2) is pre-asymptotically stable, then there exists $\rho>0$ such that

$$\left\{\begin{array}{lll} x\in\left(T^M_{C_q}(0)\right)_\rho & \dot{x} & \in\left(\mathrm{con}F_q^{M,d}\right)_\rho(x)\\ x\in\left(T^M_{D_q}(0)\right)_\rho & \begin{pmatrix}q^+\\ x^+\end{pmatrix} & \in\left(G_q^{M,0}\right)_\rho(x)\end{array}\right. \tag{7.4}$$

has the origin pre-asymptotically stable. Above, $\left(T^M_{C_q}(0)\right)_\rho$, $\left(\mathrm{con}F_q^{M,d}\right)_\rho$, and $\left(T^M_{D_q}(0)\right)_\rho$ are obtained from $T^M_{C_q}(0)$, $\mathrm{con}F_q^{M,d}$, and $T^M_{D_q}(0)$, respectively, just like C_ρ, F_ρ, and D_ρ are obtained from C, F, and D, respectively, in Definition 4.4, while

$$\left(G_q^{M,0}\right)_\rho(x)=G_q^{M,0}\left((x+\rho M(\omega(x))\mathcal{B})\cap T^M_{D_q}(0)\right)+\{0\}\times\rho M(\omega(x))\mathcal{B}.$$

To prove this, the arguments parallel to those in the proof of Theorem 4.9 can be given. These arguments use the fact that the system

$$\left\{\begin{array}{lll} x\in T^M_{C_q}(0)\setminus\mathrm{int}\,\mathcal{B} & \dot{x} & \in\mathrm{con}F_q^{M,d}(x)\\ x\in T^M_{D_q}(0)\setminus\mathrm{int}\,\mathcal{B} & \begin{pmatrix}q^+\\ x^+\end{pmatrix} & \in G_q^{M,0}(x)\end{array}\right.$$

satisfies the basic assumptions on $\mathbb{R}^{n+1}$, is pre-asymptotically stable, and thus there exists a $\mathcal{KL}$ function β and for each $\varepsilon>0$, each $K>0$, there exists $\delta>0$ such that (4.3) holds for all solutions (q,ϕ) to it. The bound (4.3) comes from Proposition 6.4 and Theorems 6.5, 6.6 in [14], since the concept of pre-asymptotic stability used here translates, via the trick used in the proof of Lemma 4.8, to asymptotic stability of the compact set $Q \times \{0\}$ in the sense of [14]. These arguments also use the fact that the system (7.4) is homogeneous, in the sense mentioned above. Indeed, the justification for $\left(T^M_{C_q}(0)\right)_\rho$, $\left(\mathrm{con}F^{M,d}_q\right)_\rho$, and $\left(T^M_{D_q}(0)\right)_\rho$ is the same as in Proposition 4.5. For $\left(G^{M,0}_q\right)_\rho$, one needs to consider, for each $q \in Q$, the mapping $x \mapsto \left\{y \mid (q,y) \in \left(G^{M,0}_q\right)_\rho(x)\right\}$, and then rely on Proposition 4.5.

The proof can be finished, as for Theorem 3.16, by noting that, for each $q \in Q$, the data of (7.4) contains, locally around 0, the data of (7.1).

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