APPENDIX 1

Matrices and Other Useful Mathematical Results

A1.1 Matrices and Matrix Algebra

The following presentation represents a very elementary and condensed discussion of matrices and matrix operations. If you seek a more comprehensive introduction to the subject, consult the books listed in the references indicated at the end of Chapter 11.

We will define a *matrix* as a rectangular array (arrangement) of real numbers and will indicate specific matrices symbolically with bold capital letters. The numbers in the matrix, *elements*, appear in specific row-column positions, all of which are filled. The number of rows and columns may vary from one matrix to another, so we conveniently describe the size of a matrix by giving its *dimensions*—that is, the number of its rows and columns. Thus matrix **A**

$$\underset{2\times 3}{\mathbf{A}} = \begin{bmatrix} 6 & 0 & -1 \\ 4 & 2 & 7 \end{bmatrix}$$

possesses dimensions 2×3 because it contains two rows and three columns. Similarly, for

$$\underset{4\times 1}{\mathbf{B}} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 7 \end{bmatrix} \quad \text{and} \quad \underset{2\times 2}{\mathbf{C}} = \begin{bmatrix} 2 & 0 \\ -1 & 4 \end{bmatrix}$$

the dimensions of **B** and **C** are 4×1 and 2×2, respectively. Note that the row dimension always appears first and that the dimensions may be written below the identifying symbol of the matrix as indicated for matrices **A, B**, and **C**.

As in ordinary algebra, an element of a matrix may be indicated by a symbol, $a, b, \ldots,$ and its row-column position identified by means of a double subscript. Thus a_{21} would be the element in the second row, first column. Rows are numbered in order from top to bottom and columns from left to right. In matrix **A**, $a_{21} = 4$, $a_{13} = -1$, and so on.

Elements in a particular row are identified by their column subscript and hence are numbered from left to right. The first element in a row is on the left. Likewise, elements in a particular column are identified by their row subscript and therefore are identified from the top element in the column to the bottom. For example, the first element in column 2 of matrix **A** is 0, the second is 2. The first, second, and third elements of row 1 are 6, 0, and -1, respectively.

The term *matrix algebra* involves, as the name implies, an algebra dealing with matrices, much as the ordinary algebra deals with real numbers or symbols representing real numbers. Hence, we will wish to state rules for the addition and multiplication of matrices as well as to define other elements of an algebra. In so doing we will point out the similarities as well as the dissimilarities between matrix and ordinary algebra. Finally, we will use our matrix operations to state and solve a very simple *matrix equation*. This, as you may suspect, will be the solution that we desire for the least squares equations.

A1.2 Addition of Matrices

Two matrices, say **A** and **B**, can be added *only* if they are of the same dimensions. The sum of the two matrices will be a matrix obtained by adding *corresponding* elements of matrices **A** and **B**—that is, elements in corresponding positions. This being the case, the resulting sum will be a matrix of the same dimensions as **A** and **B**.

EXAMPLE **A1.1** Find the indicated sum of matrices **A** and **B**:

$$\underset{2\times 3}{\mathbf{A}} = \begin{bmatrix} 2 & 1 & 4 \\ -1 & 6 & 0 \end{bmatrix} \qquad \underset{2\times 3}{\mathbf{B}} = \begin{bmatrix} 0 & -1 & 1 \\ 6 & -3 & 2 \end{bmatrix}$$

Solution

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 2 & 1 & 4 \\ -1 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1 \\ 6 & -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} (2+0) & (1-1) & (4+1) \\ (-1+6) & (6-3) & (0+2) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 5 \\ 5 & 3 & 2 \end{bmatrix}. \quad \blacksquare$$

EXAMPLE **A1.2** Find the sum of the matrices

$$\underset{3\times 3}{\mathbf{A}} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & 4 \\ 2 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \underset{3\times 3}{\mathbf{B}} = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & 6 \\ 3 & 1 & 4 \end{bmatrix}.$$

Solution

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & -1 & 10 \\ 5 & 0 & 4 \end{bmatrix}. \quad \blacksquare$$

Note that $(\mathbf{A}+\mathbf{B}) = (\mathbf{B}+\mathbf{A})$, as in ordinary algebra, and remember that we never add matrices of unlike dimensions.

A1.3 Multiplication of a Matrix by a Real Number

We desire a rule for multiplying a matrix by a real number, for example, $3\mathbf{A}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 4 & 6 \\ -1 & 0 \end{bmatrix}.$$

Certainly we would want $3\mathbf{A}$ to equal $(\mathbf{A}+\mathbf{A}+\mathbf{A})$, to conform with the addition rule. Hence, $3\mathbf{A}$ would mean that each element in the $\mathbf{A}$ matrix must be multiplied by the multiplier 3, and

$$3\mathbf{A} = \begin{bmatrix} 3(2) & 3(1) \\ 3(4) & 3(6) \\ 3(-1) & 3(0) \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 12 & 18 \\ -3 & 0 \end{bmatrix}.$$

In general, given a real number c and a matrix $\mathbf{A}$ with elements a_{ij}, the product $c\mathbf{A}$ will be a matrix whose elements are equal to ca_{ij}.

A1.4 Matrix Multiplication

The rule for matrix multiplication requires "row-column multiplication," which we will define subsequently. The procedure may seem a bit complicated to the novice but should not prove too difficult after practice. We will illustrate with an example.

Let **A** and **B** be

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}.$$

An element in the *ith row* and *jth column* of the product **AB** is obtained by multiplying the *ith row of* **A** by the *jth column of* **B**. Thus the element in the first row, first column of **AB** is obtained by multiplying the first row of **A** by the first column of **B**. Likewise, the element in the first row, second column would be the product of the first row of **A** and the second column of **B**. Notice that we always use the rows of **A** and the columns of **B**, where **A** is the matrix to the left of **B** in the product **AB**.

Row-column multiplication is relatively easy. Obtain the products, first-row element by first-column element, second-row element by second-column element, third by third, and so on, and then sum. Remember that row and column elements are marked from left to right and top to bottom, respectively.

Applying these rules to our example, we obtain

$$\underset{2\times 2}{\mathbf{A}}\ \underset{2\times 2}{\mathbf{B}} = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 1 & 14 \end{bmatrix}.$$

The first-row-first-column product would be $(2)(5)+(0)(-1) = 10$, which is located (and circled) in the first row, first column of **AB**. Likewise, the element in the first row, second column is equal to the product of the first row of **A** and the second column of **B**, or $(2)(2)+(0)(3) = 4$. The second-row-first-column product is $(1)(5)+(4)(-1) = 1$ and is located in the second row, first column of **AB**. Finally, the second-row-second-column product is $(1)(2) + (4)(3) = 14$.

EXAMPLE **A1.3** Find the products **AB** and **BA**, where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & -1 & -1 \\ 2 & 0 & 2 \end{bmatrix}.$$

Solution

$$\underset{3\times 2}{\mathbf{A}}\ \underset{2\times 3}{\mathbf{B}} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 4 & -1 & -1 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 10 & -2 & 0 \\ 2 & -1 & -3 \\ 8 & 0 & 8 \end{bmatrix}$$

and

$$\underset{2\times 3}{\mathbf{B}}\ \underset{3\times 2}{\mathbf{A}} = \begin{bmatrix} 4 & -1 & -1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 4 & 10 \end{bmatrix}.$$

■

Note that in matrix algebra, unlike ordinary algebra, **AB** does not equal **BA**. Because **A** contains three rows and **B** contains three columns, we can form $(3)(3) = 9$ row-column combinations and hence nine elements for **AB**. In contrast, **B** contains only two rows, **A** two columns, and hence the product **BA** will possess only $(2)(2) = 4$ elements, corresponding to the four different row-column combinations.

Furthermore, we observe that row-column multiplication is predicated on the assumption that the rows of the matrix on the left contain the same number of elements as the columns of the matrix on the right, so that corresponding elements will exist for the row-column multiplication. What do we do when this condition is not satisfied? We agree never to multiply two matrices, say **AB**, where the rows of **A** and the columns of **B** contain an unequal number of elements.

An examination of the dimensions of the matrices will tell whether they can be multiplied as well as give the dimensions of the product. Writing the dimensions underneath the two matrices,

$$\underset{m\times p}{\mathbf{A}}\;\underset{p\times q}{\mathbf{B}} = \underset{m\times q}{\mathbf{AB}}$$

we observe that the inner two numbers, giving the number of elements in a row of **A** and column of **B**, respectively, must be equal. The outer two numbers, indicating the number of rows of **A** and columns of **B**, give the dimensions of the product matrix. You may verify the operation of this rule for Example A1.3.

EXAMPLE **A1.4** Obtain the product **AB**:

$$\underset{1\times 3}{\mathbf{A}}\;\underset{3\times 2}{\mathbf{B}} = [\,2 \quad 1 \quad 0\,]\begin{bmatrix} 2 & 0 \\ 0 & 3 \\ -1 & 0 \end{bmatrix} = [\,4 \quad 3\,]$$

Note that product **AB** is (1×2) and that **BA** is undefined because of the respective dimensions of **A** and **B**.

EXAMPLE **A1.5** Find the product **AB**, where

$$\mathbf{A} = [\,1 \quad 2 \quad 3 \quad 4\,] \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solution

$$\underset{1\times 4}{\mathbf{A}}\;\underset{4\times 1}{\mathbf{B}} = [\,1 \quad 2 \quad 3 \quad 4\,]\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = [\,30\,].$$

Note that this example produces a different method for writing a sum of squares. ■

A1.5 Identity Elements

The identity elements for addition and multiplication in ordinary algebra are 0 and 1, respectively. In addition, 0 plus any other element, say a, is identically equal to a; that is,

$$0 + 2 = 2, \qquad 0 + (-9) = -9.$$

Similarly, the multiplication of the identity element 1 by any other element, say a, is equal to a; that is,

$$(1)(5) = 5, \qquad (1)(-4) = -4.$$

In matrix algebra two matrices are said to be equal when all corresponding elements are equal. With this in mind we will define the identity matrices in a manner similar to that employed in ordinary algebra. Hence, if **A** is any matrix, a matrix **B** will be an identity matrix for addition if

$$\mathbf{A} + \mathbf{B} = \mathbf{A} \qquad \text{and} \qquad \mathbf{B} + \mathbf{A} = \mathbf{A}.$$

It easily can be seen that the identity matrix for addition is one in which every element is equal to zero. This matrix is of interest but of no practical importance in our work.

Similarly, if **A** is any matrix, the identity matrix for multiplication is a matrix **I** that satisfies the relation

$$\mathbf{AI} = \mathbf{A} \qquad \text{and} \qquad \mathbf{IA} = \mathbf{A}.$$

This matrix, called the *identity matrix*, is the *square matrix*

$$\underset{n\times n}{\mathbf{I}} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

That is, all elements in the *main diagonal* of the matrix, running from top left to bottom right, are equal to 1; all other elements equal zero. Note that the identity matrix is always indicated by the symbol **I**.

Unlike ordinary algebra, which contains only one identity element for multiplication, matrix algebra must contain an infinitely large number of identity matrices. Thus we must have matrices with dimensions 1×1, 2×2, 3×3, 4×4, and so on, so as to provide an identity of the correct dimensions to permit multiplication. All will be of this pattern.

That the **I** matrix satisfies the relation

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}$$

can be shown by an example.

EXAMPLE **A1.6** Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 6 & 3 \end{bmatrix}.$$

Show that $\mathbf{IA} = \mathbf{A}$ and $\mathbf{AI} = \mathbf{A}$.

Solution

$$\underset{2\times2}{\mathbf{I}}\;\underset{2\times3}{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ -1 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 6 & 3 \end{bmatrix} = \mathbf{A}$$

and

$$\underset{2\times3}{\mathbf{A}}\;\underset{3\times3}{\mathbf{I}} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 6 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 6 & 3 \end{bmatrix} = \mathbf{A}.$$

■

A1.6 The Inverse of a Matrix

For matrix algebra to be useful, we must be able to construct and solve matrix equations for a matrix of unknowns in a manner similar to that employed in ordinary algebra. This, in turn, requires a method of performing division.

For example, we would solve the simple equation in ordinary algebra,

$$2x = 6$$

by dividing both sides of the equation by 2 and obtaining $x = 3$. Another way to view this operation is to define the reciprocal of each element in an algebraic system and to think of division as multiplication by the reciprocal of an element. We could solve the equation $2x = 6$ by multiplying both sides of the equation by the reciprocal of 2. Because every element in the real number system possesses a reciprocal, with the exception of 0, the multiplication operation eliminates the need for division.

The reciprocal of a number c in ordinary algebra is a number b that satisfies the relation

$$cb = 1$$

that is, the product of a number by its reciprocal must equal the identity element for multiplication. For example, the reciprocal of 2 is $1/2$ and $(2)(1/2) = 1$.

A reciprocal in matrix algebra is called the *inverse* of a matrix and is defined as follows:

DEFINITION **A1.1**

Let $\mathbf{A}_{n\times n}$ be a square matrix. If a matrix $\mathbf{A}^{-1}$ can be found such that

$$\mathbf{AA}^{-1} = \mathrm{I} \qquad \text{and} \qquad \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

then $\mathbf{A}^{-1}$ is called the *inverse* of $\mathbf{A}$.

Note that the requirement for an inverse in matrix algebra is the same as in ordinary algebra—that is, the product of $\mathbf{A}$ by its inverse must equal the identity matrix for multiplication. Furthermore, the inverse is undefined for nonsquare matrices, and hence many matrices in matrix algebra do not have inverses (recall that 0 was the only element in the real number system without an inverse). Finally, we state without proof that many square matrices do not possess inverses. Those that do will be identified in Section A1.9, and a method will be given for finding the inverse of a matrix.

A1.7 The Transpose of a Matrix

We have just discussed a relationship between a matrix and its inverse. A second useful matrix relationship defines the *transpose* of a matrix.

DEFINITION A1.2

Let $A_{p\times q}$ be a matrix of dimensions $p \times q$. Then $\mathbf{A}'$, called the *transpose* of $\mathbf{A}$, is defined to be a matrix obtained by interchanging corresponding rows and columns of $\mathbf{A}$; that is, first with first, second with second, and so on.

For example, let

$$\underset{3\times 2}{\mathbf{A}} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 4 & 3 \end{bmatrix}.$$

Then

$$\underset{2\times 3}{\mathbf{A}'} = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 3 \end{bmatrix}.$$

Note that the first and second rows of $\mathbf{A}'$ are identical with the first and second columns, respectively, of $\mathbf{A}$.

As a second example, let

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Then $\mathbf{Y}' = [y_1 \quad y_2 \quad y_3]$. As a point of interest, we observe that $\mathbf{Y}'\mathbf{Y} = \sum_{i=1}^{3} y_i^2$.

Finally, if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 3 \\ 1 & 6 & 9 \end{bmatrix}$$

then

$$\mathbf{A}' = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 6 \\ 4 & 3 & 9 \end{bmatrix}.$$

A1.8 A Matrix Expression for a System of Simultaneous Linear Equations

We will now introduce you to one of the very simple and important applications of matrix algebra. Let

$$2v_1 + v_2 = 5$$
$$v_1 - v_2 = 1$$

be a pair of simultaneous linear equations in the two variables, v_1 and v_2. We will then define three matrices:

$$\underset{2\times 2}{\mathbf{A}} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \qquad \underset{2\times 1}{\mathbf{V}} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \underset{2\times 1}{\mathbf{G}} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Note that **A** is the matrix of coefficients of the unknowns when the equations are each written with the variables appearing in the same order, reading left to right, and with the constants on the right-hand side of the equality sign. The **V** matrix gives the unknowns in a column and in the same order as they appear in the equations. Finally, the **G** matrix contains the constants in a column exactly as they occur in the set of equations.

The simultaneous system of two linear equations may now be written in matrix notation as

$$\mathbf{AV} = \mathbf{G}$$

a statement that can easily be verified by multiplying **A** and **V** and then comparing the answer with **G**.

$$\mathbf{AV} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 + v_2 \\ v_1 - v_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} = G.$$

Observe that corresponding elements in **AV** and **G** are equal—that is, $2v_1 + v_2 = 5$ and $v_1 - v_2 = 1$. Therefore, $\mathbf{AV} = \mathbf{G}$.

The method for writing a pair of linear equations in two unknowns as a matrix equation can easily be extended to a system of r equations in r unknowns. For example, if the equations are

$$\begin{aligned} a_{11}v_1 + a_{12}v_2 + a_{13}v_3 + \cdots + a_{1r}v_r &= g_1 \\ a_{21}v_1 + a_{22}v_2 + a_{23}v_3 + \cdots + a_{2r}v_r &= g_2 \\ a_{31}v_1 + a_{32}v_2 + a_{33}v_3 + \cdots + a_{3r}v_r &= g_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots &= \vdots \\ a_{r1}v_1 + a_{r2}v_2 + a_{r3}v_3 + \cdots + a_{rr}v_r &= g_r \end{aligned}$$

define

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1r} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2r} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3r} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & a_{r3} & \cdots & a_{rr} \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_r \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_r \end{bmatrix}.$$

Observe that, once again, **A** is a square matrix of variable coefficients, whereas **V** and **G** are column matrices containing the variables and constants, respectively. Then $\mathbf{AV} = \mathbf{G}$.

Regardless of how large the system of equations, if we possess n linear equations in n unknowns, the system may be written as the simple matrix equation $\mathbf{AV} = \mathbf{G}$.

You will observe that the matrix **V** contains all the unknowns, whereas **A** and **G** are constant matrices.

Our objective, of course, is to solve for the matrix of unknowns, **V**, where the equation $\mathbf{AV} = \mathbf{G}$ is similar to the equation

$$2v = 6$$

in ordinary algebra. This being true, we would not be too surprised to find that the methods of solutions are the same. In ordinary algebra both sides of the equation are multiplied by the reciprocal of 2; in matrix algebra both sides of the equation are multiplied by $\mathbf{A}^{-1}$. Then

$$\mathbf{A}^{-1}(\mathbf{AV}) = \mathbf{A}^{-1}\mathbf{G}$$

or

$$\mathbf{A}^{-1}\mathbf{AV} = \mathbf{A}^{-1}\mathbf{G}.$$

But $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{IV} = \mathbf{V}$. Therefore, $\mathbf{V} = \mathbf{A}^{-1}\mathbf{G}$. In other words, the solutions to the system of simultaneous linear equations can be obtained by finding $\mathbf{A}^{-1}$ and then obtaining the product $\mathbf{A}^{-1}\mathbf{G}$. The solutions values of $v_1, v_2, v_3, \ldots, v_r$ will appear in sequence in the column matrix $\mathbf{V} = \mathbf{A}^{-1}\mathbf{G}$.

A1.9 Inverting a Matrix

We have indicated in Section A1.8 that the key to the solutions of a system of simultaneous linear equations by the method of matrix algebra rests on the acquisition of the inverse of the **A** matrix. Many methods exist for inverting matrices. The method that we present is not the best from a computational point of view, but it works very well for the matrices associated with most experimental designs and it is one of the easiest to present to the novice. It depends upon a theorem in matrix algebra and the use of *row operations*.

Before defining *row operations* on matrices, we must state what is meant by the addition of two rows of a matrix and the multiplication of a row by a constant. We will illustrate with the **A** matrix for the system of two simultaneous linear equations,

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

Two rows of a matrix may be added by adding corresponding elements. Thus if the two rows of the **A** matrix are added, one obtains a new row with elements $[(2+1)\ (1-1)] = [3\ \ 0]$. Multiplication of a row by a constant means that each element in the row is multiplied by the constant. Twice the first row of the **A** matrix would generate the row $[4\ \ 2]$. With these ideas in mind, we will define three ways to operate on a row in a matrix:

1. A row may be multiplied by a constant.
2. A row may be multiplied by a constant and added to or subtracted from another row (which is identified as the one upon which the operation is performed).
3. Two rows may be interchanged.

Given matrix **A**, it is quite easy to see that we might perform a series of row operations that would yield some new matrix **B**. In this connection we state without proof a surprising and interesting theorem from matrix algebra; namely, there exists some matrix **C** such that

$$\mathbf{CA} = \mathbf{B}.$$

In other words, a series of row operations on a matrix **A** is equivalent to multiplying **A** by a matrix **C**. We will use this principle to invert a matrix.

Place the matrix **A**, which is to be inverted, alongside an identity matrix of the same dimensions:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \quad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then perform the same row operations on **A** *and* **I** *in such a way that* **A** *changes to an identity matrix*. In doing so, we must have multiplied **A** by a matrix **C** so that $\mathbf{CA} = \mathbf{I}$. Therefore, **C** must be the inverse of **A**! The problem, of course, is to find the unknown matrix **C** and, fortunately, this proves to be of little difficulty. Because we performed the same row operations on **A** and **I**, the identity matrix must have changed to $\mathbf{CI} = \mathbf{C} = \mathbf{A}^{-1}$.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\downarrow \text{ (same row operations) } \downarrow$$

$$\mathbf{CA} = \mathbf{I} \qquad \mathbf{CI} = \mathbf{C} = \mathbf{A}^{-1}$$

We will illustrate with the following example.

EXAMPLE **A1.7** Invert the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

Solution

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Step 1. Operate on row 1 by multiplying row 1 by 1/2. (Note: It is helpful to the beginner to identify the row upon which he or she is operating because all other rows will remain unchanged, even though they may be used in the operation. We will star the row upon which the operation is being performed.)

$$\begin{matrix} * \\ \end{matrix}\begin{bmatrix} 1 & 1/2 \\ 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Step 2. Operate on row 2 by subtracting row 1 from row 2.

$$\begin{matrix} \\ * \end{matrix}\begin{bmatrix} 1 & 1/2 \\ 0 & -3/2 \end{bmatrix} \qquad \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}.$$

(Note that row 2 is simply used to operate on row 1 and hence remains unchanged.)

Step 3. Multiply row 2 by $(-2/3)$.

$$\begin{matrix} \\ * \end{matrix}\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1/2 & 0 \\ 1/3 & -2/3 \end{bmatrix}.$$

Step 4. Operate on row 1 by multiplying row 2 by 1/2 and subtracting from row 1.

$$*\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix}.$$

(Note that row 2 is simply used to operate on row 1 and hence remains unchanged.) Hence the inverse of $\mathbf{A}$ must be

$$\mathbf{A}^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix}.$$

A ready check on the calculations for the inversion procedure is available because $\mathbf{A}^{-1}A$ must equal the identity matrix $\mathbf{I}$. Thus

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$ ■

EXAMPLE **A1.8** Invert the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

and check the results.

Solution

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step 1. Multiply row 1 by 1/2.

$$*\begin{bmatrix} 1 & 0 & 1/2 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step 2. Operate on row 2 by subtracting row 1 from row 2.

$$\begin{matrix} \\ * \\ \\ \end{matrix}\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & -1 & 3/2 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step 3. Operate on row 3 by subtracting row 1 from row 3.

$$\begin{matrix} \\ \\ * \end{matrix}\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & -1 & 3/2 \\ 0 & 0 & -1/2 \end{bmatrix} \qquad \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}.$$

Step 4. Operate on row 2 by multiplying row 3 by 3 and adding to row 2.

$$\begin{matrix} \\ * \\ \\ \end{matrix}\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & -1 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \qquad \begin{bmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 3 \\ -1/2 & 0 & 1 \end{bmatrix}.$$

Step 5. Multiply row 2 by (-1).

$$* \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \quad \begin{bmatrix} 1/2 & 0 & 0 \\ 2 & -1 & -3 \\ -1/2 & 0 & 1 \end{bmatrix}.$$

Step 6. Operate on row 1 by adding row 3 to row 1.

$$* \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & -3 \\ -1/2 & 0 & 1 \end{bmatrix}.$$

Step 7. Multiply row 3 by (-2).

$$* \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & -3 \\ 1 & 0 & -2 \end{bmatrix} = \mathbf{A}^{-1}.$$

The seven row operations have changed the **A** matrix to the identity matrix and, barring errors of calculation, have changed the identity to $\mathbf{A}^{-1}$.

Checking, we have

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & -3 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We see that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and hence that the calculations are correct. ■

Note that the sequence of row operations required to convert **A** to **I** is not unique. One person might achieve the inverse by using five row operations whereas another might require ten, but the end result will be the same. However, in the interests of efficiency it is desirable to employ a system.

Observe that the inversion process utilizes row operations to change off-diagonal elements in the **A** matrix to 0s and the main diagonal elements to 1s. One systematic procedure is as follows. Change the top left element into a 1 and then perform row operations to change all other elements in the *first* column to 0. Then move to the diagonal element in the second row, second column, change it into a 1, and change all elements in the *second* column *below* the main diagonal to 0. This process is repeated, moving down the main diagonal from top left to bottom right, until all elements below the main diagonal have been changed to 0s. To eliminate nonzero elements above the main diagonal, operate on all elements in the last column, changing each to 0; then move to the next to last column and repeat the process. Continue this procedure until you arrive at the first element in the first column, which was the starting point. This procedure is indicated diagrammatically in Figure A1.1.

Matrix inversion is a tedious process, at best, and requires every bit as much labor as the solutions of a system of simultaneous equations by elimination or substitution. You will be pleased to learn that we do not expect you to develop a facility for matrix inversion. Fortunately, most matrices associated with designed experiments follow patterns and are easily inverted.

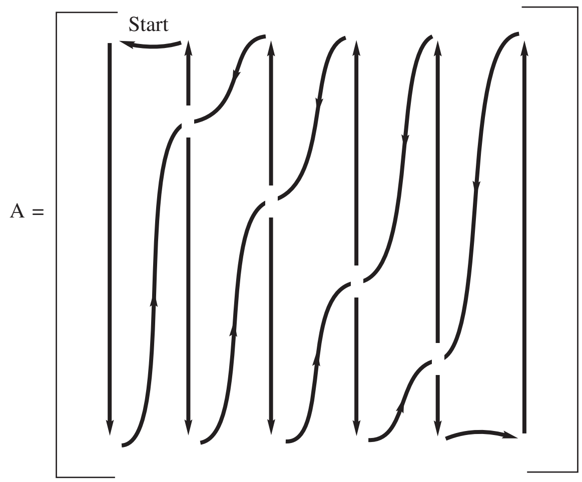

FIGURE **A1.1** Procedure for matrix inversion

It will be beneficial to you to invert a few 2×2 and 3×3 matrices. Matrices lacking pattern, particularly large matrices, are inverted most efficiently and economically by using a computer. (Programs for matrix inversion have been developed for most computers.)

We emphasize that obtaining the solutions for the least squares equations (Chapter 11) by matrix inversion has distinct advantages that may or may not be apparent. Not the least of these is the fact that the inversion procedure is systematic and hence is particularly suitable for electronic computation. However, the major advantage is that the inversion procedure will automatically produce the variances of the estimators of all parameters in the linear model.

Before leaving the topic of matrix inversion, we ask how one may identify a matrix that has an inverse. Reference to a discussion of linear equations in ordinary algebra should reveal the answer.

Clearly, a unique solutions for a system of simultaneous linear equations cannot be obtained unless the equations are independent. Thus if one of the equations is a linear combination of the others, the equations are dependent. Coefficient matrices associated with dependent systems of linear equations do not possess an inverse.

A1.10 Solving a System of Simultaneous Linear Equations

We have finally obtained all the ingredients necessary for solving a system of simultaneous linear equations,

$$
\begin{aligned}
2v_1 + v_2 &= 5 \\
v_1 - v_2 &= 1
\end{aligned}
$$

Recalling that the matrix solutions to the system of equations $\mathbf{AV} = \mathbf{G}$ is $\mathbf{V} = \mathbf{A}^{-1}\mathbf{G}$, we obtain

$$\mathbf{V} = \mathbf{A}^{-1}\mathbf{G} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Hence the solutions is

$$\mathbf{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

that is, $v_1 = 2$ and $v_2 = 1$, a fact that may be verified by substitution of these values in the original linear equations.

EXAMPLE **A1.9** Solve the system of simultaneous linear equations

$$\begin{aligned} 2v_1 + v_3 &= 4 \\ v_1 - v_2 + 2v_3 &= 2 \\ v_1 &= 1. \end{aligned}$$

Solution The coefficient matrix for these equations,

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

appeared in Example A1.8. In that example we found that

$$\mathbf{A}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & -3 \\ 1 & 0 & -2 \end{bmatrix}.$$

Solving, we obtain

$$\mathbf{V} = \mathbf{A}^{-1}\mathbf{G} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & -3 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

Thus $v_1 = 1$, $v_2 = 3$ and $v_3 = 2$ give the solutions to the set of three simultaneous linear equations. ■

A1.11 Other Useful Mathematical Results

The purpose of this section is to provide the reader with a convenient reference to some of the key mathematical results that are used frequently in the body of the text.