COUPLED SINGULAR PERTURBATIONS
AND HOMOGENIZATION IN PHASE TRANSITIONS

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Abstract

This thesis addresses the characterization of the limiting energy of families of functionals of the type

$$J_\varepsilon(u) := \frac{1}{\varepsilon} E_\varepsilon(u)$$

with $E_\varepsilon$ of the form

$$E_\varepsilon(u) := \int_\Omega f_\varepsilon(x, u(x), \varepsilon \nabla u(x)) \, dx.$$  

Motivated by issues in fluid-fluid phase transition problems, and the corresponding development of mathematical techniques to treat (anisotropic) singular perturbations, here the focus will be on special classes of non-negative energy densities $f_\varepsilon$ satisfying $f_\varepsilon(x, u, 0) = 0$ if and only if $u \in \{a, b\}$.

The first part of this work addresses the case where the singular perturbation is coupled with the two-well energy, and the second part considers (decoupled) periodically oscillating two-well energy density and second order penalization terms.
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1 Introduction

The asymptotic behavior of functionals of the type

$$E_\varepsilon(u) := \int_\Omega \left( \frac{1}{\varepsilon} W(u(x)) + \varepsilon |\nabla u(x)|^2 \right) dx \quad (1.1)$$

has received much attention in the last two decades in the context of fluid-fluid phase transitions. If $\Omega$ is an open, bounded domain in $\mathbb{R}^N$, with Lipschitz boundary, and if $W$ is a nonnegative bulk energy density with \{W = 0\} = \{a, b\}, then Gibbs’ criterion for equilibria leads to the study of the problem

$$(P) \quad \text{minimize} \int_\Omega W(u(x)) dx \quad \text{subject to the constraint} \int_\Omega u(x) dx = m.$$

If $m = \theta a + (1 - \theta)b$, $0 < \theta < \mathcal{L}^N(\Omega)$, then the minimum problem $(P)$ admits infinitely many solutions. In order to select physically preferred solutions to this problem, and following the ideas of the gradient theory of phase transitions proposed in 1893 by van der Waals, Cahn and Hilliard [14] introduced a model where to each configuration $u$ of the two-fluid system an energy $E_\varepsilon$ which penalizes the original energy of the system $u \mapsto \int_\Omega W(u(x)) dx$ through a term containing the gradient of $u$ and a small parameter $\varepsilon > 0$, i.e. $u \mapsto \int_\Omega W(u(x)) + \varepsilon^2 |\nabla u|^2 dx$. The competing effects of the resulting two integrals favor separation of phases (i.e. those configurations where $u$ takes values close to $a$ and $b$), while penalizing spatial inhomogeneities of $u$ and, consequently, the introduction of too many transition regions.

The connection between the classical theory of phase transition based on Gibbs’ criterion and the gradient theory is due to Gurtin [30], [31], who conjectured in 1983 that solutions of

$$(P_\varepsilon) \quad \text{minimize} E_\varepsilon(u) \quad \text{subject to the constraint} \int_\Omega u(x) dx = m$$

converge to minimizers of $(P)$ having minimal interfacial energy. Gurtin’s conjecture was proved by Carr, Gurtin, and Slemrod [15] in the scalar case ($N = 1$), and independently by Modica [38] and Sternberg [43], in the higher dimensional case $N \geq 2$. It was shown in [38] (see also [1] and [12]) that $\Gamma = \lim_{\varepsilon \to 0} I_\varepsilon(u) = I(u)$ with

$$I(u) := \begin{cases} K_0 \text{Per}_\Omega(E) & \text{if } u = \chi_E a + (1 - \chi_E)b, \ \mathcal{L}^N(E) = \theta \mathcal{L}^N(\Omega), \ u \in BV(\Omega; \{a, b\}) \\ +\infty & \text{otherwise}, \end{cases} \quad (1.2)$$

and where $K_0 := 2 \int_a^b \sqrt{W(s)} ds$, thus showing that distributions of the two phases with minimal interfacial area and given volume fraction $\theta$ are selected in the limit as $\varepsilon \to 0^+$. The approach in [38] and [43] uses the notion of $\Gamma$-convergence, due to De Giorgi [19] (see also [1], [12], [17]), and follows the ideas of Modica and Mortola [39] who studied a similar functional proposed by De Giorgi in a completely different physical context.
The vector-valued case, where \( u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^d \) \( (d, N \geq 2) \) was considered by Fonseca and Tartar [27], Sternberg [44], and Barroso and Fonseca [8]. In [27] \( K_0 \) becomes

\[
\bar{K} := \inf \left\{ \int_{-L}^{L} W(g(s)) + |g'(s)|^2 ds : L > 0, g \text{ piecewise } C^1, \; g(-L) = a, \; g(L) = b \right\}. \tag{1.3}
\]

The case where \( W \) has more than two wells was addressed by Baldo [6] (see also Sternberg [44]), and later generalized by Ambrosio [2]. We refer also to Bouchitté [10], and Owen and Sternberg [40] for the study of the coupled problem, where the integrand of \( E_\varepsilon \) has the form \( \varepsilon^{-1} f(x, v(x), \varepsilon \nabla u(x)) \), and to the work of Kohn and Sternberg [35] where the study of local minimizers for \( E_\varepsilon \) was undertaken in the case where \( f \) is convex in the last variable.

In this thesis we consider two situations in the vectorial setting, with underlying energy

\[
u \mapsto \frac{1}{\varepsilon} \int_\Omega f_\varepsilon(x, u(x), \varepsilon \nabla u(x)) \, dx,
\]

where we take \( f_\varepsilon(x, u, \varepsilon \nabla u) := f(x, u, \varepsilon \nabla u) \) in Chapter 3, and \( f_\varepsilon(x, u, \varepsilon \nabla u) := W \left( \frac{x}{\varepsilon}, u \right) + \varepsilon^2 |\nabla u|^2 \) in Chapter 4.

In Chapter 3 we study the vector-valued case in a framework similar to that considered in [10] and [40] in the scalar case. Precisely, here

\[
E_\varepsilon(u) := \frac{1}{\varepsilon} \int_\Omega f(x, u(x), \varepsilon \nabla u(x)) \, dx,
\]

where \( f : \Omega \times \mathbb{R}^d \times M^{d \times N} \to [0, +\infty) \) is a continuous function satisfying the following hypotheses:

(H1) \( f(x, u, 0) = 0 \) if and only if \( u \in \{a, b\} \);

(H2) there exists a continuous function \( g : \Omega \times \mathbb{R}^d \to [0, +\infty) \) such that

\[
\frac{1}{C} (g(x, u) + |\xi|^2) \leq f(x, u, \xi) \leq C (g(x, u) + |\xi|^2)
\]

for all \((x, u, \xi) \in \Omega \times \mathbb{R}^d \times M^{d \times N}\), and

\[
\frac{1}{C} |u|^q - C \leq g(x, u) \leq C(1 + |u|^q)
\]

for some \( q \geq 2, \; C > 0 \), and for all \((x, u, \xi) \in \Omega \times \mathbb{R}^d \times M^{d \times N}\).

(H3) For any \( x_0 \in \Omega \) and any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( |x - x_0| < \delta \) implies that

\[
|f(x, u, \xi) - f(x_0, u, \xi)| \leq \varepsilon f(x, u, \xi)
\]

for every \((u, \xi) \in \mathbb{R}^d \times M^{d \times N}\).

Before stating our \( \Gamma \)-convergence result, we need to introduce some notation.

Let \( Q \subset \mathbb{R}^N \) be the open unit cube centered at the origin, and given \( \nu \in S^{N-1} := \{ x \in \mathbb{R}^N : \|x\| = 1 \} \), we denote by \( Q_\nu \) the cube centered at the origin with two of its faces normal to \( \nu \). Precisely, if \( \{\nu_1, ..., \nu_{N-1}, \nu\} \) is an orthonormal basis of \( \mathbb{R}^N \), then

\[
Q_\nu := \left\{ x \in \mathbb{R}^N : |x \cdot \nu_i| < \frac{1}{2}, \; |x \cdot \nu| < \frac{1}{2}, \; i = 1, ..., N - 1 \right\}.
\]
Let \((a, b, \nu) \in \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}\), and define the class of admissible functions
\[
\mathcal{A}(a, b, \nu) := \left\{ \xi \in W^{1, \infty}_{\text{loc}}(S_\nu; \mathbb{R}^d) : \xi(y) = a \text{ if } y \cdot \nu = -\frac{1}{2}, \quad \xi(y) = b \text{ if } y \cdot \nu = \frac{1}{2}, \right. \\
\left. \quad \text{and } \xi(y) = \xi(y + k\nu), \text{ for all } y \in S_\nu, \ i \in \{1, \ldots, N - 1\}, \text{ and } k \in \mathbb{Z} \right\},
\]
where \(S_\nu\) is the strip
\[
S_\nu := \left\{ y \in \mathbb{R}^N : |y \cdot \nu| < \frac{1}{2} \right\}.
\]
We introduce the surface energy density \(K : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1} \to [0, +\infty)\) defined by
\[
K(x, a, b, \nu) := \inf_{s > 0} \left\{ \int_{Q_{\nu}} \frac{1}{s} f(x, \xi(y), s\nabla \xi(y)) dy : \xi \in \mathcal{A}(a, b, \nu(x)) \right\}.
\]

The main result of Chapter 3 is the following theorem.

**Theorem 1.1** ([25, Theorem 1.1]) Assume that (H1)-(H3) hold, and for every \(\varepsilon > 0\) let \(J_\varepsilon : L^1(\Omega; \mathbb{R}^d) \to [0, +\infty]\) be the functional defined by
\[
J_\varepsilon(u) := \begin{cases} 
E_\varepsilon(u) & \text{if } u \in H^1(\Omega; \mathbb{R}^d), \\
+\infty & \text{otherwise}.
\end{cases}
\]

Then \(J_\varepsilon \Gamma(L^1(\Omega))-\)converges to the functional \(J_0 : L^1(\Omega; \mathbb{R}^d) \to [0, +\infty]\) defined by
\[
J_0(u) := \begin{cases} 
\int_{\Omega \cap \partial^* A_0} K(x, a, b, \nu(x)) d\mathcal{H}^{N-1}(x) & \text{if } u \in \text{BV}(\Omega; \{a, b\}), \\
+\infty & \text{otherwise},
\end{cases}
\]
where \(A_0 := \{ x \in \Omega : u(x) = a \}\), and \(\nu(x)\) stands for the measure theoretic inner unit normal to the reduced boundary \(\partial^* A_0\) at \(x\).

**Remark 1.2**

(i) The \(\Gamma\)-convergence results obtained in the scalar case by Bouchitté [10] and by Owen and Sternberg [40] assume that \(f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)\) is convex in the last variable, and, in addition, require some regularity of \(f\). Precisely, in [40] \(f\) is assumed to be of class \(C^4(\Omega \times \mathbb{R} \times \mathbb{R}^N)\), thus extending the result of Owen, who treated the case of radial perturbations (\(f\) is of class \(C^4\) and depends radially on \(\xi\)), while [10] requires differentiability of \(f(x, u, \cdot)\) at the origin where it is assumed that \(f\) achieves a strict minimum. The proof in [10] is based on duality techniques, and the expression of the \(\Gamma\)-limit involves the conical envelope of \(f(x, u, \cdot)\), defined by \(f_\varepsilon(x, u, \cdot) := \inf_{t > 0} \frac{1}{t} f(x, u, t)\). Here we do not assume any convexity condition on \(f\), and although we need to impose some technical growth hypothesis (H2) (and here we remind that convex problems usually do not require growth conditions, while nonconvex, vector-valued problems use them in an essential way), our regularity assumption (H3) is milder. For the proof of Theorem 1.1 we need to use different methods due to the technical nature of the vector-valued case, and to the fact that the lack of convexity does not allow us to invoke duality arguments.

(ii) Taking \(f(x, u, \xi) := W(u) + \varepsilon h^2(x, \xi)\), we recover the result of Barroso and Fonseca (see [8]) concerning the \(\Gamma(L^1)\)-limit of
\[
u \mapsto \int_\Omega \left( \frac{1}{\varepsilon} W(u(x)) dx + \varepsilon h^2(x, \nabla u(x)) \right) dx,
\]
where $W$ has two isolated (global) minimum points at $a, b \in \mathbb{R}^d$. Although apriori (H5) of [8] is weaker than (H3) of this paper, Lemma 2.8 and hypothesis (H5) in [8] imply that $W + (h^\infty)^2(\cdot, \cdot)$ satisfies our hypothesis (H3), where $(h^\infty)$ is the recession function defined by $h^\infty(x, \xi) := \limsup_{t \to \infty} \frac{h(tx, t\xi)}{t}$. It can be shown that under (H4) of [8] we have
\[
\inf \left\{ \lim \inf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_\Omega (W(u_\varepsilon(x)) + \varepsilon^2 |\nabla u_\varepsilon(x)|^2) \, dx : u_\varepsilon \to u \text{ in } L^1(\Omega; \mathbb{R}^d), \quad u_\varepsilon \in H^1(\Omega; \mathbb{R}^d) \right\}
\]
and now, in view of the above considerations, we are in position to apply our result. We remark that the technique used here to prove Theorem 1.1 is different from that in [8], in particular in what concerns the existence of a recovery subsequence for the $\Gamma$-limit, where our approach is based on localization methods for $\Gamma$-convergence, De Giorgi’s slicing method, and a blow-up argument, a technique that was only incipient at the time when [8] was written.

In Chapter 4 of the thesis we study a homogenization problem within the context of the gradient theory of phase transitions, still in the vector-valued setting. Here $f_\varepsilon(x, u, \nabla u) := W(x_\varepsilon, u) + \varepsilon^2 |\nabla u|^2$, with $W : \mathbb{R}^N \times \mathbb{R}^d \to [0, +\infty)$ a continuous function satisfying the following hypotheses:

(A1) $W(\cdot, u)$ is $Q$-periodic for every $u \in \mathbb{R}^d$;

(A2) $W(x, u) = 0$ if and only if $u \in \{a, b\}$;

(A3) there exist $C > 0$ and $q \geq 2$ such that
\[
\frac{1}{C} |u|^q - C \leq W(x, u) \leq C(1 + |u|^q)
\]
for all $(x, u) \in \Omega \times \mathbb{R}^d$,

and let $I_\varepsilon : L^1(\Omega; \mathbb{R}^d) \to [0, +\infty]$ be defined by
\[
I_\varepsilon(u) := \begin{cases}
\int_\Omega \left( \frac{1}{\varepsilon} W(x_\varepsilon, u(x)) + \varepsilon |\nabla u(x)|^2 \right) \, dx & \text{if } u \in H^1(\Omega; \mathbb{R}^d) \\
+\infty & \text{otherwise}.
\end{cases}
\]

The main result of Chapter 4 is the following theorem

**Theorem 1.3** ([26, Theorem 1.1]) Assume that (A1)-(A3) hold, let $\nu \in S^{N-1}$, $\rho : \mathbb{R} \to [0, +\infty)$ be a mollifier, and let $\rho_{T, \nu}(x) := T^N \rho(Tx \cdot \nu)$. Define
\[
K_1(\nu) = \lim_{T \to \infty} \frac{1}{T^{N-1}} \inf \left\{ \int_{TQ_\nu} (W(y, u(y)) + |\nabla u(y)|^2) \, dy : u \in H^1(TQ_\nu; \mathbb{R}^d), \quad u = \rho_{T, \nu} * u_0 \text{ on } \partial(TQ_\nu) \right\}
\]
with
\[ u_0(x) = \begin{cases} b & \text{if } x \cdot \nu > 0, \\ a & \text{if } x \cdot \nu < 0. \end{cases} \]

Consider the functional \( I_0 : L^1(\Omega; \mathbb{R}^d) \to [0, +\infty] \) defined by
\[
I_0(u) := \begin{cases} \int_{\partial^* A_0 \cap \Omega} K_1(\nu(x))d\mathcal{H}^{N-1}(x) & \text{if } u \in BV(\Omega; \{a, b\}), \\ +\infty & \text{otherwise}, \end{cases}
\]
where \( A_0 := \{ x \in \Omega : u(x) = a \} \). Then

(i) \( \Gamma(L^1(\Omega; \mathbb{R}^d)) - \liminf_{\varepsilon \to 0} I_\varepsilon = I_0 \);

(ii) Assume that the set \( A_0 \) is polyhedral, and that the outward unit normal \( \nu(x) \) to the reduced boundary \( \partial^* A_0 \) is such that \( \nu(x) \in \{ \pm e_1, \cdots, \pm e_N \} \), for \( \mathcal{H}^{N-1} \)-a.e. \( x \in (\partial^* A_0) \cap \Omega \). Then
\[
\Gamma(L^1(\Omega; \mathbb{R}^d)) - \lim_{\varepsilon \to 0} I_\varepsilon = I_0.
\]

Remark 1.4 Without the additional assumption in part (ii) of Theorem 1.3, some of the techniques used in Section 4.3 to construct a recovering sequence for the \( \Gamma \)-limit would only go through under the (far too strong) requirement that \( W(R, u) \) be \( Q \)-periodic for all rotations \( R \in SO(N) \), and \( u \in \mathbb{R}^d \). Future work will address the general case. The geometry of \( A_0 \) is important here, as it can be seen in (4.51), where the periodicity of \( W(\cdot, u) \) with respect to the directions orthogonal to \( \nu(x_0) \) is strongly used.
2 Preliminaries

We begin this chapter by recalling some facts about functions of bounded variations (we refer the reader to [4] for details). A function $u \in L^1(\Omega; \mathbb{R}^d)$ is said to be of bounded variation, and we write $u \in BV(\Omega; \mathbb{R}^d)$, if for all $i = 1, \cdots, d$, and $j = 1, \cdots, N$, there exists a Radon measure $\mu_{ij}$ such that
\[
\int_\Omega u_i(x) \frac{\partial u}{\partial x_j}(x) \, dx = - \int_\Omega v(x) \, d\mu_{ij}
\]
for every $v \in C^1_c(\Omega; \mathbb{R})$. The distributional derivative $Du$ is the matrix-valued measure with components $\mu_{ij}$. Given $u \in BV(\Omega; \mathbb{R}^d)$ the approximate upper and lower limit of each component $u_i$, $i = 1, \cdots, d$, are given by
\[
 u_i^+(x) := \inf \left\{ t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \mathcal{L}^N \{ y \in \Omega \cap Q(x, \varepsilon) : u_i(y) > t \} = 0 \right\}
\]
and
\[
 u_i^-(x) := \sup \left\{ t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \mathcal{L}^N \{ y \in \Omega \cap Q(x, \varepsilon) : u_i(y) < t \} = 0 \right\},
\]
while the jump set of $u$, or singular set, is defined by
\[
 S(u) := \bigcup_{i=1}^d \{ x \in \Omega : u_i^-(x) < u_i^+(x) \}.
\]
It is well known that $S(u)$ is $N - 1$ rectifiable, i.e.
\[
 S(u) = \bigcup_{n=1}^\infty K_n \cup E,
\]
where $\mathcal{H}^{N-1}(E) = 0$ and $K_n$ is a compact subset of a $C^1$ hypersurface. If $x \in \Omega \setminus S(u)$ then $u(x)$ is taken to be the common value of $(u_1^+(x), \cdots, u_d^+(x))$ and $(u_1^-(x), \cdots, u_d^-(x))$. It can be shown that $u(x) \in \mathbb{R}^d$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \setminus S(u)$. Furthermore, for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$ there exists a unit vector $\nu_u(x) \in S^{N-1}$, normal to $S(u)$ at $x$, and two vectors $u^-(x), u^+(x) \in \mathbb{R}^d$ (the traces of $u$ on $S(u)$ at the point $x$) such that
\[
 \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\{ y \in Q(x, \varepsilon) : (y-x) \cdot \nu_u(x) > 0 \}} |u(y) - u^+(x)|^{N/(N-1)} \, dy = 0
\]
and
\[
 \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\{ y \in Q(x, \varepsilon) : (y-x) \cdot \nu_u(x) < 0 \}} |u(y) - u^-(x)|^{N/(N-1)} \, dy = 0.
\]
Note that, in general, $(u_i^+ \neq u_i^+)$ and $(u_i^- \neq u_i^-)$. We denote the jump of $u$ across $S(u)$ by $[u] := u^+ - u^-$. The distributional derivative $Du$ may be decomposed as
\[
 Du = \nabla u \mathcal{L}^N + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1}\{ S(u) + C(u) \},
\]
where $\nabla u$ is the density of the absolutely continuous part of $Du$ with respect to the $N$-dimensional Lebesgue measure $\mathcal{L}^N$ and $C(u)$ is the Cantor part of $Du$. These three measures are mutually singular, and the total variation of $u$,
\[
 \| Du \|_1(\Omega) := \sup \left\{ \int_\Omega u \, \text{div} \phi(x) \, dx : \phi \in C^1_c(\Omega; \mathbb{R}^N), \| \phi \|_\infty \leq 1 \right\},
\]
is now

\[ |Du|(\Omega) = \int_{\Omega} |\nabla u| \, dx + \int_{\partial^* A} |u^+ - u^-| \, d\mathcal{H}^{N-1} + |C(u)|(\Omega). \]

We recall that if \( \{u_n\} \subset BV(\Omega; \mathbb{R}^d) \) and \( u_n \to u \) in \( L^1(\Omega; \mathbb{R}^d) \), then

\[ |Du|(\Omega) \leq \liminf_{n \to \infty} |Du_n|(\Omega). \]

We say that a set \( E \subset \Omega \) is of finite perimeter if \( \chi_E \in BV(\Omega; \mathbb{R}) \), and we denote by \( \text{Per}_\Omega(E) := |D\chi_E|(\Omega) \) given by

\[ \text{Per}_\Omega(E) := \sup \left\{ \int_E \text{div} \phi(x) \, dx : \phi \in C_1^1(\Omega; \mathbb{R}^N), \|\phi\| \leq 1 \right\}. \]

**Definition 2.1** Let \( A \subset \mathbb{R}^N \) be a set of locally finite perimeter and let \( x_0 \in \mathbb{R}^N \). We say that \( x_0 \) belongs to the reduced boundary of \( A \) (and we write \( x_0 \in \partial^* A \)) if, with \( D\chi_A = -\nu|D\chi_A|, \nu \in L^1_{\text{loc}}(\mathbb{R}^N; \mathbb{S}^{N-1}) \) with respect to the Radon measure \( |D\chi_A| \), we have

(i) \( |D\chi_A|(B(x_0, \varepsilon)) > 0 \) for all \( \varepsilon > 0 \);

(ii) \( \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \mathcal{L}^N(B(x_0, \varepsilon)) \int_{B(x_0, \varepsilon)} \nu(x) \, d|D\chi_A|(x) = \nu(x_0) \);

(iii) \( \|\nu(x_0)\| = 1 \).

\( \nu \) is said to be the outward unit normal to the boundary of \( A \) at \( x_0 \).

**Theorem 2.2** (see [21], [28]) If \( x \in \partial^* A \) then

\[ \lim_{\delta \to 0^+} \frac{1}{\delta} \mathcal{L}^N(\{y \in B(x, \delta) \setminus A : (y - x) \cdot \nu(x) < 0\}) = 0, \]

\[ \lim_{\delta \to 0^+} \frac{1}{\delta} \mathcal{L}^N(\{y \in B(x, \delta) \cap A : (y - x) \cdot \nu(x) > 0\}) = 0. \]

It can be shown (see [24]) that if \( \text{Per}_\Omega(A) < +\infty \) then for \( \mathcal{H}^{N-1} \)-a.e. \( x \in \Omega \cap \partial^* A \)

\[ \lim_{\delta \to 0^+} \frac{1}{\delta} \mathcal{H}^{N-1}(\{\Omega \cap \partial^* A \cap (x + \delta Q_{\nu(x)})\}) = 1. \] (2.1)

**Theorem 2.3** (see [6, Lemma 3.1]) Let \( A \) be a subset of \( \Omega \) such that \( \text{Per}_\Omega(A) < +\infty \). There exists a sequence of polyhedral sets \( \{A_k\} \) (i.e. \( A_k \) are bounded, Lipschitz domains with \( \partial A_k = H_1 \cup H_2 \cup \ldots \cup H_p \), where each \( H_i \) is a closed subset of a hyperplane \( \{x \in \mathbb{R}^N : x \cdot \nu_i = \alpha_i\} \)) satisfying the following properties:

(i) \( \lim_{k \to \infty} \mathcal{L}^N([(A_k \cap \Omega) \setminus A) \cup (A \setminus (A_k \cap \Omega))] = 0; \)

(ii) \( \lim_{k \to \infty} \text{Per}_\Omega(A_k) = \text{Per}_\Omega(A); \)

(iii) \( \mathcal{H}^{N-1}(\partial A_k \cap \partial \Omega) = 0; \)

(iv) \( \mathcal{L}^N(A_k) = \mathcal{L}^N(A). \)
Let $\varepsilon_n \to 0^+$. A functional

$$I : L^1(\Omega; \mathbb{R}^d) \to [0, \infty]$$

is called the $\Gamma$-liminf (resp. $\Gamma$-limsup) of a sequence of functionals $\{I_{\varepsilon_n}\}$ with respect to the strong convergence in $L^1(\Omega; \mathbb{R}^d)$ if for every $u \in L^1(\Omega; \mathbb{R}^d)$

$$I(u) = \inf \left\{ \liminf_{n \to \infty} (\text{resp. } \limsup_{n \to \infty}) I_{\varepsilon_n}(u_n) : u_n \in L^1(\Omega; \mathbb{R}^d), \ u_n \to u \text{ in } L^1(\Omega; \mathbb{R}^d) \right\},$$

and we write

$$I = \Gamma - \liminf_{n \to \infty} I_{\varepsilon_n} \quad (\text{resp. } I = \Gamma - \limsup_{n \to \infty} I_{\varepsilon_n}).$$

We say that the sequence $\{I_{\varepsilon_n}\}$ $\Gamma$-converges to $I$ if the $\Gamma$-liminf and the $\Gamma$-limsup coincide, and we write

$$I = \Gamma - \lim_{n \to \infty} I_{\varepsilon_n}.$$

The functional $I$ is said to be the $\Gamma$-liminf (resp. $\Gamma$-limsup) of the family of functionals $\{I_{\varepsilon}\}$ with respect to the strong convergence in $L^1(\Omega; \mathbb{R}^d)$ if for every sequence $\varepsilon_n \to 0^+$ we have that

$$I = \Gamma - \lim_{\varepsilon \to 0} I_{\varepsilon} \quad (\text{resp. } I = \Gamma - \limsup_{\varepsilon \to 0} I_{\varepsilon}),$$

and we write

$$I = \Gamma - \lim_{\varepsilon \to 0} I_{\varepsilon}.$$

Finally, if $\Gamma$-liminf and $\Gamma$-limsup coincide, we say that $I$ is the $\Gamma$-limit of the family of functionals $\{I_{\varepsilon}\}$, and we write

$$I = \Gamma - \lim_{\varepsilon \to 0} I_{\varepsilon}.$$

The following lemma will be used many times throughout the thesis.

**Lemma 2.4** (Riemann-Lebesgue Lemma) Let $1 \leq p \leq \infty$, and let $u \in L^p(Q)$ be a $Q$-periodic function. Set

$$u_{\varepsilon}(x) := u \left( \frac{x}{\varepsilon} \right) \text{ for } L_N^N - \text{a.e. } x \in \mathbb{R}^N.$$

Then, if $p < \infty$, as $\varepsilon \to 0$,

$$u_{\varepsilon} \rightharpoonup \int_Q u(y)dy \text{ weakly in } L^p(\omega),$$

for any bounded open $\omega \subset \mathbb{R}^N$. If $p = \infty$, one has

$$u_{\varepsilon} \rightharpoonup \int_Q u(y)dy \text{ weakly * in } L^\infty(\mathbb{R}^N).$$

The following lemma is very useful in many diagonalization arguments.

**Lemma 2.5** (Lemma 7.1 in [13]) Let $\{a_{k,j}\}$ be a doubly indexed sequence of real numbers. If

$$\lim_{k \to \infty} \lim_{j \to \infty} a_{k,j} = L,$$

then there exists an increasing subsequence $\{k(j)\} \nearrow \infty$ such that $\lim_{j \to \infty} a_{k(j),j} = L.$
Let $C(\Omega; \mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued continuous functions on $\Omega$ and define

$$C_0(\Omega; \mathbb{R}^d) := \left\{ \varphi \in C(\Omega; \mathbb{R}^d) : \text{ for every } \varepsilon > 0 \text{ there exists a compact set } K \subset \Omega \text{ such that } |\varphi(x)| \leq \varepsilon \text{ if } x \in \Omega \setminus K \right\},$$

where $|\cdot|$ stands for the Euclidian norm in $\mathbb{R}^d$. Endowed with the supremum norm, $C_0(\Omega; \mathbb{R}^d)$ is a separable Banach space. In view of Riesz’s Theorem, the dual space $[C_0(\Omega; \mathbb{R}^d)]'$ can be identified with the space $\mathcal{M}(\Omega; \mathbb{R}^d)$ of bounded $\mathbb{R}^d$-valued Radon measures on $\Omega$ with the norm $\|\mu\| := |\mu|(\Omega)$, via the duality pairing

$$\langle \mu, \varphi \rangle = \int_{\Omega \times \mathbb{R}^d} \varphi(x) \frac{d\mu}{d|\mu|}(x) d|\mu|(x),$$

where $|\mu|$ stands for the total variation of $\mu$ and is a non-negative, finite Radon measure on $\Omega$. If $d = 1$ we write $C(\Omega)$ and $C_0(\Omega)$ and $\mathcal{M}(\Omega)$ instead of $C(\Omega; \mathbb{R})$, $C_0(\Omega; \mathbb{R})$, and $\mathcal{M}(\Omega; \mathbb{R})$, respectively.

**Definition 2.6** We say that a sequence $\{y_j\}$ in $L^1(\Omega)$ is equi-integrable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{j \in \mathbb{N}} \int_E |y_j| dx < \varepsilon$$

whenever $E$ is a Borel subset of $\Omega$ such that $\mathcal{L}^N(E) < \delta$.

Young measures were originally introduced in Optimal Control Theory by L.C. Young in connection with nonconvex problems, thus providing the appropriate framework for the description of generalized minimizers in the Calculus of Variations (see [48], [49]). Later, Tartar developed the use of Young measures in the PDE framework, and Kinderlehrer and Pedregal introduced and studied the concept of gradient Young measures (see [5], [7], [9], [33], [34], [36], [41], [45], [46], [47]). In what follows we recall the definition and some of the relevant results about Young measures.

**Definition 2.7** (i) A non-negative Radon measure $\mu$ on $\Omega \times \mathbb{R}^d$ with the property

$$\mu(B \times \mathbb{R}^d) = \mathcal{L}^N(B) \text{ for all Borel subsets of } \Omega,$$

is called a Young measure. The set of Young measures on $\Omega \times \mathbb{R}^d$ is denoted by $Y(\Omega \times \mathbb{R}^d)$.

(ii) A Young measure $\mu$ for which there exists a $\mathcal{L}^N$-measurable mapping $V : \Omega \to \mathbb{R}^d$ such that

$$\int_{\Omega \times \mathbb{R}^d} f d\mu = \int_{\Omega} f(x, V(x)) dx \text{ for all } f \in C_0(\Omega \times \mathbb{R}^d),$$

is called an elementary Young measure. We write

$$\mu = \mathcal{E}V := \int_{\Omega} \delta_x \otimes \delta_{V(x)} dx,$$

where $\delta_x$ and $\delta_{V(x)}$ are the Dirac measures on $\Omega$ concentrated at $x$ and on $\mathbb{R}^d$ concentrated at $V(x)$, respectively.
(iii) A product measure \( (\mathcal{L}^N(\Omega) \otimes \tilde{\mu}) \) on \( \Omega \times \mathbb{R}^d \), where \( \tilde{\mu} \) is a probability measure on \( \mathbb{R}^d \), is called a homogeneous Young measure.

**Remark 2.8** The definition of Young measures in Definition 2.7 (i) follows that of Berliocchi and Lasry (see [9]). It can be shown (cf. [37]) to be equivalent to the original definition of L.C. Young [48], and the concepts usually adopted in the literature (e.g., [5], [7], [41]).

The following Proposition is a special case of a result in [11] (Proposition 13, pp. 39-40). See also [4].

**Proposition 2.9** Let \( \mu \) be a Young measure on \( \Omega \times \mathbb{R}^d \). Then there exists a mapping \( x \mapsto \mu_x \) from \( \Omega \) into the set of non-negative, finite Radon measures on \( \mathbb{R}^d \), such that

\[
(\text{i}) \quad \mu = \int \delta_x \otimes \mu_x \, dx, \quad \text{i.e. for any Borel function } f : \Omega \times \mathbb{R}^d \to \mathbb{R}, \text{ the function }
\]

\[
x \mapsto \int_{\mathbb{R}^d} f(x, A) \, d\mu_x(A)
\]

is \( \mathcal{L}^N \)-measurable and

\[
\int \mathbb{R}^d \int \mathbb{R}^d f(x, A) \, d\mu_x(A) \, dx = \int \Omega \times \mathbb{R}^d f(x, A) \, d\mu_x(A) \, dx,
\]

(2.2)

\[(\text{ii}) \quad \mu_x(\mathbb{R}^d) = 1, \text{ for } \mathcal{L}^N \text{-a.e. } x \in \Omega.
\]

Moreover, if \( x \mapsto \nu_x \) is another such mapping then \( \nu_x = \mu_x \mathcal{L}^N \text{-a.e. } x \in \Omega \).

**Remark 2.10** If \( \{V_n \} \) is a sequence of measurable mappings of \( \Omega \) into \( \mathbb{R}^d \), then the corresponding sequence \( \{E_{V_n} \} \) of elementary Young measures is bounded in \( [C_0(\Omega \times \mathbb{R}^d)]' \) and thus, by the fact that bounded sets in \( [C_0(\Omega \times \mathbb{R}^d)]' \) are sequential weak* compact (note that the space \( C_0(\Omega \times \mathbb{R}^d) \) is separable), there exists a subsequence \( \{V_{n_k} \} \) and a measure \( \mu \in [C_0(\Omega \times \mathbb{R}^d)]' \) such that

\[
E_{V_{n_k}} \rightarrow \mu \text{ weakly* in } [C_0(\Omega \times \mathbb{R}^d)]'.
\]

(2.3)

A necessary and sufficient condition for \( \mu \) to be a Young measure is that

\[
\lim_{R \to \infty} \sup_{n \in \mathbb{N}} \mathcal{L}^N (\{x \in \Omega : |V_{n_k}(x)| \geq R \}) = 0,
\]

(2.4)

or, equivalently (see [32], [36]),

there exists a Borel function \( g : \mathbb{R}^d \to [0, +\infty] \) such that \( \lim_{|A| \to +\infty} g(A) = +\infty \),

and \( \sup_{n \in \mathbb{N}} \int \Omega g(V_{n_k}(x)) \, dx < +\infty \).

**Definition 2.11** If (2.3) and (2.4) hold, then we say that the Young measure \( \mu \) is generated by the sequence \( \{V_{n_k} \} \).

**Definition 2.12** (i) A function \( f : \Omega \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\} \) is called a normal integrand if \( f \) is Borel measurable and \( f(x, \cdot) : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\} \) is lower semicontinuous for every \( x \in \Omega \).
(ii) A real-valued function \( f : \Omega \times \mathbb{R}^d \to \mathbb{R} \) is called a Carathéodory integrand if both \( f \) and \(-f\) are normal integrands.

A map \( \nu : \Omega \to \mathcal{M}(\mathbb{R}^d) \) is said to be weak* measurable if \( x \mapsto \langle \nu(x), \varphi \rangle \) is measurable for every \( \varphi \in C_0(\Omega) \). In the sequel we will denote \( \nu(x) \) by \( \nu_x \). We summarize the main properties of Young measures in the following theorem.

**Theorem 2.13** (The Fundamental Theorem on Young measures [7], [9], [45]) Let \( E \subset \mathbb{R}^N \) be a measurable set of finite measure and let \( \{y_j\} \) be a sequence of measurable functions, \( y_j : E \to \mathbb{R}^d \). Then there exist a subsequence \( \{y_{j_k}\} \subset \{y_j\} \) and a weak* measurable map \( \nu : E \to \mathcal{M}(\mathbb{R}^d) \) such that the following hold:

(i) \( \nu_x \geq 0, \quad \|\nu_x\|_{\mathcal{M}(\mathbb{R}^d)} := \int_{\mathbb{R}^d} d\nu_x \leq 1, \) for \( \mathcal{L}^N \)-a.e. \( x \in E \);

(ii) \( \|\nu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1 \) if and only if either

\[
\lim_{R \to -\infty} \sup_{k \in \mathbb{N}} \mathcal{L}^N \left( \{x \in E : |y_{j_k}(x)| \geq R \} \right) = 0,
\]

or there exists a Borel function \( g : \mathbb{R}^d \to [0, +\infty] \) such that \( \lim_{|A| \to +\infty} g(A) = +\infty \), and

\[
\sup_{k \in \mathbb{N}} \int_E g(y_{j_k}(x)) dx < +\infty,
\]

hold. In this case, \( \nu \) is a Young measure generated by the sequence \( \{y_{j_k}\} \);

(iii) if \( K \subset \mathbb{R}^d \) is compact and \( \text{dist}(y_{j_k}, K) \to 0 \) in measure, then \( \text{supp}\nu_x \subset K \) for \( \mathcal{L}^N \)-a.e. \( x \in E \). The converse also holds provided that \( \|\nu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1 \);

(iv) if \( \nu \) is generated by \( \{y_{j_k}\} \), if \( f : E \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\} \) is a normal integrand and \( \{f^-(\cdot, y_{j_k})\} \) is equi-integrable, then

\[
\int_E \int_{\mathbb{R}^d} f(x, \Lambda) d\nu_x(\Lambda) dx \leq \liminf_{k \to +\infty} \int_E f(x, y_{j_k}(x)) dx,
\]

where \( f^- := -\min\{f, 0\} \);

(v) if \( \nu \) is generated by \( \{y_{j_k}\} \) and if \( f \) is a Carathéodory integrand then \( \{f(\cdot, y_{j_k})\} \) is equi-integrable if and only if

\[
\int_E \int_{\mathbb{R}^d} f(x, \Lambda) d\nu_x(\Lambda) dx = \lim_{k \to +\infty} \int_E f(x, y_{j_k}(x)) dx.
\]

The following results are well known, and may be found e.g. in [41].

**Proposition 2.14** Let \( 1 \leq p < +\infty \), \( \{v_n\} \) be a sequence bounded in \( L^p(\Omega; \mathbb{R}^d) \) such that \( \{|v_n|^p\} \) is weakly convergent in \( L^1(\Omega; \mathbb{R}^d) \), and let \( \mu = \{\mu_x\}_{x \in \Omega} \) be the Young measure generated by \( \{v_n\} \) (a subsequence, not relabelled, of) \( \{v_n\} \). Then, \( v_n \rightharpoonup v \) strongly in \( L^p(\Omega; \mathbb{R}^d) \) if and only if \( \mu_x = \delta_{v(x)} \) for \( \mathcal{L}^N \)-a.e \( x \in \Omega \).

**Proposition 2.15** Let \( w_n = (u_n, v_n) : \Omega \to \mathbb{R}^d \times \mathbb{R}^m \) be a bounded sequence in \( L^p(\Omega; \mathbb{R}^d \times \mathbb{R}^m) \) such that \( u_n \rightharpoonup u \) strongly in \( L^p(\Omega; \mathbb{R}^d) \). If \( \nu = \{\nu_x\}_{x \in \Omega} \) is the Young measure generated by \( \{v_n\} \), then \( \{w_n\} \) generates \( \{\delta_{u(x)} \otimes \nu_x\}_{x \in \Omega} \).
3 Coupled singular perturbations for phase transitions

This chapter is devoted to the proof of Theorem 1.1. We start by recalling the standing hypotheses and the statement of the theorem.

Let \( f : \Omega \times \mathbb{R}^d \times M^{d \times N} \rightarrow [0, +\infty) \) be a continuous function satisfying the following hypotheses:

(H1) \( f(x, u, 0) = 0 \) if and only if \( u \in \{a, b\} \);

(H2) there exists a continuous function \( g : \Omega \times \mathbb{R}^d \rightarrow [0, +\infty) \) such that

\[
\frac{1}{C} \left( g(x, u) + |\xi|^2 \right) \leq f(x, u, \xi) \leq C \left( g(x, u) + |\xi|^2 \right)
\]

for all \((x, u, \xi) \in \Omega \times \mathbb{R}^d \times M^{d \times N}, \) and

\[
\frac{1}{C} |u|^q - C \leq g(x, u) \leq C(1 + |u|^q)
\]

for some \( q \geq 2, \ C > 0, \) and for all \((x, u, \xi) \in \Omega \times \mathbb{R}^d \times M^{d \times N} \).

(H3) For any \( x_0 \in \Omega \) and any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( |x - x_0| < \delta \) implies that

\[
|f(x, u, \xi) - f(x_0, u, \xi)| \leq \varepsilon f(x, u, \xi)
\]

for every \((u, \xi) \in \mathbb{R}^d \times M^{d \times N} \).

**Theorem 1.1** Assume that (H1)-(H3) hold, and for every \( \varepsilon > 0 \) let \( J_\varepsilon : L^1(\Omega; \mathbb{R}^d) \rightarrow [0, +\infty] \) be the functional defined by

\[
J_\varepsilon(u) := \begin{cases}
E_\varepsilon(u) & \text{if } u \in H^1(\Omega; \mathbb{R}^d), \\
+\infty & \text{otherwise}.
\end{cases}
\]

Then \( J_\varepsilon \Gamma(L^1(\Omega)) \)-converges to the functional \( J_0 : L^1(\Omega; \mathbb{R}^d) \rightarrow [0, +\infty] \) defined by

\[
J_0(u) := \begin{cases}
\int_{\Omega \cap \partial^* A_0} K(x, a, b, \nu(x))dH^{N-1}(x) & \text{if } u \in BV(\Omega; \{a, b\}), \\
+\infty & \text{otherwise},
\end{cases}
\]

where \( A_0 := \{x \in \Omega : u(x) = a\} \), and \( \nu(x) \) stands for the measure theoretic inner unit normal to the reduced boundary \( \partial^* A_0 \) at \( x \).

In order to prove Theorem 1.1, it is enough to show that every sequence \( \{\varepsilon_n\} \) of positive numbers converging to zero has a subsequence \( \{\varepsilon_{n_k}\} \) such that \( J_{\varepsilon_{n_k}} \Gamma(L^1(\Omega; \mathbb{R}^d)) \)-converges to \( J_0 \) (see [17], [18]). We divide the proof of Theorem 1.1 into two parts. The first part is dealt with in Section 3.1 and the second part is left for Section 3.2.

### 3.1 The \( \Gamma \)-liminf inequality

In this section we prove

**Proposition 3.1** Let (H1)-(H3) hold and let \( u \in L^1(\Omega; \mathbb{R}^d) \) be given. If \( \varepsilon_n \rightarrow 0^+ \) and if \( \{u_n\} \subset H^1(\Omega; \mathbb{R}^d) \) is such that \( u_n \rightharpoonup u \) in \( L^1(\Omega; \mathbb{R}^d) \), then

\[
\liminf_{n \rightarrow +\infty} \frac{1}{\varepsilon_n} \int_\Omega f(x, u_n(x), \varepsilon_n \nabla u_n(x))dx \geq J_0(u).
\]
The proof relies on the following lemma:

**Lemma 3.2** Assume that (H1)-(H3) hold, let \( \nu \) be a unit vector and let

\[
  u_0(x) := \begin{cases} 
    b & \text{if } x \cdot \nu > 0, \\
    a & \text{if } x \cdot \nu < 0.
  \end{cases}
\]

Let \( \rho : \mathbb{R} \to \mathbb{R} \) be a symmetric mollifier and set \( v_n := \rho_{1/\varepsilon_n, \nu} * u_0 \), where \( \rho_{1/\varepsilon_n, \nu}(x) := \frac{1}{\varepsilon_n} \rho \left( \frac{x - u}{\varepsilon_n} \right) \), and \( \{\varepsilon_n\} \) is a sequence of real numbers such that \( \varepsilon_n \to 0^+ \). If \( \{u_n\} \) is a sequence in \( H^1(Q_\nu; \mathbb{R}^d) \) converging in \( L^1(Q_\nu; \mathbb{R}^d) \) to \( u_0 \), then there exists a sequence \( \{w_n\} \) in \( H^1(Q_\nu; \mathbb{R}^d) \) such that \( w_n \to u_0 \) in \( L^1(Q_\nu; \mathbb{R}^d) \), \( w_n = v_n \) on \( \partial Q_\nu \), and

\[
  \limsup_{n \to +\infty} \int_{Q_\nu} \frac{1}{\varepsilon_n} f(x, w_n(x), \varepsilon_n \nabla w_n(x)) \, dx \leq \liminf_{n \to +\infty} \int_{Q_\nu} \frac{1}{\varepsilon_n} f(x, u_n(x), \varepsilon_n \nabla u_n(x)) \, dx.
\]

**Proof.** Step 1. Assume, without loss of generality, that

\[
  \liminf_{n \to +\infty} \int_{Q_\nu} \frac{1}{\varepsilon_n} f(x, u_n(x), \varepsilon_n \nabla u_n(x)) \, dx = \lim_{n \to +\infty} \int_{Q_\nu} \frac{1}{\varepsilon_n} f(x, u_n(x), \varepsilon_n \nabla u_n(x)) \, dx < +\infty, \quad (3.1)
\]

and that \( u_n(x) \to u_0(x) \) \( \mathcal{L}^d \)-a.e. \( x \in Q_\nu \). By (3.1)

\[
  \int_{Q_\nu} f(x, u_n(x), \varepsilon_n \nabla u_n(x)) \, dx = \varepsilon_n \int_{Q_\nu} \frac{1}{\varepsilon_n} f(x, u_n(x), \varepsilon_n \nabla u_n(x)) \, dx \to 0 \quad \text{as } n \to +\infty. \quad (3.2)
\]

By (H2) we have that

\[
  |u_n(x) - u_0(x)|^q \leq C \left( f(x, u_n(x), \varepsilon_n \nabla u_n(x)) + 1 \right),
\]

and by Fatou's Lemma and (3.2)

\[
  C |Q_\nu| - \limsup_{n \to +\infty} \int_{Q_\nu} |u_n - u_0|^q \, dx
\]

\[
  = \liminf_{n \to +\infty} \int_{Q_\nu} (C f(x, u_n(x), \varepsilon_n \nabla u_n(x)) + C - |u_n - u_0|^q) \, dx
\]

\[
  \geq \int_{Q_\nu} \liminf_{n \to +\infty} \left[ C f(x, u_n(x), \varepsilon_n \nabla u_n(x)) + C - |u_n - u_0|^q \right] \, dx
\]

\[
  \geq C |Q_\nu|.
\]

Therefore,

\[
  \limsup_{n \to +\infty} \int_{Q_\nu} |u_n - u_0|^q \, dx = 0, \quad (3.3)
\]

and in particular, since \( q \geq 2 \), we conclude that \( u_n \to u_0 \) in \( L^2(Q_\nu; \mathbb{R}^d) \) as \( n \to +\infty \).
Step 2. For simplicity, assume that $\nu = e_N$ and denote $Q_\nu = Q$. Note that
\[ v_n(x) = \begin{cases} b & \text{if } x_N > \varepsilon_n, \\ a & \text{if } x_N < -\varepsilon_n, \end{cases} \]
and
\[ v_n \in \mathcal{A}(a, b, e_N), \quad \|\nabla v_n\|_\infty = O(1/\varepsilon_n), \quad \text{supp}\nabla v_n \subset \{x \in Q : |x_N| < \varepsilon_n\}, \quad \text{and } v_n \to u_0 \text{ in } L^q(Q; \mathbb{R}^d). \]

For each $k \in \mathbb{N}$ define
\[ L_k := \left\{ x \in Q : \text{dist}(x, \partial Q) \leq \frac{1}{k} \right\}. \]
Consider $n$ sufficiently large ($n \geq n(k)$ for some $n(k)$), and divide $L_k$ into $M_{k,n}$ layers $L_{k,n}^{(i)} (i = 1, ..., M_{k,n})$ of width $\varepsilon_n \|u_n - v_n\|_{L^2(Q; \mathbb{R}^d)}^{1/2}$, so that $M_{k,n} \varepsilon_n \|u_n - v_n\|_{L^2(Q; \mathbb{R}^d)}^{1/2} = O(1/k)$.

We have that
\[
\sum_{i=1}^{M_{k,n}} \int_{L_{k,n}^{(i)}} \left( 1 + |u_n|^q + |v_n|^q + \varepsilon_n^2 |\nabla u_n|^2 + \frac{|u_n - v_n|^2}{\|u_n - v_n\|_{L^2(Q; \mathbb{R}^d)}} \right) \, dx \\
= \int_{L_k} \left( 1 + |u_n|^q + |v_n|^q + \varepsilon_n^2 |\nabla u_n|^2 + \frac{|u_n - v_n|^2}{\|u_n - v_n\|_{L^2(Q; \mathbb{R}^d)}} \right) \, dx,
\]
and thus there exists $i = i(k,n) \in \{1, ..., M_{k,n}\}$ such that
\[
\int_{L_{k,n}^{(i)}} \left( 1 + |u_n|^q + |v_n|^q + \varepsilon_n^2 |\nabla u_n|^2 + \frac{|u_n - v_n|^2}{\|u_n - v_n\|_{L^2(Q; \mathbb{R}^d)}} \right) \, dx \\
\leq \frac{1}{M_{k,n}} \int_{L_k} \left( 1 + |u_n|^q + |v_n|^q + \varepsilon_n^2 |\nabla u_n|^2 + \frac{|u_n - v_n|^2}{\|u_n - v_n\|_{L^2(Q; \mathbb{R}^d)}} \right) \, dx. \tag{3.5}
\]

Choose cut-off functions $\varphi_{k,n} \in C^\infty_c(Q; [0, 1])$ such that $\varphi_{k,n} = 0$ on $\bigcup_{j=i+1}^{M_{k,n}} L_{k,n}^{(j)} =: A_{k,n}$, $\varphi_{k,n} = 1$ on $(Q \setminus L_k) \cup \bigcup_{j=1}^{i-1} L_{k,n}^{(j)} =: B_{k,n}$. Define
\[ w_{k,n} := \varphi_{k,n} u_n + (1 - \varphi_{k,n}) v_n. \]
We have that
\[ \lim_{k \to \infty} \lim_{n \to \infty} \|w_{k,n} - u_0\|_{L^1(Q; \mathbb{R}^d)} = 0. \]

Also
\[
\limsup_{k \to \infty} \limsup_{n \to \infty} \int_Q \frac{1}{\varepsilon_n} f(x, w_{k,n}(x), \varepsilon_n \nabla w_{k,n}(x)) \, dx \tag{3.6}
\leq \limsup_{k \to \infty} \limsup_{n \to \infty} \int_{A_{k,n}} \frac{1}{\varepsilon_n} f(x, v_n(x), \varepsilon_n \nabla v_n(x)) \, dx
\]

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for any sequence $\varepsilon_n \to 0^+$ and for any $\{u_n\} \subset H^1(\Omega; \mathbb{R}^d)$ such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^d)$, we have
\[
\int_\Omega \frac{1}{\varepsilon_n} f(x, u_n(x), \varepsilon_n \nabla u_n(x)) \, dx \to +\infty. \tag{3.7}
\]
Indeed, if for some sequences $\varepsilon_n \to 0^+$, and $u_n \to u$ in $L^1(\Omega; \mathbb{R}^d)$
\[
\sup_{n \in \mathbb{N}} \int_{\Omega} \frac{1}{\varepsilon_n} f(x, u_n(x), \varepsilon_n \nabla u_n(x)) dx < +\infty,
\]
then by (H2) and Fatou’s lemma \[
\int_{\Omega} g(x, u) dx = 0,
\]
and thus $u(x) \in \{a, b\}$ for $\mathcal{L}^N$-a.e. $x \in \Omega$, that is a contradiction.

**Step 2.** Consider now the case where $u = \chi_{A_0}(x) a + (1 - \chi_{A_0}(x)) b$ and $u \notin BV(\Omega; \mathbb{R}^d)$, i.e. $\text{Per}_\Omega(A_0) = +\infty$. Again, we show that (3.7) is satisfied. We argue by contradiction. Suppose that there exists a subsequence (not relabelled) such that
\[
\sup_{n \in \mathbb{N}} \int_{\Omega} \frac{1}{\varepsilon_n} f(x, u_n(x), \varepsilon_n \nabla u_n(x)) dx \leq C.
\]
Then, by (H2), we have
\[
\sup_{n \in \mathbb{N}} \int_{\Omega} \left[ \frac{1}{\varepsilon_n} g(x, u_n(x)) + \varepsilon_n |\nabla u_n(x)|^2 \right] dx \leq C,
\]
which, by the Cauchy-Schwarz inequality, implies that
\[
\sup_{n \in \mathbb{N}} \int_{\Omega} \sqrt{g(x, u_n(x)) |\nabla u_n(x)|} dx \leq C. \tag{3.8}
\]
Setting $\overline{g}(u) := \min_{x \in \Omega} g(x, u)$, we note that $\overline{g}(u) = 0$ if and only if $u \in \{a, b\}$, with $\overline{g}(u) \geq C |u|$ for suitable $C > 0$ and $|u|$ sufficiently large. In view of Lemma 3.7 in [27], for suitable $M > 0$ the function
\[
\Phi(u) := \inf \left\{ \int_{-1}^{1} \sqrt{\min \{\overline{g}(\gamma(s)), M\}} |\gamma'(s)| ds : \gamma \text{ is piecewise } C^1, \gamma(-1) = a, \gamma(1) = u \right\}
\]
is Lipschitz continuous and $|\nabla (\Phi \circ v)(x)| \leq \sqrt{\overline{g}(v(x))} |\nabla v(x)|$ for any $v \in H^1(\Omega; \mathbb{R}^d)$, and $\mathcal{L}^N$-a.e. $x \in \Omega$. Thus
\[
\sup_{n \in \mathbb{N}} \|\nabla (\Phi \circ u_n)\|_{L^1(\Omega; \mathbb{R}^d)} < +\infty.
\]
Therefore $|D(\Phi \circ u)(\Omega)| < +\infty$, and since $\Phi \circ u = (1 - \chi_{A_0}) \Phi(b)$ we obtain that $\text{Per}_\Omega(A_0) < +\infty$.

**Step 3.** We look now at the case where $u = \chi_{A_0}(x) a + (1 - \chi_{A_0}(x)) b$ with $\text{Per}_\Omega(A_0) < +\infty$. Assume, without loss of generality, that
\[
\liminf_{n \to \infty} \int_{\Omega} \frac{1}{\varepsilon_n} f(x, u_n(x), \varepsilon_n \nabla u_n(x)) dx = \liminf_{n \to \infty} \int_{\Omega} \frac{1}{\varepsilon_n} f(x, u_n(x), \varepsilon_n \nabla u_n(x)) dx < +\infty.
\]
We must show that
\[
\lim_{n \to \infty} \int_{\Omega} \frac{1}{\varepsilon_n} f(x, u_n(x), \varepsilon_n \nabla u_n(x)) dx \geq \int_{\Omega \cap \partial^* A_0} K(x, a, b, \nu(x)) \, d\mathcal{H}^{N-1}(x). \tag{3.9}
\]
Since the integrands $\frac{1}{\varepsilon_n} f(x, u_n(x), \varepsilon_n \nabla u_n(x))$ form a sequence of nonnegative functions bounded in $L^1(\Omega; \mathbb{R}^d)$, there exists a subsequence (not relabelled) and a nonnegative Radon measure $\mu$ such that

$$\frac{1}{\varepsilon_n} f(\cdot, u_n(\cdot), \varepsilon_n \nabla u_n(\cdot)) \to \mu \text{ weakly * in the sense of measures}. \quad (3.10)$$

Using the Radon-Nikodym Theorem, we may write $\mu$ as a sum of two mutually singular nonnegative measures $\mu = \mu_0 H^{N-1}[\Omega \cap \partial^* A_0] + \mu_s$, where $\mu_0$ is a suitable Borel function on $\Omega \cap \partial^* A_0$. We claim that

$$\mu_0(x) \geq K(x, a, b, \nu(x)) \text{ for } H^{N-1}-\text{a.e. } x \in \Omega \cap \partial^* A_0. \quad (3.11)$$

Assuming that (3.11) holds, we consider an increasing sequence of smooth cut-off functions, $\varphi_k \in C_c(\Omega; [0, 1])$, and we obtain

$$\lim_{n \to \infty} \frac{1}{\varepsilon_n} \int_{\Omega} f(x, u_n(x), \varepsilon_n \nabla u_n(x)) \, dx \geq \lim_{n \to \infty} \frac{1}{\varepsilon_n} \int_{\Omega} \varphi_k(x) f(x, u_n(x), \varepsilon_n \nabla u_n(x)) \, dx
\begin{align*}
&= \int_{\Omega} \varphi_k(x) \, d\mu(x) \\
&\geq \int_{\Omega} \varphi_k(x) \, d\mu_0 H^{N-1}[\Omega \cap \partial^* A_0](x) \\
&\geq \int_{\Omega \cap \partial^* A_0} \varphi_k(x) K(x, a, b, \nu(x)) \, dH^{N-1}(x).
\end{align*}$$

Letting $k \to +\infty$ and using the Monotone Convergence Theorem we deduce (3.9). It remains to show that (3.11) holds.

By Theorem 2.2 for $H^{N-1}$-a.e. $x \in \Omega \cap \partial^* A_0$ we have

$$\lim_{\rho \to 0^+} \frac{1}{\rho^N} \mathcal{L}^N(\{y \in B(x, \rho) \cap A_0 : (y - x) \cdot \nu(x) > 0\}) = 0, \quad (3.12)$$

$$\lim_{\rho \to 0^+} \frac{1}{\rho^N} \mathcal{L}^N(\{y \in B(x, \rho) \setminus A_0 : (y - x) \cdot \nu(x) < 0\}) = 0, \quad (3.13)$$

$$\mu_0(x) = \lim_{\rho \to 0^+} \frac{\mu(x + \rho Q_{\nu(x)})}{H^{N-1}[\Omega \cap \partial^* A_0](x + \rho Q_{\nu(x)})},$$

and by (2.1), (3.10), and choosing $\rho_k \to 0^+$ such that $\mu(\partial(x + \rho_k Q_{\nu(x)})) = 0$, we have

$$\mu_0(x) = \lim_{\rho \to 0^+} \frac{\mu(x + \rho Q_{\nu(x)})}{H^{N-1}[\Omega \cap \partial^* A_0](x + \rho Q_{\nu(x)})},$$
Thus, by (3.15), (3.16), (3.17), and taking into account hypothesis (H3), we have

Applying Lemma 3.2 to the sequences

Choose

Since

Let

Since \( u_n \to u \) in \( L^1(\Omega; \mathbb{R}^d) \), and in view of (3.12) and (3.13), we have that

\[
\lim_{k \to \infty} \lim_{n \to \infty} \|w_{n,k} - u_0\|_{L^1(Q_{\nu(x)}; \mathbb{R}^d)} = \lim_{k \to \infty} \lim_{n \to \infty} \int_{Q_{\nu(x)}} |w_{n,k}(x) - u_0(x)| \, dy
\]

\[
= \lim_{k \to \infty} \lim_{n \to \infty} \int_{Q_{\nu(x)} \cap \{ y : y \cdot \nu(x) > 0 \}} |u_n(x + \rho_k y) - b| \, dy
\]

\[
+ \int_{Q_{\nu(x)} \cap \{ y : y \cdot \nu(x) < 0 \}} |u_n(x + \rho_k y) - a| \, dy
\]

\[
= \lim_{k \to \infty} \left[ \int_{Q_{\nu(x)} \cap \{ y : y \cdot \nu(x) > 0 \}} |u(x + \rho_k y) - b| \, dy + \int_{Q_{\nu(x)} \cap \{ y : y \cdot \nu(x) < 0 \}} |u(x + \rho_k y) - a| \, dy \right]
\]

\[
= \int_{(x + \rho_k Q_{\nu(x)}) \cap \{ y : (y - x) \cdot \nu(x) > 0 \} \cap A_0} |a - b| \, dy + \int_{(x + \rho_k Q_{\nu(x)}) \cap \{ y : (y - x) \cdot \nu(x) < 0 \} \cap A_0} |b - a| \, dy = 0.
\]

Since \( \nabla w_{n,k}(y) = \rho_k \nabla u_{n}(x + \rho_k y) \), (3.14) yields

\[
\mu_n(x) = \lim_{k \to \infty} \lim_{n \to \infty} \int_{Q_{\nu(x)}} \frac{\rho_k}{\varepsilon_n} \int_{Q_{\nu(x)}} \frac{\rho_k}{\varepsilon_n} f \left( x + \rho_k y, w_{n,k}(y), \frac{\varepsilon_n}{\rho_k} \nabla w_{n,k}(y) \right) \, dy.
\]  (3.15)

Choose \( n_k \in \mathbb{N} \) large enough so that \( \frac{\varepsilon_n}{\rho_k} \to 0 \), \( \lim_{k \to \infty} \|w_{n,k} - u_0\|_{L^1(Q_{\nu(x)}; \mathbb{R}^d)} = 0 \), and

\[
\lim_{k \to \infty} \int_{Q_{\nu(x)}} \frac{\rho_k}{\varepsilon_n} \int_{Q_{\nu(x)}} \frac{\rho_k}{\varepsilon_n} f \left( x + \rho_k y, w_{n,k}(y), \frac{\varepsilon_n}{\rho_k} \nabla w_{n,k}(y) \right) \, dy
\]

\[
= \lim_{k \to \infty} \lim_{n \to \infty} \int_{Q_{\nu(x)}} \frac{\rho_k}{\varepsilon_n} \int_{Q_{\nu(x)}} \frac{\rho_k}{\varepsilon_n} f \left( x + \rho_k y, w_{n,k}(y), \frac{\varepsilon_n}{\rho_k} \nabla w_{n,k}(y) \right) \, dy.
\]  (3.16)

Applying Lemma 3.2 to the sequences \( \{w_{n,k}\} \) and \( \{\alpha_k\} := \{\frac{\varepsilon_n}{\rho_k}\} \), we conclude that there exists a sequence \( \{\xi_k\} \subset H^1(Q_{\nu(x)}; \mathbb{R}^d) \) such that \( \xi_k \to u_0 \) in \( L^1(Q_{\nu(x)}; \mathbb{R}^d) \), \( \xi_k \in \mathcal{A}(a, b, \nu(x)) \), and

\[
\limsup_{k \to \infty} \int_{Q_{\nu(x)}} \frac{1}{\alpha_k} f(x, \xi_k(z), \alpha_k \nabla \xi_k(z)) \, dz \leq \lim_{k \to \infty} \int_{Q_{\nu(x)}} \frac{1}{\alpha_k} f(x, w_{n,k}(z), \alpha_k \nabla w_{n,k}(z)) \, dz.
\]  (3.17)

Thus, by (3.15), (3.16), (3.17), and taking into account hypothesis (H3), we have
\[ \mu_a(x) \geq \lim_{k \to \infty} \int_{Q_\varepsilon(x)} \frac{1}{\alpha_k} f(x, w_{n_k,k}(z), \alpha_k \nabla w_{n_k,k}(z)) \, dz \]

\[ - \limsup_{k \to \infty} \int_{Q_\varepsilon(x)} \frac{1}{\alpha_k} [f(x, w_{n_k,k}(y), \alpha_k \nabla w_{n_k,k}(y)) - f(x + \rho_k y, w_{n_k,k}(y), \alpha_k \nabla w_{n_k,k}(y))] \, dy \geq K(x, a, b, \nu(x)). \]

### 3.2 The construction of a recovering sequence for the \( \Gamma \)-limit

In this section we show that \( \Gamma - \limsup_{\varepsilon \to 0^+} J_\varepsilon \leq J_0 \). In view of Steps 1 and 2 in the proof of Proposition 3.1, it suffices to prove \( \varepsilon - 0^+ \)

**Proposition 3.3** Assume the hypotheses (H1)-(H3) hold. Given any \( u \in BV(\Omega; \{a, b\}) \) and any sequence \( \varepsilon_n \to 0^+ \), there exists a sequence \( \{u_n\} \subset H^1(\Omega; \mathbb{R}^d) \) such that \( u_n \to u \) in \( L^1(\Omega; \mathbb{R}^d) \) and

\[ \lim_{n \to +\infty} \int_{\Omega} \frac{1}{\varepsilon_n} f(x, u_n(x), \varepsilon_n \nabla u_n(x)) \, dx = J_0(u). \]  

(3.18)

We will achieve this by showing that given any sequence \( \varepsilon_n \to 0^+ \), (3.18) holds for a subsequence \( \{\varepsilon_R^n\} \) of \( \{\varepsilon_n\} \). Indeed, recalling the main result of the previous section (Proposition 3.1) we then obtain that the \( \Gamma(\mathbb{L}^1) \)-limit of \( J_{\varepsilon_R^n}(u) \) is \( J_0(u) \), which is independent on the specific subsequence \( \{\varepsilon_R^n\} \). In light of Proposition 7.11 in [12], we deduce that, in fact, \( J_\varepsilon(u) \Gamma(\mathbb{L}^1) \)-converges to \( J_0(u) \).

We begin by considering the particular case where \( u := \chi_{A_0}(x)a + (1 - \chi_{A_0}(x))b \) has planar interface and \( f \) and \( K \) do not depend explicitly on \( x \).

**Lemma 3.4** Assume that (H1)-(H3) hold, \( \eta > 0 \), let \( u_0 \) be as in Lemma 3.2, and

\[ u(x) := \begin{cases} b & \text{if } (x - a_0) \cdot \nu > 0, \\ a & \text{if } (x - a_0) \cdot \nu < 0, \end{cases} \]

for some \( a_0 \in \mathbb{R}^N \). Assume that \( f \) does not depend on \( x \). Then, for every sequence \( \varepsilon_n \to 0^+ \), there exists a sequence \( \{u_n\} \subset H^1(a_0 + \eta Q_\nu; \mathbb{R}^d) \) such that \( u_n \to u \) in \( L^1(a_0 + \eta Q_\nu; \mathbb{R}^d) \), and

\[ \lim_{n \to +\infty} \int_{a_0 + \eta Q_\nu} \frac{1}{\varepsilon_n} f(u_n(x), \varepsilon_n \nabla u_n(x)) \, dx = \eta^{N-1} K(b, a, \nu) = J_0(u). \]

**Proof.** For simplicity, we assume that \( \nu = e_N \) and we denote \( Q_\nu \) by \( Q \). Let \( Q' \) be the projection of \( Q \) on \( \mathbb{R}^{N-1} \), \( Q' := \{x \in Q : x_N = 0\} \).

**Case 1.** Suppose first that \( a_0 = 0 \) and \( \eta = 1 \). Let \( L_n > 0 \) and \( \xi_n \in \mathcal{A}(a, b, e_N) \) be such that

\[ \lim_{n \to \infty} \int_Q \frac{1}{L_n} f(\xi_n(x), L_n \nabla \xi_n(x)) \, dx = K(b, a, e_N). \]  

(3.19)

Define

\[ v_n^k(x) := \begin{cases} b & \text{if } x_N > \frac{\varepsilon_n L_k}{2}, \\ \xi_k \left( \frac{x}{\varepsilon_n L_k} \right) & \text{if } |x_N| \leq \frac{\varepsilon_n L_k}{2}, \\ a & \text{if } x_N < -\frac{\varepsilon_n L_k}{2}. \end{cases} \]
Clearly, $v_n^k \in A(a, b, e_N)$ for all $k, n \in \mathbb{N}$, and $v_n^k \to u$ in $L^1(Q; \mathbb{R}^d)$ as $n \to \infty$. On the other hand,

$$\int_Q \frac{1}{\varepsilon_n} f(v_n^k(x), \varepsilon_n \nabla v_n^k(x)) \, dx \leq \frac{1}{\varepsilon_n} \int_{Q'} L_k f \left( \frac{x'}{\varepsilon_n L_k}, x_N \right) dx' \, dx_N.$$

Therefore, in view of the Riemann-Lebesgue Lemma, we obtain by (3.19)

$$\lim_{n \to \infty} \int_Q \frac{1}{\varepsilon_n} f(v_n^k(x), \varepsilon_n \nabla v_n^k(x)) \, dx = \lim_{k \to \infty} \int Q \frac{1}{L_k} \nabla \xi_k(x) \, dx = K(a, b, e_N).$$

Since we also have $\lim_{k \to \infty} \lim_{n \to \infty} \|v_n^k - u\|_{L^p} = 0$, a diagonalization argument (see Lemma 7.1 in [13]) produces a subsequence $\{k(n)\} \subset \{k\}$ such that

$$\lim_{n \to \infty} \int_Q \frac{1}{\varepsilon_n} f(v_n^{k(n)}(x), \varepsilon_n \nabla v_n^{k(n)}(x)) \, dx = K(a, b, e_N),$$

and

$$\lim_{n \to \infty} \|v_n^{k(n)} - u\|_{L^p} = 0.$$

It suffices to set $u_n := v_n^{k(n)}$.

**Case 2.** We now consider the general case where $a_0 \in \mathbb{R}^N$, and for $\eta > 0$ we define

$$f_\eta(u, A) := f \left( u, \frac{A}{\eta} \right).$$

Set

$$u_0(x) := \begin{cases} b & \text{if } x \cdot e_N > 0, \\ a & \text{if } x \cdot e_N < 0. \end{cases}$$

Given $\varepsilon_n \to 0^+$, by Case 1 we obtain a sequence $\{v_n\} \subset A(a, b, e_N)$ such that $v_n \to u_0$ in $L^1(Q; \mathbb{R}^d)$, and

$$\lim_{n \to \infty} \int_Q \frac{1}{\varepsilon_n} f_\eta(v_n(x), \varepsilon_n \nabla v_n(x)) \, dx = K_\eta(a, b, e_N)$$

where

$$K_\eta(a, b, e_N) := \inf_{L > 0} \left\{ L f_\eta \left( \xi(x), \frac{1}{L} \nabla \xi(x) \right) \, dx : \xi \in A(a, b, e_N) \right\}.$$

Note that

$$K_\eta(a, b, e_N) = \frac{1}{\eta} K(a, b, e_N).$$

(3.20)
For \( x \in a_0 + \eta Q \), define \( u_n \in H^1(a_0 + \eta Q; \mathbb{R}^d) \) by
\[
 u_n(x) := v_n\left(\frac{x - a_0}{\eta}\right).
\]

We have
\[
\int_{a_0 + \eta Q} |u_n(x) - u(x)| \, dx = \int_{a_0 + \eta Q} \left| v_n \left( \frac{x - a_0}{\eta} \right) - u(x) \right| \, dx
\]
\[
= \eta^N \int_Q |v_n(x) - u(a_0 + \eta x)| \, dx
\]
\[
= \eta^N \int_Q |v_n(x) - u_0(x)| \, dx \to 0 \text{ as } n \to \infty.
\]

Also, in view of (3.20),
\[
\lim_{n \to \infty} \int_{a_0 + \eta Q} \frac{1}{\varepsilon_n} f(u_n(x), \varepsilon_n \nabla u_n(x)) \, dx = \lim_{n \to \infty} \int_{a_0 + \eta Q} \frac{1}{\varepsilon_n} f\left( v_n \left( \frac{x - a_0}{\eta} \right), \frac{\varepsilon_n}{\eta} \nabla v_n \left( \frac{x - a_0}{\eta} \right) \right) \, dx
\]
\[
= \lim_{n \to \infty} \eta^N \int_Q \frac{1}{\varepsilon_n} f(v_n(x), \frac{\varepsilon_n}{\eta} \nabla v_n(x)) \, dx
\]
\[
= \lim_{n \to \infty} \eta^N \int_Q \frac{1}{\varepsilon_n} f(\xi(y), \varepsilon_n \nabla \xi(y)) \, dy
\]
\[
= \eta^N K_\eta(a, b, e_N) = \eta^{N-1} K(a, b, e_N).
\]

The last step of the proof of Proposition 3.3 uses the upper semicontinuity property of \( K \). Precisely

**Proposition 3.5** If (H2) holds, then

(i) \( 0 \leq K(x, a, b, \nu) \leq C(1 + |a|^q + |b|^q) \) for all \( (x, a, b, \nu) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1} \);

(ii) \( K(\cdot, a, b, \cdot) \) is upper semicontinuous.

**Proof.** (i) Fix \( (x, a, b, \nu) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1} \), and let
\[
\xi(y) := (b - a)(y \cdot \nu) + \frac{a + b}{2}.
\]

Since \( \xi \in \mathcal{A}(a, b, \nu) \), \( q \geq 2 \), and in view of (H2), we have
\[
0 \leq K(x, a, b, \nu) \leq \int_{Q_\nu} f(x, \xi(y), \nabla \xi(y)) \, dy
\]
\[
\leq \int_{Q_\nu} C(g(x, \xi(y)) + ||\nabla \xi(y)||^2) \, dy
\]
\[
\leq \int_{Q_\nu} C(1 + ||\xi(y)||^q + ||\nabla \xi(y)||^2) \, dy
\]
\[
\leq C(1 + |a|^q + |b|^q).
\]

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(ii) First, it is clear that
\[ K(x, a, b, \nu) := \inf_{s > 0} \left\{ \frac{1}{s} \int_Q f(x, u(y), s\nabla u(y)R^T) \, dy : u \in \mathcal{A}(a, b, e_N), \ R \in SO(N), \ Re_N = \nu \right\}. \]

Let \((x_n, \nu_n) \to (x, \nu)\). Given \(\varepsilon > 0\) choose a rotation \(R_{\varepsilon,x}\) such that \(R_{\varepsilon,x}e_N = \nu, \xi_{\varepsilon,x} \in \mathcal{A}(a, b, e_N) , \) and \(s_{\varepsilon,x} > 0\) such that
\[
\left| K(x, a, b, \nu) - \int_Q \frac{1}{s_{\varepsilon,x}} f(x, \xi_{\varepsilon,x}(y), s_{\varepsilon,x} \nabla \xi_{\varepsilon,x}(y)R^T_{\varepsilon,x}) \, dy \right| < \varepsilon. \tag{3.21}
\]

Let \(R_n \in SO(N)\) be such that \(R_n e_N = \nu_n (n \in \mathbb{N}), \) and \(R_n \to R_{\varepsilon,x}\) as \(n \to +\infty.\) We have
\[
K(x_n, a, b, \nu_n) \leq \int_Q \frac{1}{s_{\varepsilon,x}} f(x_n, \xi_{\varepsilon,x}(y), s_{\varepsilon,x} \nabla \xi_{\varepsilon,x}(y)R^T_n) \, dy \tag{3.22}
\]

Since \(f\) is continuous, and \(\xi_{\varepsilon,x} \in \mathcal{A}(a, b, e_N),\) we have
\[
\frac{1}{s_{\varepsilon,x}} f(x_n, \xi_{\varepsilon,x}(y), s_{\varepsilon,x} \nabla \xi_{\varepsilon,x}(y)R^T_n) \to \frac{1}{s_{\varepsilon,x}} f(x, \xi_{\varepsilon,x}(y), s_{\varepsilon,x} \nabla \xi_{\varepsilon,x}(y)R^T_{\varepsilon,x}), \]
pointwise, as \(n \to \infty,\) and
\[
\left| \frac{1}{s_{\varepsilon,x}} f(x_n, \xi_{\varepsilon,x}(y), s_{\varepsilon,x} \nabla \xi_{\varepsilon,x}(y)R^T_n) \right| \leq \frac{1}{s_{\varepsilon,x}} \max \left\{ f(z, u, \xi) : |z| \leq |x| + 1, \ |u| \leq \|
abla \xi_{\varepsilon,x}\|_{\infty} \right\}, \]
for \(y \in Q,\) and \(n \in \mathbb{N}\) sufficiently large. Thus, by Lebesgue’s Dominated Convergence Theorem, we obtain that
\[
\lim_{n \to \infty} \int_Q \frac{1}{s_{\varepsilon,x}} f(x_n, \xi_{\varepsilon,x}(y), s_{\varepsilon,x} \nabla \xi_{\varepsilon,x}(y)R^T_n) \, dy = \int_Q \frac{1}{s_{\varepsilon,x}} f(x, \xi_{\varepsilon,x}(y), s_{\varepsilon,x} \nabla \xi_{\varepsilon,x}(y)R^T_{\varepsilon,x}) \, dy.
\]

Passing to \(\limsup\) as \(n \to \infty\) in (3.22), and taking into account (3.21), we have
\[
\limsup_{n \to \infty} K(x_n, a, b, \nu_n) \leq K(x, a, b, \nu) + \varepsilon.
\]

We conclude the proof by letting \(\varepsilon \to 0^+.\)

Proof of Proposition 3.3. We divide the proof into two steps:

Step 1: \(A_0\) is polyhedral.

Let \(C\) be the family of all open cubes in \(\Omega\) with faces parallel to the axes, centered at points \(x \in \Omega \cap \mathbb{Q}^N\) and with rational edge length. Denote by \(\mathcal{R}\) the countable subfamily of \(A(\Omega)\) obtained by taking all finite unions of elements of \(C,\) i.e.,
\[
\mathcal{R} := \{ \bigcup_{i=1}^k C_i : k \in \mathbb{N}, \ C_i \in \mathcal{C} \}. \]
Let $\varepsilon_n \to 0^+$. Since $L^1(\Omega; \mathbb{R}^d)$ is a separable metric space, using Kuratowski’s Compactness Theorem (see, e.g. [17]), a diagonalization argument, and in the spirit of $\Gamma$-convergence (see Proposition 7.9 in [12]), we can assert that there exists a subsequence $\{\varepsilon_{n_j}^R\}$ of $\{\varepsilon_n\}$ such that, upon setting

$$F_{\varepsilon_{n_j}^R}(u; A) := \inf \left\{ \liminf_{n \to \infty} \frac{1}{\delta_n} \int_A f(x, v_n(x), \delta_n \nabla v_n(x)) \, dx : v_n \to u \text{ in } L^1(A; \mathbb{R}^d) \right\},$$

for every $u \in L^1(\Omega; \mathbb{R}^d)$ and $C \in \mathcal{R}$, there exists a sequence $\{u_{\varepsilon_{n_j}^R}\} \subset H^1(C; \mathbb{R}^d)$ such that

$$u_{\varepsilon_{n_j}^R} \to u \text{ in } L^1(C; \mathbb{R}^d)$$

and

$$F_{\varepsilon_{n_j}^R}(u; C) = \lim_{n \to \infty} \frac{1}{\varepsilon_{n_j}^R} \int_C f \left( x, u_{\varepsilon_{n_j}^R}(x), \varepsilon_{n_j}^R \nabla u_{\varepsilon_{n_j}^R}(x) \right) \, dx.$$

Claim 1: $F_{\varepsilon_{n_j}^R}(u; \cdot)$ is a finite nonnegative Radon measure, absolutely continuous with respect to $\mathcal{H}^{N-1} \mid \partial^* A_0$.

Claim 2: The following inequality holds:

$$\frac{dF_{\varepsilon_{n_j}^R}(u; \cdot)}{d\mathcal{H}^{N-1} \mid \partial^* A_0}(x_0) \leq K(x_0, a, b, \nu(x_0)) \text{ for } \mathcal{H}^{N-1} \text{-a.e. } x_0 \in \Omega \cap \partial^* A_0.$$

Assuming that Claims 1 and 2 hold, we obtain

$$F_{\varepsilon_{n_j}^R}(u; \Omega) = \int_{\Omega} \frac{dF_{\varepsilon_{n_j}^R}(u; \cdot)}{d\mathcal{H}^{N-1} \mid \partial^* A_0}(x) d\mathcal{H}^{N-1}(x) = \int_{\Omega \cap \partial^* A_0} \frac{dF_{\varepsilon_{n_j}^R}(u; \cdot)}{d\mathcal{H}^{N-1} \mid \partial^* A_0}(x) d\mathcal{H}^{N-1}(x)$$

$$\leq \int_{\Omega \cap \partial^* A_0} K(x, a, b, \nu(x)) d\mathcal{H}^{N-1}(x).$$

In view of Proposition 3.1, we deduce that, in fact,

$$F_{\varepsilon_{n_j}^R}(u; \Omega) = \int_{\Omega \cap \partial^* A_0} K(x, a, b, \nu(x)) d\mathcal{H}^{N-1}(x),$$

and the conclusion in this case follows by a simple diagonalization argument. To finish Step 1 it remains to prove the claims.

Proof of Claim 1: For each $k \in \mathbb{N}$, let $\{v_n^k\} \subset H^1(\Omega; \mathbb{R}^d)$ be such that $\lim_{n \to \infty} \|v_n^k - u\|_{L^1(\Omega, \mathbb{R}^d)} = 0$, and

$$F_{\varepsilon_{n_j}^R}(u; \Omega) \leq \liminf_{n \to \infty} \frac{1}{\varepsilon_{n_j}^R} \int_{\Omega} f(x, v_n^k(x), \varepsilon_{n_j}^R \nabla v_n^k(x)) \, dx \leq F_{\varepsilon_{n_j}^R}(u; \Omega) + \frac{1}{k}.$$

Extract $\{n(j, k)\} \subset \{n\}$ such that

$$\liminf_{n \to \infty} \frac{1}{\varepsilon_{n_j}^R} \int_{\Omega} f(x, v_n^k(x), \varepsilon_{n_j}^R \nabla v_n^k(x)) \, dx = \lim_{j \to \infty} \frac{1}{\varepsilon_{n(j, k)}^R} \int_{\Omega} f(x, v_{n(j, k)}^k(x), \varepsilon_{n(j, k)}^R \nabla v_{n(j, k)}^k(x)) \, dx.$$
We have
\[
\lim_{k \to \infty} \lim_{j \to \infty} \frac{1}{\varepsilon_n^{k(j,k)}} \int_{\Omega} f(x, v_{n(j,k)}^k(x), \varepsilon_n^{k(j,k)} \nabla v_{n(j,k)}^k(x)) \, dx = \lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{\varepsilon_n} \int_{\Omega} f(x, v_n^k(x), \varepsilon_n \nabla v_n^k(x)) \, dx = \mathcal{F}(\varepsilon_n)(u; \Omega).
\]

We can extract a subsequence \( \{j(k)\} \subset \{j\} \) such that
\[
\lim_{k \to \infty} \frac{1}{\varepsilon_n^k} \int_{\Omega} f(x, v_k(x), \varepsilon_n^k \nabla v_k(x)) \, dx = \mathcal{F}(\varepsilon_n)(u; \Omega),
\]
where we have denoted \( n_k := n(j(k), k) \) and \( v_k := v_{n(j(k), k)}^k \).

The sequence of measures \( \{\mu_k\} \), where \( \mu_k := \frac{1}{\varepsilon_n^k} f(x, v_k(x), \varepsilon_n^k \nabla v_k(x)) \mathcal{L}^N \mid \Omega \), is bounded in \( \mathcal{M}(\Omega) \). Thus, there exists a nonnegative Radon measure \( \mu \) such that, up to a subsequence (not relabelled), \( \mu_k \rightharpoonup \mu \) weakly* in \( \mathcal{M}(\Omega) \). We want to show that \( \mu(A) = \mathcal{F}(\varepsilon_n)(u; A) \) for all \( A \in \mathcal{A}(\Omega) \).

To this end, we will verify conditions (i)-(iv) of Lemma 7.3 in [13] (see also [23]), with \( \pi : \mathcal{A}(\Omega) \to [0, \infty) \) defined by
\[
\pi(A) := \mathcal{F}(\varepsilon_n)(u; A).
\]

Precisely, for any \( A, B, C \in \mathcal{A}(\Omega) \),

(i) if \( \overline{C} \subseteq B \subseteq A \), then \( \pi(A) \leq \pi(A \setminus \overline{C}) + \pi(B) \),

(ii) for any \( \varepsilon > 0 \), there exists \( C_\varepsilon \in \mathcal{A}(\Omega) \) with \( \overline{C_\varepsilon} \subseteq A \) and \( \pi(A \setminus \overline{C_\varepsilon}) \leq \varepsilon \),

(iii) \( \pi(\Omega) \geq \mu(\mathbb{R}^N) \),

(iv) \( \pi(A) \leq \mu(A) \).

Note first that (iii) follows immediately from
\[
\mu(\mathbb{R}^N) \leq \lim_{k \to \infty} \mu_k(\mathbb{R}^N) = \lim_{k \to \infty} \int_{\Omega} \frac{1}{\varepsilon_n^k} f(x, v_k(x), \varepsilon_n^k \nabla v_k(x)) \, dx = \mathcal{F}(\varepsilon_n)(u; \Omega) = \pi(\Omega).
\]

The sequence \( \{\pi_n\} \subset H^1(A; \mathbb{R}^d) \), where \( \pi_n := v_k \) if \( n = n_k \) for some \( k \in \mathbb{N} \), and \( \pi_n := u \ast \rho \frac{1}{\varepsilon_n^k} \) if \( n \notin \{n_k : k \in \mathbb{N}\} \), is admissible for the definition of \( \mathcal{F}(\varepsilon_n)(u; A) \). We obtain that
\[
\mathcal{F}(\varepsilon_n)(u; A) \leq \liminf_{n \to \infty} \frac{1}{\varepsilon_n} \int_A f(x, \pi_n(x), \varepsilon_n \nabla \pi_n(x)) \, dx \leq \liminf_{k \to \infty} \frac{1}{\varepsilon_n^k} \int_A f(x, v_k(x), \varepsilon_n^k \nabla v_k(x)) \, dx \leq \limsup_{k \to \infty} \mu_k(A) \leq \mu(A),
\]

thus asserting (iv).

To prove (ii), we first show that there exists a constant \( C > 0 \), such that
\[
\mathcal{F}(\varepsilon_n)(u; A) \leq C \mathcal{H}^{N-1}(A \cap \partial^* A_0) \text{ for all } A \in \mathcal{A}(\Omega). \tag{3.23}
\]

Let \( \rho : \mathbb{R}^N \to [0, +\infty) \) be a symmetric mollifier, and define
\[
\rho \frac{1}{\varepsilon_n}(x) := \frac{1}{(\varepsilon_n)^N} \rho \left( \frac{x}{\varepsilon_n} \right). \tag{3.24}
\]
Let \( u_n := \rho \frac{1}{n} * u \ (n \in \mathbb{N}) \) we have, by means of the growth conditions in (H2), and since \( A_0 \) is polyhedral,

\[
\mathcal{F}(\varepsilon_n) (u; A) \leq \liminf_{n \to \infty} \frac{1}{\varepsilon_n} \int_A f(x, u_n(x), \varepsilon_n \nabla u_n(x)) \, dx
\]

\[
= \liminf_{n \to \infty} \frac{1}{\varepsilon_n} \int_{x \in A: \text{dist}(x, \partial^* A_0) \leq \varepsilon_n} f(x, u_n(x), \varepsilon_n \nabla u_n(x)) \, dx
\]

\[
\leq \liminf_{n \to \infty} \frac{C}{\varepsilon_n} \int_{x \in A: \text{dist}(x, \partial^* A_0) \leq \varepsilon_n} (1 + |u_n|^q + (\varepsilon_n R)^2 |\nabla u_n|^2) \, dx
\]

\[
\leq C \liminf_{n \to \infty} \mathcal{L}^N \left( \{ x \in A: \text{dist}(x, \partial^* A_0) \leq \varepsilon_n \} \right) = C \mathcal{H}^{N-1}(A \cap \partial^* A_0).
\]

In view of (3.23), and using the inner regularity of the Radon measure \( C \mathcal{H}^{N-1}(\partial^* A_0) \), we deduce (ii).

It remains to show that (i) holds. To this aim, let \( A, B, C \in \mathcal{A}(\Omega) \) be such that \( C \subset B \subset A \). For \( \delta > 0 \), let \( B^\delta \) and \( D^\delta \) be two elements of \( \mathcal{R} \) such that \( B^\delta \subset B \), \( D^\delta \subset A \setminus C \), and

\[
\mathcal{H}^{N-1}((A \setminus (B^\delta \cup D^\delta)) \cap \partial^* A_0) < \delta. \quad (3.25)
\]

Let \( \{ u_{B^\delta}^{x_n} \} \) and \( \{ u_{D^\delta}^{x_n} \} \) be sequences in \( H^1(B^\delta; \mathbb{R}^d) \) and \( H^1(D^\delta; \mathbb{R}^d) \), respectively, such that \( u_{B^\delta}^{x_n} \to u \) in \( L^1(B^\delta; \mathbb{R}^d) \), \( u_{D^\delta}^{x_n} \to u \) in \( L^1(D^\delta; \mathbb{R}^d) \),

\[
\lim_{n \to \infty} \frac{1}{\varepsilon_n} \int_{B^\delta} f \left( x, u_{B^\delta}^{x_n}(x), \varepsilon_n \nabla u_{B^\delta}^{x_n}(x) \right) \, dx = \mathcal{F}(x_n)(u; B^\delta) < +\infty, \quad (3.26)
\]

and

\[
\lim_{n \to \infty} \frac{1}{\varepsilon_n} \int_{D^\delta} f \left( x, u_{D^\delta}^{x_n}(x), \varepsilon_n \nabla u_{D^\delta}^{x_n}(x) \right) \, dx = \mathcal{F}(x_n)(u; D^\delta) < +\infty. \quad (3.27)
\]

We may assume, without loss of generality, that

\[
u_{B^\delta}^{x_n} = \rho \frac{1}{n} * u \ \text{on} \ \partial B^\delta, \ \nu_{D^\delta}^{x_n} \to u \ \text{in} \ L^q(B^\delta; \mathbb{R}^d) \ \text{and} \ \mathcal{L}^N \ - \ a.e. \ x \in B^\delta.
\]

The proof of this fact follows along the lines of Step 2 of the proof of Lemma 3.2, where we replace \( Q \) by \( B^\delta \), and \( \nu_n \) by \( \rho \frac{1}{n} * u \) (now \( \rho \frac{1}{n} \) is given by (3.24)). We also note that in this case \( \text{supp} \ (\rho \frac{1}{n} * u) \subset \{ x : \text{dist}(x, \partial^* A_0) < \varepsilon_n \} \), and for each \( k \in \mathbb{N} \), the layer \( L_k \) in Step 2 of the proof of Lemma 3.2 should now be taken to be \( L_k := \{ x \in B^\delta : \text{dist}(x, \partial B^\delta) \leq \frac{1}{k} \} \).

Similarly, we may assume that

\[
u_{D^\delta}^{x_n} = \rho \frac{1}{n} * u \ \text{on} \ \partial D^\delta, \ \nu_{D^\delta}^{x_n} \to u \ \text{in} \ L^q(D^\delta; \mathbb{R}^d) \ \text{and} \ \mathcal{L}^N \ - \ a.e. \ x \in D^\delta.
\]

Extend \( u_{B^\delta}^{x_n} \) as \( \rho \frac{1}{n} * u \) outside \( B^\delta \), and \( u_{D^\delta}^{x_n} \) as \( \rho \frac{1}{n} * u \) outside \( D^\delta \). Note that, in view of (3.3),

\[
\lim_{n \to \infty} \| u_{B^\delta}^{x_n} - u \|_{L^q(A; \mathbb{R}^d)} = \lim_{n \to \infty} \| u_{D^\delta}^{x_n} - u \|_{L^q(A; \mathbb{R}^d)} = 0. \quad (3.28)
\]
Write $B \setminus C$ as a union of $M_n$ layers $L_n^{(i)}$ ($i = 1, \ldots, M_n$) of width $\varepsilon_n \|u^{B}_{\varepsilon_n} - u^{D}_{\varepsilon_n}\|_{L^2(A;\mathbb{R}^d)}^{1/2}$ so that

$$M_n \cdot \varepsilon_n \|u^{B}_{\varepsilon_n} - u^{D}_{\varepsilon_n}\|_{L^2(A;\mathbb{R}^d)}^{1/2} = O(1).$$

(3.29)

We have

$$\sum_{i=1}^{M_n} \int_{L_n^{(i)}} \left( 1 + |u^{B}_{\varepsilon_n}|^q + |u^{D}_{\varepsilon_n}|^q + (\varepsilon_n^2)|\nabla u^{B}_{\varepsilon_n}|^2 + (\varepsilon_n^2)|\nabla u^{D}_{\varepsilon_n}|^2 + \frac{|u^{B}_{\varepsilon_n} - u^{D}_{\varepsilon_n}|^2}{\|u^{B}_{\varepsilon_n} - u^{D}_{\varepsilon_n}\|_{L^2(A;\mathbb{R}^d)}} \right) dx$$

$$= \int_{B \setminus C} \left( 1 + |u^{B}_{\varepsilon_n}|^q + |u^{D}_{\varepsilon_n}|^q + (\varepsilon_n^2)|\nabla u^{B}_{\varepsilon_n}|^2 + (\varepsilon_n^2)|\nabla u^{D}_{\varepsilon_n}|^2 + \frac{|u^{B}_{\varepsilon_n} - u^{D}_{\varepsilon_n}|^2}{\|u^{B}_{\varepsilon_n} - u^{D}_{\varepsilon_n}\|_{L^2(A;\mathbb{R}^d)}} \right) dx,$$

and thus there exists $i_0 \in \{1, \ldots, M_n\}$ such that

$$\int_{L_n^{(i_0)}} \left( 1 + |u^{B}_{\varepsilon_n}|^q + |u^{D}_{\varepsilon_n}|^q + (\varepsilon_n^2)|\nabla u^{B}_{\varepsilon_n}|^2 + (\varepsilon_n^2)|\nabla u^{D}_{\varepsilon_n}|^2 + \frac{|u^{B}_{\varepsilon_n} - u^{D}_{\varepsilon_n}|^2}{\|u^{B}_{\varepsilon_n} - u^{D}_{\varepsilon_n}\|_{L^2(A;\mathbb{R}^d)}} \right) dx \leq \frac{1}{M_n} \int_{B \setminus C} \left( 1 + |u^{B}_{\varepsilon_n}|^q + |u^{D}_{\varepsilon_n}|^q + (\varepsilon_n^2)|\nabla u^{B}_{\varepsilon_n}|^2 + (\varepsilon_n^2)|\nabla u^{D}_{\varepsilon_n}|^2 + \frac{|u^{B}_{\varepsilon_n} - u^{D}_{\varepsilon_n}|^2}{\|u^{B}_{\varepsilon_n} - u^{D}_{\varepsilon_n}\|_{L^2(A;\mathbb{R}^d)}} \right) dx.$$

(3.30)

We remark that by (3.26), (3.27), (3.28), and (H2),

$$\sup_{n \in \mathbb{N}} \int_{B \setminus C} \left( 1 + |u^{B}_{\varepsilon_n}|^q + |u^{D}_{\varepsilon_n}|^q + (\varepsilon_n^2)|\nabla u^{B}_{\varepsilon_n}|^2 + (\varepsilon_n^2)|\nabla u^{D}_{\varepsilon_n}|^2 + \frac{|u^{B}_{\varepsilon_n} - u^{D}_{\varepsilon_n}|^2}{\|u^{B}_{\varepsilon_n} - u^{D}_{\varepsilon_n}\|_{L^2(A;\mathbb{R}^d)}} \right) dx$$

$$=: a_0 < +\infty.$$  

(3.31)

Consider cut-off functions $\varphi_n \in C^\infty_0(\mathbb{R}^N; [0,1])$ such that

$$\varphi_n(x) = 0 \text{ if } x \in \left( \bigcup_{j=1}^{i_0-1} L_n^{(j)} \right) \cup (A \setminus B),$$

$$\varphi_n(x) = 1 \text{ if } x \in \left( \bigcup_{j=1}^{i_0-1} L_n^{(j)} \right) \cup C,$$

and

$$\|\nabla \varphi_n\|_\infty = O \left( \frac{1}{\varepsilon_n \|u^{B}_{\varepsilon_n} - u^{D}_{\varepsilon_n}\|_{L^2(A;\mathbb{R}^d)}^{1/2}} \right).$$

Define

$$u_n := \varphi_n u^{B}_{\varepsilon_n} + (1 - \varphi_n) u^{D}_{\varepsilon_n} + \chi_{(A \setminus (B \cup D^s))} \left( \rho_{\varepsilon_n} * u \right).$$

We have that $u_n \to u$ in $L^1(A;\mathbb{R}^d)$ as $n \to \infty$, and in view of (3.26), (3.27),

$$F(\varepsilon_n)(u; A) \leq \liminf_{n \to \infty} \frac{1}{\varepsilon_n} \int_A f(x, u_n(x), \varepsilon_n \nabla u_n(x)) \, dx$$
By (3.29), (3.30), and the growth conditions in (H2), we obtain that
\[
\leq \limsup_{n \to \infty} \frac{1}{\varepsilon_n} \int_{A \setminus (B^\delta \cup D^\delta)} f \left( x, u_n(x), \varepsilon_n \nabla u_n(x) \right) \, dx 
+ \frac{1}{\varepsilon_n} \int_{B^\delta} f \left( x, u_n \varepsilon_n \nabla u_n \right) \, dx 
+ \frac{1}{\varepsilon_n} \int_{D^\delta} f \left( x, u_n \varepsilon_n \nabla u_n \right) \, dx 
\leq \limsup_{n \to \infty} \frac{1}{\varepsilon_n} \int_{A \setminus (B^\delta \cup D^\delta)} f \left( x, u_n(x), \varepsilon_n \nabla u_n(x) \right) \, dx 
+ \mathcal{F}_{\varepsilon_n}(u; B^\delta) + \limsup_{n \to \infty} \frac{1}{\varepsilon_n} \int_{L^1(\varepsilon)} \frac{f(x, u_n, \varepsilon_n \nabla u_n)}{\nu(x)} \, dx
\]
\begin{equation}
\leq \mathcal{H}^{N-1} \left( (A \setminus (B^\delta \cup D^\delta)) \cap \partial^* A_0 \right) + \mathcal{F}_{\varepsilon_n}(u; D^\delta) + \mathcal{F}_{\varepsilon_n}(u; B^\delta)
+ \limsup_{n \to \infty} \frac{1}{\varepsilon_n} \int_{L^1(\varepsilon)} \frac{f(x, u_n, \varepsilon_n \nabla u_n)}{\nu(x)} \, dx
\end{equation}
(3.32)

By (3.29), (3.30), and the growth conditions in (H2), we obtain that
\[
\frac{1}{\varepsilon_n} \int_{L^1(\varepsilon)} \frac{f(x, u_n(x), \varepsilon_n \nabla u_n(x))}{x} \, dx 
\leq \frac{C}{\varepsilon_n} \int_{L^1(\varepsilon)} \left( 1 + |u_n B^\delta| + |u_n D^\delta| + (\varepsilon_n)^2 |\nabla u_n B^\delta| + (\varepsilon_n)^2 |\nabla u_n D^\delta| \right)
+ (\varepsilon_n)^2 \|\nabla \varphi_n\|_\infty |u_n B^\delta| - |u_n D^\delta| \right) \, dx
\leq \frac{C}{\varepsilon_n} \int_{\mathbb{R}^d} \left( 1 + |u_n B^\delta| + |u_n D^\delta| + (\varepsilon_n)^2 |\nabla u_n B^\delta| + (\varepsilon_n)^2 |\nabla u_n D^\delta| \right) \, dx
\leq c_0 \frac{C}{\varepsilon_n} \left( |u_n B^\delta| - |u_n D^\delta| \right)^{1/2} \|L(A, \mathbb{R}^d)\)
\]
in view of (3.31). Thus,
\[
\limsup_{n \to \infty} \frac{1}{\varepsilon_n} \int_{L^1(\varepsilon)} \frac{f(x, u_n(x), \varepsilon_n \nabla u_n(x))}{x} \, dx = 0
\]
and we deduce from (3.25) and (3.32) that
\[
\mathcal{F}_{\varepsilon_n}(u; A) \leq \delta + \mathcal{F}_{\varepsilon_n}(u; B^\delta) + \mathcal{F}_{\varepsilon_n}(u; D^\delta) \leq \delta + \mathcal{F}_{\varepsilon_n}(u; B) + \mathcal{F}_{\varepsilon_n}(u; A \setminus \overline{C})
\]
Letting \( \delta \to 0^+ \) we obtain that (i) holds. This concludes the proof of Claim 1.

Proof of Claim 2: Since nearby \( \mathcal{H}^{N-1} \)-a.e. \( x_0 \in \partial A_0, u \) has a planar interface we can apply Lemma 3.4 to deduce that for \( \varepsilon > 0 \) sufficiently small there exists \( \{ u_n(\varepsilon) \} \subset H^1(Q_{\varepsilon}(x_0), e); \mathbb{R}^d \) such that \( u_n(\varepsilon) \to u \) in \( L^1(Q_{\varepsilon}(x_0), e); \mathbb{R}^d \), and
\[
\varepsilon^{N-1} K(x_0, a, b, \nu(x_0)) = \lim_{n \to \infty} \frac{1}{\varepsilon_n} \int_{Q_{\varepsilon}(x_0), \varepsilon} f \left( x_0, u_n(x), \varepsilon \nabla u_n(x) \right) \, dx
\]
(3.33)
Taking into account (2.1) and (3.33), we obtain

\[
\frac{dF(ε_k) (u; ·)}{dH^{N-1}} \left| \partial^* A_0 \right| (x_0) = \lim_{ε \to 0^+} \frac{F(ε_k) (u; Q(ν_n)(x_0, ε))}{ε^{N-1}} \\
\leq \lim_{ε \to 0^+} \frac{1}{ε^{N-1}} \liminf_{n \to \infty} \frac{1}{ε^N} \int f \left( x_0, u_n(ε)(x), ε^n R u_n(ε)(x) \right) dx \\
= K(x_0, a, b, ν(x_0)).
\]

**Step 2.** We are now ready to consider the general case. In view of Theorem 2.3, there exist polyhedral sets $A_k$ such that $χ_{A_k} \to χ_{A_0}$ in $L^1(Ω)$, $\text{Per}_Ω(A_k) \to \text{Per}_Ω(A_0)$, $\mathcal{L}^N(A_k) = \mathcal{L}^N(A_0)$ and $\mathcal{H}^{N-1}(∂^* A_k \cap ∂ Ω) = 0$. By Step 1, for every $k ∈ N$, there exist sequences $u_n^{(k)} → χ_{A_k} a + (1 − χ_{A_k}) b$ as $n → ∞$ in $L^1(Ω; R^d)$, such that

\[
\lim_{n \to ∞} \frac{1}{ε_n} \int f \left( x, u_n^{(k)}(x), ε^n R u_n^{(k)}(x) \right) dx = \int f \left( x, a, b, ν_k(x) \right) dH^{N-1}(x),
\]

where $ν_k(x)$ is the measure theoretic unit normal to $∂^* A _k$ at $x$. Clearly,

\[
\lim_{k → ∞} \lim_{n → ∞} \|u_n^{(k)}(ε_n) − u\|_{L^1(Ω; R^d)} = 0,
\]

and for every continuous function $h : Ω × R^N → [0, +∞)$, we have (see [22], [29], [42]):

\[
\lim_{k → ∞} \int_{∂^* A_k \cap Ω} h(x, ν_k(x)) dH^{N-1}(x) = \int_{∂ A_0 \cap Ω} h(x, ν(x)) dH^{N-1}(x).
\]

As $K(·, a, b, ·)$ is upper semicontinuous (see Proposition 3.5 (ii)), there exist continuous functions $h_m : Ω × R^N → [0, +∞)$ such that

\[
K(x, a, b, ξ) ≤ h_m(x, ξ) ≤ C|ξ|
\]

and

\[
K(x, a, b, ξ) = \inf_m h_m(x, ξ)
\]

for every $(x, ξ) ∈ Ω × R^N$, where we have extended $K(x, a, b, ·)$ as a homogeneous function of degree one (see [24]). Thus, for all $m ∈ N$,

\[
\limsup_{k → ∞} \lim_{n → ∞} \frac{1}{ε_n} \int f \left( x, u_n^{(k)}(x), ε^n R u_n^{(k)}(x) \right) dx = \limsup_{k → ∞} \int_{∂^* A_k \cap Ω} K(x, a, b, ν_k(x)) dH^{N-1}(x) \\
\leq \limsup_{k → ∞} \int_{∂^* A_k \cap Ω} h_m(x, ν_k(x)) dH^{N-1}(x) = \int_{∂ A_0 \cap Ω} h_m(x, ν(x)) dH^{N-1}(x).
\]

Taking the limit as $m → +∞$ and using Lebesgue’s Monotone Convergence Theorem, we deduce that

\[
\limsup_{k → ∞} \lim_{n → ∞} \frac{1}{ε_n} \int f \left( x, u_n^{(k)}(x), ε^n R u_n^{(k)}(x) \right) dx ≤ \int_{∂^* A_0 \cap Ω} K(x, a, b, ν(x)) dH^{N-1}(x).
\]

In view of Proposition 3.1, and by means of a standard diagonalization procedure, we conclude the proof.
4 A homogenization result in the gradient theory of phase transitions

In this chapter we prove Theorem 1.3. We recall that $W : \mathbb{R}^N \times \mathbb{R}^d \to [0, +\infty)$ is a continuous function satisfying the following hypotheses:

(A1) $W(\cdot, u)$ is $Q$-periodic for every $u \in \mathbb{R}^d$;

(A2) $W(x, u) = 0$ if and only if $u \in \{a, b\}$;

(A3) there exist $C > 0$ and $q \geq 2$ such that

$$\frac{1}{C}|u|^q - C \leq W(x, u) \leq C(1 + |u|^q)$$

for all $(x, u) \in \Omega \times \mathbb{R}^d$,

and let $I_\varepsilon : L^1(\Omega; \mathbb{R}^d) \to [0, +\infty]$ be defined by

$$I_\varepsilon(u) := \begin{cases} \int \left( \frac{1}{\varepsilon} W \left( \frac{y}{\varepsilon}, u(y) \right) + \varepsilon|\nabla u(y)|^2 \right) dy & \text{if } u \in H^1(\Omega; \mathbb{R}^d) \\ +\infty & \text{otherwise.} \end{cases}$$

**Theorem 1.3** Assume that (A1)-(A3) hold, let $\nu \in S^{N-1}$, $\rho : \mathbb{R} \to [0, +\infty)$ be a mollifier, and let $\rho_{T, \nu}(x) := T\rho(Tx \cdot \nu)$. Define

$$K_1(\nu) = \lim_{T \to \infty} \frac{1}{T^{N-1}} \inf \left\{ \int_{TQ_{\nu}} (W(y, u(y)) + |\nabla u(y)|^2) dy : u \in H^1(TQ_{\nu}; \mathbb{R}^d), \quad u = \rho_{T, \nu} \ast u_0 \text{ on } \partial(TQ_{\nu}) \right\}$$

with

$$u_0(x) = \begin{cases} b & \text{if } x \cdot \nu > 0, \\ a & \text{if } x \cdot \nu < 0. \end{cases}$$

Consider the functional $I_0 : L^1(\Omega; \mathbb{R}^d) \to [0, +\infty]$ defined by

$$I_0(u) := \begin{cases} \int_{\partial^* A_0 \cap \Omega} K_1(\nu(x)) d\mathcal{H}^{N-1}(x) & \text{if } u \in BV(\Omega; \{a, b\}), \\ +\infty & \text{otherwise,} \end{cases}$$

where $A_0 := \{x \in \Omega : u(x) = a\}$. Then

(i) $\Gamma(L^1(\Omega; \mathbb{R}^d)) - \liminf_{\varepsilon \to 0} I_\varepsilon = I_0$;

(ii) Assume that the set $A_0$ is polyhedral, and that the outward unit normal $\nu(x)$ to the reduced boundary $\partial^* A_0$ is such that $\nu(x) \in \{\pm e_1, \cdots, \pm e_N\}$, for $\mathcal{H}^{N-1}$-a.e. $x \in (\partial^* A_0) \cap \Omega$. Then

$$\Gamma(L^1(\Omega; \mathbb{R}^d)) - \lim_{\varepsilon \to 0} I_\varepsilon = I_0.$$
Lemma 4.1 For all $\nu \in S^{N-1}$ the limit
\[
\lim_{T \to \infty} \frac{1}{T^{N-1}} \inf_{TQ} \left\{ \int_{TQ} (W(y,u(y)) + |\nabla u(y)|^2) \, dy \quad u \in H^1(TQ;\mathbb{R}^d), u = \rho_{T,\nu} * u_0 \quad \text{on } \partial(TQ_{\nu}) \right\}
\]
exists.

Proof. Assume, without loss of generality that $\nu = e_N$, and write $\rho_T$ for $\rho_{T,e_N}$. For any $T > 0$, define
\[
g(T) := \frac{1}{T^{N-1}} \inf_{TQ} \left\{ \int_{TQ} (W(y,u(y)) + |\nabla u(y)|^2) \, dy \quad u \in H^1(TQ;\mathbb{R}^d), u = \rho_T * u_0 \quad \text{on } \partial(TQ) \right\},
\]
and let $u_T \in H^1(TQ;\mathbb{R}^d)$ be such that $u_T = \rho_T * u_0$ on $\partial(TQ)$, and
\[
\frac{1}{T^{N-1}} \int_{TQ} (W(y,u_T(y)) + |\nabla u_T(y)|^2) \, dy \leq g(T) + \frac{1}{T}. \tag{4.1}
\]
Let $S > T+3$, and let $E_{T,S}, E^*_{T,S} \subset \left(S - \frac{1}{T}\right)Q \cap \{x \in \mathbb{R}^N : x_N = 0\}$, $M_{S,T} := \left[\left(\frac{S - \frac{1}{T}}{\sqrt{N}}\right)^{N-1}\right] \in \mathbb{N}$, and $z_i \in \mathbb{Z}^{N-1} \times \{0\} (i = 1, \cdots , M_{S,T})$ be such that
\[
\left(S - \frac{1}{T}\right)Q \cap \{x \in \mathbb{R}^N : x_N = 0\} = \left(\bigcup_{i=1}^{M_{S,T}} (z_i + ([T] + 2)Q) \cap \{x \in \mathbb{R}^N : x_N = 0\}\right) \cup E_{T,S}
\]
\[
= \left(\bigcup_{i=1}^{M_{S,T}} (z_i + TQ) \cap \{x \in \mathbb{R}^N : x_N = 0\}\right) \cup E^*_{T,S}.
\]
We have
\[
\mathcal{L}^{N-1}(E_{T,S}) = \left(S - \frac{1}{T}\right)^{N-1} - M_{S,T}([T] + 2)^{N-1}
\]
and so, since
\[
E^*_{T,S} = E_{T,S} \cup \left(\bigcup_{i=1}^{M_{S,T}} ((z_i + ([T] + 2)Q) \setminus (z_i + TQ)) \cap \{x \in \mathbb{R}^N : x_N = 0\}\right),
\]
we obtain
\[
\mathcal{L}^{N-1}(E^*_{T,S}) = \left(S - \frac{1}{T}\right)^{N-1} - M_{S,T} T^{N-1}. \tag{4.2}
\]
Consider cut-off functions $\varphi_{S,T} \in C_c(SQ;[0,1])$ and, for $2 \leq m < T$, $i \in \{1, \cdots , M_{S,T}\}$, $\varphi_{m,i} \in C_c \left(z_i + (T + \frac{1}{m})Q;[0,1]\right)$ such that
\[
\varphi_{S,T}(x) = 0 \text{ if } x \in \partial(SQ), \quad \varphi_{S,T}(x) = 1 \text{ if } x \in \left(S - \frac{1}{T}\right)Q, \quad \|\nabla \varphi_{S,T}\|_{\infty} \leq CT,
\]
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In view of (A1) and (4.1), and because $z_i \in Z^N$, we get $W(\cdot + z_i, \cdot) = W(\cdot, \cdot)$, and
Finally, using (A2), (A3), and in view of (4.2), we have

\[
I_1(S, T) = \frac{1}{SN^{-1}} M_{ST} \int_{T^Q} (W(x, u_T(x)) + |\nabla u_T(x)|^2) \, dx \leq \frac{1}{SN^{-1}} M_{ST} T^{N-1} \left( g(T) + \frac{1}{T} \right) \\
\leq g(T) + \frac{1}{T}.
\] (4.4)

Using (A2), (A3), and the facts that \((\rho_T * u_0)(x) \in \{a, b\}\) if \(|x_N| \geq \frac{1}{m}\) and \((\rho_m * u_0)(x) \in \{a, b\}\) if \(|x_N| \geq \frac{1}{m}\), we obtain that

\[
I_2(S, T, m) \leq \frac{C}{SN^{-1}} \sum_{i=1}^{M_{ST}} \int_{((z_i + (T + \frac{1}{m}))Q) \cap \{x \in \mathbb{R}^N : |x_N| < \frac{1}{m}\}} \left( 1 + |\rho_T * u_0|^q \right. \\
+ \left. \|\nabla (\rho_T * u_0)\|_\infty^2 \right) \, dx \\
+ \frac{C}{SN^{-1}} \sum_{i=1}^{M_{ST}} \int_{((z_i + (T + \frac{1}{m}))Q) \cap \{x \in \mathbb{R}^N : |x_N| < \frac{1}{m}\}} \left( 1 + |\rho_T * u_0|^q + |\rho_m * u_0|^q \right. \\
+ \left. \|\nabla \varphi_{m, i}\|_\infty^2 + \|\nabla (\rho_m * u_0)\|_\infty^2 \right) \, dx \\
\leq \frac{C}{SN^{-1}} M_{ST} \left( \left( T + \frac{1}{m}\right)^{N-1} - T^{N-1} \right) \frac{1 + T^2}{T} \\
+ \frac{C}{SN^{-1}} M_{ST} \left( \left( T + \frac{1}{m}\right)^{N-1} - T^{N-1} \right) \frac{1 + m^2}{m} \\
\leq C \left( \frac{1}{m} \cdot \frac{T^{N-2} + T^N}{T^N} + \frac{1 + m^2}{m^2} \cdot \frac{1}{T} \right).
\] (4.5)

Using again (A3), and in view of (4.2), we have

\[
I_3(S, T, m) \leq \frac{C}{SN^{-1}} \int_{E_T \times (-\frac{1}{m}, \frac{1}{m})} \left( 1 + |(\rho_m * u_0)(x)|^q + \|\nabla (\rho_m * u_0)(x)\|^2 \right) \, dx \\
\leq \frac{C}{SN^{-1}} \frac{1 + m^2}{m} \left( \left( S - \frac{1}{T}\right)^{N-1} - T^{N-1} \left( \left( \frac{S - 1}{T} \right)^{N-1} - 1 \right) \right) \\
= C \left( \frac{1 + m^2}{m} \right) \left( 1 - \frac{1}{ST} \right)^{N-1} - \left( \frac{T}{|T| + 2} \right)^{N-1} \left( 1 - \frac{1}{ST} \right)^{N-1} + \left( \frac{T}{S} \right)^{N-1} \right).
\] (4.6)

Finally,

\[
I_4(S, T, m) \leq \frac{C}{SN^{-1}} \int_{(SQ \setminus (S - \frac{1}{m})Q) \cap \{x \in \mathbb{R}^N : |x_N| < \frac{1}{m}\}} \left( 1 + |\rho_S * u_0|^q + \|\nabla (\rho_S * u_0)\|_\infty^2 \right) \, dx
\]
\[ + \frac{C}{S_{N-1}} \int_{(S\mathbb{Q}(S-\frac{1}{T}) \cap \{x \in \mathbb{R}^N : |x_N| < \frac{1}{T}\})} \left(1 + |\rho_S * u_0|^q + |\rho_m * u_0|^q\right) \]
\[ + \|\nabla \varphi_{S,T}\|_\infty^2 + \|\nabla (\rho_m * u_0)\|_\infty^2 \right) dx \]
\[ \leq C(1 + S^2) \left(S_{N-1} - \left(S - \frac{1}{T}\right)^{N-1}\right) \frac{1}{S} \]
\[ + C(1 + T^2 + m^2) \left(S_{N-1} - \left(S - \frac{1}{T}\right)^{N-1}\right) \frac{1}{m} \]
\[ \leq C \frac{S^{N-2} + S^N}{S^N} \cdot \frac{1}{T} + C \frac{(1 + T^2 + m^2)}{Tm} \cdot \frac{1}{S}. \] (4.7)

Taking into account (4.4), (4.5), (4.6), and (4.7), we obtain
\[ \limsup_{m \to \infty} \liminf_{T \to \infty} \limsup_{S \to \infty} (I_1(S, T) + I_2(S, T, m) + I_3(S, T, m) + I_4(S, T, m)) \leq \liminf_{T \to \infty} g(T). \]

Thus, in view of (4.3), we deduce that
\[ \limsup_{S \to \infty} g(S) \leq \liminf_{T \to \infty} g(T). \]

\[ 4.1 \text{ Compactness} \]

In this section we prove the following compactness result

\[ \textbf{Theorem 4.2} \text{ Let } \varepsilon_n \to 0^+, \text{ and } \{u_n\} \subset H^1(\Omega; \mathbb{R}^d) \text{ be such that} \]
\[ \sup_{n \in \mathbb{N}} \int_\Omega \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n |\nabla u_n(x)|^2 \right) dx < +\infty. \] (4.8)

Then there exists \( u \in L^1(\Omega; \mathbb{R}^d) \), with \( u(x) \in \{a, b\} \mathcal{L}^N \text{-a.e } x \in \Omega \) such that, up to a subsequence, \( u_n \rightharpoonup u \) strongly in \( L^1(\Omega; \mathbb{R}^d) \).

\[ \textbf{Proof.} \text{ First, note that} \]
\[ \lim_{n \to \infty} \int_\Omega W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) dx = 0. \] (4.9)

By the coercivity condition in (A3), there exists a constant \( R > 0 \) such that
\[ W(y, u) \geq C|u| \text{ for } \mathcal{L}^N \text{-a.e. } y \in \mathbb{R}^N, \text{ } |u| > R. \]

Define \( w_n(x) := u_n(x) \chi_{\{x \in \Omega : |u_n(x)| > R\}}(x) \), and set \( v_n(x) := u_n(x) - w_n(x), \ x \in \Omega \). Thus,
\[ \int_\Omega |w_n(x)| dx = \int_{\{x \in \Omega : |u_n(x)| > R\}} |u_n(x)| dx \leq \frac{1}{C} \int_\Omega W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) dx, \]
which gives, in view of (4.9),
\[ w_n \to 0 \text{ strongly in } L^1(\Omega; \mathbb{R}^d). \] (4.10)

Taking into account (4.9) one more time, we also have that
\[
\int_{\Omega} W\left(\frac{x}{\varepsilon_n}, v_n(x)\right) \, dx = \int_{\{x \in \Omega : |u_n(x)| \leq R\}} W\left(\frac{x}{\varepsilon_n}, v_n(x)\right) \, dx + \int_{\{x \in \Omega : |u_n(x)| > R\}} W\left(\frac{x}{\varepsilon_n}, 0\right) \, dx
\]
\[
\leq \left(1 + \frac{C}{R}\right) \int_{\Omega} W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) \, dx \to 0 \text{ as } n \to \infty. \] (4.11)

Set \( \overline{W}(u) := \min_{y \in Q} W(y, u) \), and note that \( \overline{W} : \mathbb{R}^d \to [0, \infty) \) is continuous on \( \mathbb{R}^d \), \( \overline{W}(u) = 0 \) if and only if \( u \in \{a, b\} \), and that by the coercivity condition in (A3) there exists \( C > 0 \) such that
\[ W(u) \geq C |u|, \text{ for } |u| \text{ sufficiently large}. \]

We have
\[ 0 \leq \int_{\Omega} \overline{W}(v_n(x)) \, dx \leq \int_{\Omega} W\left(\frac{x}{\varepsilon_n}, v_n(x)\right) \, dx. \]

Thus, by (4.11),
\[
\lim_{n \to \infty} \int_{\Omega} \overline{W}(v_n(x)) \, dx = 0. \] (4.12)

In what follows, we proceed as in the proof of Theorem 4.1 in Fonseca and Tartar [27]. Since \( \{v_n\} \) is bounded in \( L^\infty(\Omega; \mathbb{R}^d) \) (note that \( |v_n(x)| \leq R \) for \( \mathcal{L}^N\)-a.e \( x \in \Omega \)), up to a subsequence (not relabelled), \( \{v_n\} \) generates a Young measure \( \{\mu_x\}_{x \in \Omega} \). Since \( \overline{W}(u) = 0 \) if and only if \( u \in \{a, b\} \), and in view of (4.12), we deduce that there exists \( 0 \leq \theta(x) \leq 1 \) for \( \mathcal{L}^N\)-a.e \( x \in \Omega \), such that
\[
\mu_x = \theta(x)\delta_a + (1 - \theta(x))\delta_b \text{ for } \mathcal{L}^N\text{-a.e. } x \in \Omega. \] (4.13)

Using the uniform bound on the energy (4.8) and the Cauchy-Schwarz inequality, we obtain that,
\[
\sup_{n \in \mathbb{N}} \int_{\Omega} \sqrt{W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) |\nabla u_n(x)|} \, dx \leq C.
\]

In view of Lemma 3.7 in [27], for suitable \( M > 0 \) the function
\[ \Phi(u) := \inf \left\{ \int_{-1}^{1} \sqrt{\min\{\overline{W}(\gamma(s)), M\}} |\gamma'(s)| \, ds : \gamma \text{ is piecewise } C^1, \gamma(-1) = a, \gamma(1) = u \right\} \]
is Lipschitz continuous and \( |\nabla (\Phi \circ v)(x)| \leq \sqrt{\overline{W}(v(x))} |\nabla v(x)| \) for any \( v \in H^1(\Omega; \mathbb{R}^d) \), and \( \mathcal{L}^N\text{-a.e. } x \in \Omega \). Thus
\[
\sup_{n \in \mathbb{N}} \int_{\Omega} |\nabla (\Phi \circ u_n)(x)| \, dx < +\infty. \] (4.14)
In addition,
\[
\int_{\Omega} |\Phi(u_n(x))| \, dx = \int_{\{ x \in \Omega : |u_n(x)| > R \}} \Phi(w_n(x)) \, dx + \int_{\{ x \in \Omega : |u_n(x)| \leq R \}} \Phi(v_n(x)) \, dx
\]
\[
\leq \int_{\Omega} |\Phi(w_n(x))| \, dx + \| \Phi \circ v_n \|_{L^\infty(\Omega)} L^N(\Omega)
\]
\[
\leq C.
\]

Thus, the sequence \( \{ \Phi \circ u_n \} \) belongs to the set
\[
\left\{ \xi \in L^1(\Omega) : \int_{\Omega} |\xi(x)| \, dx + \int_{\Omega} |\nabla \xi(x)| \, dx \leq C < +\infty \right\},
\]
which is compact in \( L^1(\Omega) \) with respect to the strong topology. Hence, we may extract a subsequence (not relabelled) such that
\[
\Phi \circ u_n \rightarrow f \in L^1(\Omega) \text{ strongly in } L^1(\Omega).
\]

By (4.10), and since \( \Phi \) is Lipschitz, we obtain that
\[
\Phi \circ v_n \rightarrow f \text{ strongly in } L^1(\Omega).
\]

Therefore, the Young measure generated by \( \{ \Phi \circ v_n \} \) is \( \delta_{f(x)} \) for \( x \in \Omega \). On the other hand, since \( \{ v_n \} \) generates \( \{ \mu_x \} \), we deduce that
\[
\theta(x) \delta_{\phi(a)} + (1 - \theta(x)) \delta_{\phi(b)} = \delta_{f(x)} \text{ for } \mathcal{L}^N - \text{a.e. } x \in \Omega.
\]

It follows that \( \theta(x) \in \{ 0, 1 \} \) for \( \mathcal{L}^N \)-a.e. \( x \in \Omega \). Set \( A := \{ x \in \Omega : \theta(x) = 1 \} \), and define \( u(x) := \chi_A(x) a + (1 - \chi_A(x)) b \) so that, by (4.13), \( \mu_x = \delta_{u(x)} \) for \( \mathcal{L}^N \)-a.e. \( x \in \Omega \).

We have constructed a function \( u \in L^1(\Omega; \mathbb{R}^d) \), with \( u(x) \in \{ a, b \} \) \( \mathcal{L}^N \)-a.e \( x \in \Omega \) such that the Young measure generated by \( \{ v_n \} \) is \( \{ \delta_{u(x)} \} \). Thus, in view of Proposition 2.14, \( v_n \rightarrow u \) strongly in \( L^1(\Omega; \mathbb{R}^d) \) (in fact, in any \( L^p(\Omega; \mathbb{R}^d) \) with \( 1 \leq p < +\infty \)). Taking into account (4.10), and since \( u_n = v_n + w_n \), we deduce that \( u_n \rightarrow u \) strongly in \( L^1(\Omega; \mathbb{R}^d) \).

\[\Box\]

\[\textbf{4.2 The } \Gamma\text{-liminf inequality}\]

In this section we prove part (i) of Theorem 1.3. Precisely,

**Proposition 4.3** Let (A1)-(A3) hold, and let \( u \in L^1(\Omega; \mathbb{R}^d) \) be given. If \( \varepsilon_n \rightarrow 0^+ \) and if \( \{ u_n \} \subset H^1(\Omega; \mathbb{R}^d) \) is such that \( u_n \rightarrow u \) in \( L^1(\Omega; \mathbb{R}^d) \), then
\[
\liminf_{n \rightarrow \infty} J_{\varepsilon_n}(u_n) \geq J_0(u).
\]

**Proof.** Step 1. If \( u \in L^1(\Omega; \mathbb{R}^d) \) and \( \mathcal{L}^N(\{ x \in \Omega : u(x) \notin \{ a, b \} \}) > 0 \) then for any sequence \( \varepsilon_n \rightarrow 0^+ \) and for any \( \{ u_n \} \subset H^1(\Omega; \mathbb{R}^d) \) such that \( u_n \rightarrow u \) in \( L^1(\Omega; \mathbb{R}^d) \), we have
\[
\int_{\Omega} \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n |\nabla u_n(x)|^2 \right) \, dx \rightarrow +\infty. \tag{4.15}
\]
Indeed, if for some sequences \( \varepsilon_n \to 0^+ \), and \( u_n \to u \) in \( L^1(\Omega; \mathbb{R}^d) \)

\[
\sup_{n \in \mathbb{N}} \int_{\Omega} \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n |\nabla u_n(x)|^2 \right) \, dx < +\infty,
\]

then

\[
\lim_{n \to \infty} \int_{\Omega} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) \, dx = 0. \tag{4.16}
\]

For \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \), consider the \( Q \)-periodic function \( v(x) = \{x\} = \{(x_1), \ldots, \{x_N\}\} \), where, for each \( i \in \{1, 2, \ldots, N\} \), \( \{x_i\} = x_i - [x_i] \) (\([y]\) stands for the integer part of the real number \( y \)), and define \( v_n(x) := v \left( \frac{x}{\varepsilon_n} \right) \). Up to a subsequence (not relabelled), \( \{v_n\} \) and \( \{u_n\} \) generate Young measures \( \{\nu_x\}_{x \in \Omega} \) and \( \{\mu_x\}_{x \in \Omega} \) respectively, where \( \{\nu_x\}_{x \in \Omega} \) is homogeneous

\[
\langle \nu_x, \varphi \rangle = \langle \nu, \varphi \rangle := \int_{Q} \varphi(y) dy \text{ for } \mathcal{L}^N - \text{a.e. } x \in \Omega,
\]

and, in view of the strong convergence of \( u_n \) to \( u \) in \( L^1(\Omega; \mathbb{R}^d) \),

\[
\mu_x = \delta_{u(x)} \text{ for } \mathcal{L}^N - \text{a.e. } x \in \Omega.
\]

Thus, by Proposition 2.15, the sequence \( (v_n, u_n) : \Omega \to \mathbb{R}^N \times \mathbb{R}^d \) generates the Young measure \( \{\nu \otimes \delta_{u(x)}\}_{x \in \Omega} \). By the Fundamental Theorem on Young measures (Theorem 2.13 (iv)), and using the periodicity of \( W \) in its first variable, we have that

\[
\lim_{n \to \infty} \int_{\Omega} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) \, dx = \liminf_{n \to \infty} \int_{\Omega} W (v_n(x), u_n(x)) \, dx
\]

\[
\geq \int_{\Omega} \int_{\mathbb{R}^N \times \mathbb{R}^d} W(A, B) d(\nu \otimes \delta_{u(x)})(A, B) \, dx = \int_{\Omega} \int_{Q} W(y, u(x)) dy \, dx.
\]

Thus, in view of (4.16),

\[
\int_{\Omega} \int_{Q} W(y, u(x)) dy \, dx \leq 0.
\]

The fact that \( W \) is nonnegative, together with (A2), implies that \( u(x) \in [a, b] \) for \( \mathcal{L}^N \)-a.e. \( x \in \Omega \), and we have reached a contradiction.

Step 2. Let \( u(x) = \chi_{A_0}(x)a + (1-\chi_{A_0}(x))b \) and assume that \( u \not\in BV(\Omega; \mathbb{R}^d) \), that is, \( \text{Per}_\Omega(A_0) = +\infty \). We will show once again that (4.15) is satisfied. We argue by contradiction. Suppose that there exists a subsequence (not relabelled) such that \( u_n \to u \) in \( L^1(\Omega; \mathbb{R}^d) \), and

\[
\sup_{n \in \mathbb{N}} \int_{\Omega} \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n |\nabla u_n(x)|^2 \right) \, dx < +\infty.
\]

Then, by Theorem 4.2, using the same notation as in the proof of that result, and in view of (4.14), we have that \( |D(\Phi \circ u)(\Omega)| < +\infty \). Since \( \Phi \circ u = \chi_{A_0} \Phi(a) + (1-\chi_{A_0}) \Phi(b) \), we obtain that \( \text{Per}_\Omega(A_0) < +\infty \), which contradicts our initial assumption on \( u \).
Step 3. It remains to prove the proposition in the case where \( u(x) = \chi_{A_0}(x)a + (1 - \chi_{A_0}(x))b \), with \( \text{Per}_\Omega(A_0) < +\infty \). Here, \( I_0(u) = \int_{\partial^* A_0 \cap \Omega} K_1(\nu(x))d\mathcal{H}^{N-1}(x) \), and it suffices to show that for any sequence \( \varepsilon_n \to 0^+ \) and for any \( \{u_n\} \subset H_1(\Omega; \mathbb{R}^d) \) such that \( u_n \to u \) in \( L^1(\Omega; \mathbb{R}^d) \), we have

\[
\int_{\partial^* A_0 \cap \Omega} K_1(\nu(x))d\mathcal{H}^{N-1}(x) \leq \liminf_{n \to \infty} \int_{\Omega} \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n|\nabla u_n(x)|^2 \right) dx. \tag{4.17}
\]

Upon extracting a subsequence (not relabelled) we may assume, without loss of generality, that

\[
\liminf_{n \to \infty} \int_{\Omega} \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n|\nabla u_n(x)|^2 \right) dx = \lim_{n \to \infty} \int_{\Omega} \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n|\nabla u_n(x)|^2 \right) dx < +\infty,
\]

and that there exists a finite Radon measure \( \mu \geq 0 \), such that

\[
\frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(\cdot) \right) + \varepsilon_n|\nabla u_n(\cdot)|^2 \rightharpoonup \mu, \tag{4.18}
\]

weakly* in the sense of measures. We claim that

\[
\frac{d\mu}{d\mathcal{H}^{N-1}[\Omega \cap \partial^* A_0]}(x) \geq K_1(\nu(x)) \quad \text{for} \quad H^{N-1} \quad \text{a.e.} \quad x \in \Omega \cap \partial^* A_0. \tag{4.19}
\]

Let \( \delta_k \to 0^+ \) be such that for \( H^{N-1} \) a.e. \( x_0 \in \Omega \cap \partial^* A_0 \) we have \( \mu(\partial Q_{\nu(x_0)}(x_0, \delta_k)) = 0 \) for all \( k \in \mathbb{N} \), and

\[
\frac{d\mu}{d\mathcal{H}^{N-1}[\Omega \cap \partial^* A_0]}(x_0) = \lim_{k \to \infty} \frac{\mu(Q_{\nu(x_0)}(x_0, \delta_k))}{\delta_k^{N-1}},
\]

where we have taken into account (2.1). Thus, in view of (4.18), we have that

\[
\frac{d\mu}{d\mathcal{H}^{N-1}[\Omega \cap \partial^* A_0]}(x_0) = \lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{\delta_k^{N-1}} \int_{Q_{\nu(x_0)}(x_0, \delta_k)} \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n|\nabla u_n(x)|^2 \right) dx.
\]

Let \( u_{k,n}(x) := u_n(x_0 + \delta_k x), \ x \in Q_{\nu(x_0)} \). Changing variables, we deduce that

\[
\frac{d\mu}{d\mathcal{H}^{N-1}[\Omega \cap \partial^* A_0]}(x_0) = \lim_{k \to \infty} \lim_{n \to \infty} \int_{Q_{\nu(x_0)}} \left( \frac{\delta_k}{\varepsilon_n} W \left( \frac{x_0 + \delta_k y}{\varepsilon_n}, u_n(x_0 + \delta_k y) \right) + \varepsilon_n \delta_k |\nabla u_n(x_0 + \delta_k y)|^2 \right) dy
\]

\[
= \lim_{k \to \infty} \lim_{n \to \infty} \int_{Q_{\nu(x_0)}} \left( \frac{\delta_k}{\varepsilon_n} W \left( \frac{x_0 + \delta_k y}{\varepsilon_n}, u_{k,n}(y) \right) + \varepsilon_n \delta_k |\nabla u_{k,n}(y)|^2 \right) dy. \tag{4.20}
\]
Let $m_n \in \mathbb{Z}^N$ and $s_n \in [0, 1)^N$ be such that $\frac{x_k}{\sqrt{n}} = m_n + s_n$. Put $x_{k,n} := -\frac{x_k}{\sqrt{n}} s_n$, and note that we have $\lim_{k \to \infty} \lim_{n \to \infty} x_{k,n} = 0$. Changing variables, and using the periodicity of $W(\cdot, u)$, we obtain that

$$
\lim_{k \to \infty} \lim_{n \to \infty} \int_{Q(x_0)} \left( \frac{\delta_k}{\varepsilon_n} W \left( \frac{x_0 + \delta_k y}{\varepsilon_n}, u_{k,n}(y) \right) + \frac{\varepsilon_n}{\delta_k} |\nabla u_{k,n}(y)|^2 \right) dy
$$

$$
= \lim_{k \to \infty} \lim_{n \to \infty} \int_{-x_k \cdot \nu + Q(x_0)} \left( \frac{\delta_k}{\varepsilon_n} W \left( \frac{x_0 + \delta_k (z + x_k,n)}{\varepsilon_n}, u_{k,n}(z + x_k,n) \right) + \frac{\varepsilon_n}{\delta_k} |\nabla u_{k,n}(z + x_k,n)|^2 \right) dz
$$

$$
= \lim_{k \to \infty} \lim_{n \to \infty} \int_{-x_k \cdot \nu + Q(x_0)} \left( \frac{\delta_k}{\varepsilon_n} W \left( \frac{\delta_k z}{\varepsilon_n}, u_{k,n}(z + x_k,n) \right) + \frac{\varepsilon_n}{\delta_k} |\nabla u_{k,n}(z + x_k,n)|^2 \right) dz. \quad (4.21)
$$

Recall that

$$
u_0(x) = \begin{cases} b & \text{if } x \cdot \nu(x_0) > 0, \\ a & \text{if } x \cdot \nu(x_0) < 0. \end{cases}
$$

We claim that

$$
\lim_{k \to \infty} \lim_{n \to \infty} \|u_{k,n} - \nu_0\|_{L^1(Q(x_0); \mathbb{R}^N)} = 0. \quad (4.22)
$$

Indeed, after changing variables, and making use of Lebesgue’s Dominated Convergence Theorem,

$$
\lim_{k \to \infty} \lim_{n \to \infty} \int_{Q(x_0)} |u_{k,n}(x) - \nu_0(x)|dx
$$

$$
= \lim_{k \to \infty} \lim_{n \to \infty} \left\{ \int_{Q(x_0)} |u_n(x_0 + \delta_k x) - b|dx + \int_{-x_k \cdot \nu + Q(x_0)} |u_n(x_0 + \delta_k x - a|dx \right\}
$$

$$
= \lim_{k \to \infty} \left\{ \int_{Q(x_0)} |u(x_0 + \delta_k x) - b|dx + \int_{Q(x_0)} |u(x_0 + \delta_k x) - a|dx \right\}
$$

$$
= \lim_{k \to \infty} \frac{1}{\delta_k^N} \left\{ \int_{Q(x_0, \delta_k) \cap \{x \in \mathbb{R}^N : x \cdot \nu(x_0) > x_0 \cdot \nu(x_0) \}} |u(x) - b|dx + \int_{Q(x_0, \delta_k) \cap \{x \in \mathbb{R}^N : x \cdot \nu(x_0) < x_0 \cdot \nu(x_0) \}} |u(x) - a|dx \right\}
$$

$$
= |b - a| \lim_{k \to \infty} \frac{\mathcal{L}^N(\{x \in Q(x_0) \cap A_0 : x \cdot \nu(x_0) > x_0 \cdot \nu(x_0) \})}{\delta_k^N} + \frac{\mathcal{L}^N(\{x \in Q(x_0) \cap \mathbb{R}^N : x \cdot \nu(x_0) < x_0 \cdot \nu(x_0) \})}{\delta_k^N} = 0.
$$
where the last equality follows by Theorem 2.2.

A diagonalization process allows us to find an increasing sequence \( \{n_k\} \) \( \not\to \infty \) such that, denoting \( \eta_k := x_{n_k} = x_k, ~ w_k(z) := u(x_k)(z + x_k), \) we have

\[
\lim_{k \to \infty} \eta_k = \lim_{n \to \infty} \lim_{k \to \infty} \frac{\varepsilon_n}{\delta_k} = 0, \\
\lim_{k \to \infty} x_k = \lim_{n \to \infty} \lim_{k \to \infty} x_{k,n} = 0,
\]

in view of (4.22),

\[
\lim_{k \to \infty} \|w_k - u_0\|_{L^1(Q(\varepsilon_n, \varepsilon_k))} = \lim_{k \to \infty} \lim_{n \to \infty} \|u_{k,n} - u_0\|_{L^1(Q(\varepsilon_n, \varepsilon_k))} = 0,
\]

and, in addition,

\[
\lim_{k \to \infty} \int_{-x_k+Q(\varepsilon_k)} \left( \frac{1}{\eta_k} W \left( \frac{z}{\eta_k}, w_k(z) \right) + \eta_k|\nabla w_k(z)|^2 \right) dz = \lim_{k \to \infty} \lim_{n \to \infty} \int_{-x_{k,n}+Q(\varepsilon_k)} \left( \frac{\delta_k}{\varepsilon_n} W \left( \frac{\delta_k z}{\varepsilon_n}, u_{k,n}(z + x_{k,n}) \right) + \frac{\varepsilon_n}{\delta_k} |\nabla u_{k,n}(z + x_{k,n})|^2 \right) dz.
\]

By (4.20) and (4.21), we obtain

\[
\frac{d\mu}{d\mathcal{H}^{N-1}[(\Omega \cap \partial^* A_0)](x_0)} = \lim_{k \to \infty} \lim_{n \to \infty} \int_{-x_k+Q(\varepsilon_k)} \left( \frac{1}{\eta_k} W \left( \frac{z}{\eta_k}, w_k(z) \right) + \eta_k|\nabla w_k(z)|^2 \right) dz.
\]

Since \( x_k \to 0 \) as \( k \to \infty \), for \( k \) sufficiently large there exists a cube \( Q_k \subset Q(\varepsilon_n) \), such that \( Q_k \subset (-x_k + Q(\varepsilon_k)) \), and \( \lim_{k \to \infty} \mathcal{L}^N(Q(\varepsilon_k) \setminus Q_k) = 0 \). In view of (4.23), we deduce that

\[
\frac{d\mu}{d\mathcal{H}^{N-1}[(\Omega \cap \partial^* A_0)](x_0)} \geq \limsup_{k \to \infty} \int_{Q_k} \left( \frac{1}{\eta_k} W \left( \frac{z}{\eta_k}, w_k(z) \right) + \eta_k|\nabla w_k(z)|^2 \right) dz.
\]

Let \( L_{k,j} := \{ x \in Q_k : \text{dist}(x, \partial Q_k) < 1/j \} \). Divide \( L_{k,j} \) into \( M_{k,j} \) equidistant layers \( L_{k,j}^{(i)} \) (\( i = 1, \ldots, M_{k,j} \)) of width \( \eta_k \|w_k - \rho_{\frac{1}{1/j}\varepsilon(x_0)} w_0\|_{L^2(Q(\varepsilon_n, \varepsilon_k))} \), so that

\[
M_{k,j} \eta_k \|w_k - \rho_{\frac{1}{1/j}\varepsilon(x_0)} w_0\|_{L^2(Q(\varepsilon_n, \varepsilon_k))} = O(1/j)
\]

Select now one of these layers \( L_{k,j}^{(i)} \) such that

\[
\int_{L_{k,j}^{(i)}} \left( 1 + |w_k|^q + |\rho_{\frac{1}{1/j}\varepsilon(x_0)} w_0|^q + \eta_k^2 |\nabla w_k|^2 + \eta_k^2 |\nabla (\rho_{\frac{1}{1/j}\varepsilon(x_0)} w_0)|^2 \right) dx \leq \frac{1}{M_{k,j}} \int_{L_{k,j}} \left( 1 + |w_k|^q + |\rho_{\frac{1}{1/j}\varepsilon(x_0)} w_0|^q + \eta_k^2 |\nabla w_k|^2 + \eta_k^2 |\nabla (\rho_{\frac{1}{1/j}\varepsilon(x_0)} w_0)|^2 \right) dx.
\]
Consider cut-off functions $\varphi_{k,j} \in C^\infty_c(Q_{v(x_0)}; [0,1])$ such that

$$\varphi_{k,j}(x) = 0 \text{ if } x \in \left( \bigcup_{i=10}^{M_{k,j}} L_{k,j}^{(i)} \right) \cup (Q_{v(x_0)} \setminus Q_k),$$

$$\varphi_{k,j}(x) = 1 \text{ if } x \in \left( \bigcup_{i=1}^{10-1} L_{k,j}^{(i)} \right) \cup (Q_k \setminus L_{k,j}),$$

and

$$\|\nabla \varphi_{k,j}\|_\infty = O \left( \frac{1}{\eta_k \|w_k - \rho_{\frac{1}{\eta_k}} v(x_0) * u_0\|_{L^2(Q_{v(x_0)}; \mathbb{R}^d)}} \right).$$

Define

$$w_{k,j} := \varphi_{k,j} w_k + (1 - \varphi_{k,j})(\rho_{\frac{1}{\eta_k}} v(x_0) * u_0).$$

We have

$$\int_{Q_k} \left( \frac{1}{\eta_k} W \left( \frac{z}{\eta_k}, w_{k,j}(z) \right) + \eta_k |\nabla w_{k,j}(z)|^2 \right) dz$$

$$= \int_{\bigcup_{i=1}^{\eta_k^{-1}} L_{k,j}^{(i)} \cup (Q_k \setminus L_{k,j})} \left( \frac{1}{\eta_k} W \left( \frac{z}{\eta_k}, w_k(z) \right) + \eta_k |\nabla w_k(z)|^2 \right) dz$$

$$+ \int_{L_{k,j}^{(\eta_k)}} \left( \frac{1}{\eta_k} W \left( \frac{z}{\eta_k}, w_{k,j}(z) \right) + \eta_k |\nabla w_{k,j}(z)|^2 \right) dz$$

$$+ \int_{\bigcup_{i=1}^{M_{k,j}} L_{k,j}^{(i)} \setminus (Q_k \setminus L_{k,j})} \left( \frac{1}{\eta_k} W \left( \frac{z}{\eta_k}, (\rho_{\frac{1}{\eta_k}} v(x_0) * u_0)(z) \right) + \eta_k |\nabla (\rho_{\frac{1}{\eta_k}} v(x_0) * u_0)(z)|^2 \right) dz$$

$$=: A_{k,j} + B_{k,j} + C_{k,j}. \tag{4.27}$$

Taking into account the growth condition in (A3), we have

$$B_{k,j} \leq C \int_{L_{k,j}^{(\eta_k)}} \left( \frac{1}{\eta_k} \left( 1 + |w_k|^q + |\rho_{\frac{1}{\eta_k}} v(x_0) * u_0|^q \right) + \eta_k \left( |\nabla w_k - \nabla (\rho_{\frac{1}{\eta_k}} v(x_0) * u_0)|^2 \right. \right.$$

$$\left. + |\nabla (\rho_{\frac{1}{\eta_k}} v(x_0) * u_0)|^2 + |\nabla \varphi_{k,j}|^2 \right) dz \left( |w_k - \rho_{\frac{1}{\eta_k}} v(x_0) * u_0|^2 + \|w_k - \rho_{\frac{1}{\eta_k}} v(x_0) * u_0\|_{L^2(Q_{v(x_0)}; \mathbb{R}^d)}^2 \right) dx$$

$$\leq \frac{C}{\eta_k} \int_{L_{k,j}^{(\eta_k)}} \left( 1 + |w_k|^q + |\rho_{\frac{1}{\eta_k}} v(x_0) * u_0|^q + \eta_k^2 |\nabla w_k|^2 + \eta_k^2 |\nabla (\rho_{\frac{1}{\eta_k}} v(x_0) * u_0)|^2 \right.$$
In view of (4.25) and (4.26) we obtain the estimate

\[ B_{k,j} \leq \frac{C}{\eta_k M_{k,j}} \int_{L_{k,j}} \left( 1 + |w_k|^q + |\rho_{\frac{1}{\eta_k} \cdot \nu(x_0)} \cdot u_0|^q + \eta_k^2 |\nabla w_k|^2 + \eta_k^2 |\nabla (\rho_{\frac{1}{\eta_k} \cdot \nu(x_0)} \cdot u_0)|^2 \right) \, dx \]

\[ = O(j) ||w_k - \rho_{\frac{1}{\eta_k} \cdot \nu(x_0)} \cdot u_0||_{L^2(Q_{\rho(x_0)})}^{1/2} \int_{L_{k,j}} \left( 1 + |w_k|^q + |\rho_{\frac{1}{\eta_k} \cdot \nu(x_0)} \cdot u_0|^q + \eta_k^2 |\nabla w_k|^2 + \eta_k^2 |\nabla (\rho_{\frac{1}{\eta_k} \cdot \nu(x_0)} \cdot u_0)|^2 \right) \, dx, \]

which gives

\[ \limsup_{j \to \infty} \limsup_{k \to \infty} B_{k,j} = 0. \quad (4.28) \]

Using again the growth condition in (A3), we have that

\[ \limsup_{j \to \infty} \limsup_{k \to \infty} C_{k,j} \]

\[ \leq \lim_{j \to \infty} \limsup_{k \to \infty} \int \frac{1}{\eta_k} \left( 1 + |(\rho_{\frac{1}{\eta_k} \cdot \nu(x_0)} \cdot u_0)(z)|^q + \eta_k^2 |\nabla (\rho_{\frac{1}{\eta_k} \cdot \nu(x_0)} \cdot u_0)(z)|^2 \right) \, dz \]

\[ \leq \limsup_{j \to \infty} \limsup_{k \to \infty} \frac{C}{\eta_k} \mathcal{L}^N \left( \bigcup_{i=0}^{M_{k,j}^{(i)}} L_{k,j}^{(i)} \right) \cap \{ x \in Q_{\rho(x_0)} : |x \cdot \nu(x_0)| < \eta_k \} = 0. \quad (4.29) \]

Similarly, and in view of our choice of the cubes \( Q_k \), we obtain that

\[ \limsup_{k \to \infty} \int_{Q_{\rho(x_0)} \setminus Q_k} \left( \frac{1}{\eta_k} W \left( \frac{z}{\eta_k}, w_{k,j}(z) \right) + \eta_k |\nabla w_{k,j}(z)|^2 \right) \, dz \]

\[ \leq \limsup_{k \to \infty} \frac{C}{\eta_k} \mathcal{L}^N \left( (Q_{\rho(x_0)} \setminus Q_k) \cap \{ x \in Q_{\rho(x_0)} : |x \cdot \nu(x_0)| < \eta_k \} \right) = 0, \]

and thus, taking into account (4.27), (4.28), and (4.29),

\[ \limsup_{j \to \infty} \limsup_{k \to \infty} \int_{Q_{\rho(x_0)}} \left( \frac{1}{\eta_k} W \left( \frac{z}{\eta_k}, w_{k,j}(z) \right) + \eta_k |\nabla w_{k,j}(z)|^2 \right) \, dz \]

\[ = \limsup_{j \to \infty} \limsup_{k \to \infty} \int_{Q_{\rho(x_0)} \setminus Q_k} \left( \frac{1}{\eta_k} W \left( \frac{z}{\eta_k}, w_{k,j}(z) \right) + \eta_k |\nabla w_{k,j}(z)|^2 \right) \, dz \]

\[ = \limsup_{j \to \infty} \limsup_{k \to \infty} \int_{Q_k} \left( \frac{1}{\eta_k} W \left( \frac{z}{\eta_k}, w_k(z) \right) + \eta_k |\nabla w_k(z)|^2 \right) \, dz. \]
In view of (4.24), we obtain that
\[
\frac{d\mu}{d\mathcal{H}^{N-1}((\Omega \cap \partial^* A_0))}(x_0) \geq \limsup_{j \to \infty} \limsup_{k \to \infty} \int_{Q_v(x_0)} \left( \frac{1}{\eta_k} W \left( \frac{z}{\eta_k}, w_{k,j}(z) \right) + \eta_k |\nabla w_{k,j}(z)|^2 \right) dz. \quad (4.30)
\]
A diagonalization procedure (see Lemma 2.5) allows us to construct an increasing subsequence \( \{k(j)\} \setminus \infty \) such that
\[
\limsup_{j \to \infty} \limsup_{k \to \infty} \int_{Q_v(x_0)} \left( \frac{1}{\eta_k} W \left( \frac{z}{\eta_k}, w_{k,j}(z) \right) + \eta_k |\nabla w_{k,j}(z)|^2 \right) dz
\]
\[
= \lim_{j \to \infty} \int_{Q_v(x_0)} \left( \frac{1}{\eta_{k(j)}} W \left( \frac{y}{\eta_{k(j)}}, w_{k(j),j}(y) \right) + \eta_{k(j)} |\nabla w_{k(j),j}(y)|^2 \right) dy
\]
\[
= \lim_{j \to \infty} \left( \frac{1}{\eta_{k(j)}} \int_{Q_v(x_0)} \left( \frac{1}{\eta_{k(j)}} W \left( z, w_{k(j),j}(\eta_{k(j)}z) \right) + \eta_{k(j)} |\nabla w_{k(j),j}(\eta_{k(j)}z)|^2 \right) dz \right), \quad (4.31)
\]
after making the change of variables \( y = \eta_{k(j)}z \).
Define \( v_j \in H^1 \left( \frac{1}{\eta_{k(j)}} Q_v(x_0); \mathbb{R}^d \right) \) by \( v_j(z) := w_{k(j),j}(\eta_{k(j)}z) \). Since \( w_{k(j),j} = \rho \frac{1}{\eta_{k(j)}} \nu(x_0) \ast u_0 \) on \( \partial Q_v(x_0) \), we have that
\[
v_j = \rho \frac{1}{\eta_{k(j)}} \nu(x_0) \ast u_0 \text{ on } \partial \left( \frac{1}{\eta_{k(j)}} Q_v(x_0) \right),
\]
and, in addition,
\[
\lim_{j \to \infty} \frac{N}{\eta_{k(j)}} \int_{\frac{1}{\eta_{k(j)}} Q_v(x_0)} \left( \frac{1}{\eta_{k(j)}} W \left( z, w_{k(j),j}(\eta_{k(j)}z) \right) + \eta_{k(j)} |\nabla w_{k(j),j}(\eta_{k(j)}z)|^2 \right) dz
\]
\[
= \lim_{j \to \infty} \frac{N-1}{\eta_{k(j)}} \int_{\frac{1}{\eta_{k(j)}} Q_v(x_0)} \left( W(z, v_j(z)) + |\nabla v_j(z)|^2 \right) dz
\]
\[
\geq \lim_{j \to \infty} \left( \frac{N-1}{\eta_{k(j)}} \inf \left\{ \int_{\frac{1}{\eta_{k(j)}} Q_v(x_0)} \left( W(z, u(z)) + |\nabla u(z)|^2 \right) dz : u \in H^1 \left( \frac{1}{\eta_{k(j)}} Q_v(x_0); \mathbb{R}^d \right), \right. \right.
\]
\[
\left. \left. \quad u = \rho \frac{1}{\eta_{k(j)}} \nu(x_0) \ast u_0 \text{ on } \partial \left( \frac{1}{\eta_{k(j)}} Q_v(x_0) \right) \right\} \right), \quad (4.32)
\]
where we have used the fact that \( v_j \) is admissible for the infimum in the definition of \( K_1(\nu(x_0)) \).
Combining (4.30), (4.31), and (4.32), we deduce that
\[
\frac{d\mu}{d\mathcal{H}^{N-1}((\Omega \cap \partial^* A_0))}(x_0) \geq \liminf_{T \to \infty} \frac{1}{T^{N-1}} \inf_{TQ_v(x_0)} \left\{ \int_{TQ_v(x_0)} \left( W(y, u(y)) + |\nabla u(y)|^2 \right) dy : u \in H^1(TQ_v(x_0); \mathbb{R}^d), u = \rho_{T,\nu(x_0)} \ast u_0 \text{ on } \partial(TQ_v(x_0)) \right\} = K_1(\nu(x_0)),
\]
Given any $\text{Lemma 4.5}$ allows us to modify competing sequences near the boundary without increasing the total energy. Let $\{\varphi_k\} \subset C_c(\Omega; [0,1])$, $\varphi_k \not\to 1$ as $k \to \infty$. In view of (4.18), we have

$$\lim_{n \to \infty} \int_\Omega \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n |\nabla u_n(x)|^2 \right) dx$$

$$\geq \lim_{n \to \infty} \int_\Omega \varphi_k(x) \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n |\nabla u_n(x)|^2 \right) dx$$

$$= \int_\Omega \varphi_k(x) d\mu = \int_\Omega \varphi_k(x) \frac{d\mu}{dH^{N-1}}(\{\Omega \cap \partial^* A_0\})(x) dH^{N-1}(x)$$

$$\geq \int_{\Omega \cap \partial^* A_0} K_1(\nu(x)) \varphi_k(x) dH^{N-1}(x).$$

Letting $k \to \infty$, and using the Monotone Convergence Theorem, we deduce that (4.17) holds, which concludes the proof. 

$\square$

### 4.3 The construction of a recovering sequence for the $\Gamma$-limit

In this section we prove part (ii) of Theorem 1.3. In view of Steps 1 and 2 in the proof of Proposition 4.3, it suffices to prove

**Proposition 4.4** Given any $u \in BV(\Omega; \{a,b\})$ and any sequence $\varepsilon_n \to 0^+$, there exists a sequence $\{u_n\} \subset H^1(\Omega; \mathbb{R}^d)$ such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^d)$ and

$$\lim_{n \to \infty} \int_\Omega \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n |\nabla u_n(x)|^2 \right) dx = \int_{\Omega \cap \partial^* A_0} K_1(\nu(x)) dH^{N-1}(x). \quad (4.33)$$

For the proof of Proposition 4.4, it will be enough to show that given any sequence $\varepsilon_n \to 0^+$, (4.33) holds for a subsequence $\{\varepsilon^R_n\}$ of $\{\varepsilon_n\}$. Indeed, recalling the main result of the previous section (Proposition 4.3) we then obtain that the $\Gamma(L^1)$-limit of $I_{\varepsilon \Gamma}$ is $I_0$, which is independent on the specific subsequence $\{\varepsilon^R_n\}$. In light of Proposition 7.11 in [12], we deduce that, in fact, $I_\varepsilon \Gamma(L^1)$-converges to $I_0$. The proof of Proposition 4.4 relies on the following result which will allow us to modify competing sequences near the boundary without increasing the total energy.

**Lemma 4.5** Assume that (A1)-(A3) hold, let $\nu$ be a unit vector and let

$$u_0(x) := \begin{cases} b & \text{if } x \cdot \nu > 0, \\ a & \text{if } x \cdot \nu < 0. \end{cases}$$

Let $\rho : \mathbb{R} \to [0, +\infty)$ be a mollifier and set $u_n := \rho_{\frac{1}{\varepsilon_n} \nu} * u_0$, where $\rho_{\frac{1}{\varepsilon_n} \nu}(x) := \left( \frac{1}{\varepsilon_n} \right)^N \rho \left( \frac{x}{\varepsilon_n} \right)$, and $\{\varepsilon_n\}$ is a sequence of real numbers such that $\varepsilon_n \to 0^+$. If $\{u_n\}$ is a sequence in $H^1(Q_\nu; \mathbb{R}^d)$ converging in $L^1(Q_\nu; \mathbb{R}^d)$ to $u_0$, then there exists a sequence $\{w_n\}$ in $H^1(Q_\nu; \mathbb{R}^d)$ such that $w_n \to u_0$ in $L^1(Q_\nu; \mathbb{R}^d)$, $w_n = v_n$ on $\partial Q_\nu$, and

$$\limsup_{n \to \infty} \int_{Q_\nu} \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n |\nabla u_n(x)|^2 \right) dx$$

$$\leq \liminf_{n \to \infty} \int_{Q_\nu} \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n |\nabla u_n(x)|^2 \right) dx.$$
Proof. Assume, without loss of generality, that
\[
\liminf_{n \to \infty} \int_{Q_{\nu}} \left( \frac{1}{\epsilon_n} W \left( \frac{x}{\epsilon_n}, u_n(x) \right) + \epsilon_n |\nabla u_n(x)|^2 \right) \, dx
\]
\[
\geq \liminf_{n \to \infty} \int_{Q_{\nu}} \left( \frac{1}{\epsilon_n} W \left( \frac{x}{\epsilon_n}, u_n(x) \right) + \epsilon_n |\nabla u_n(x)|^2 \right) \, dx < +\infty,
\]
and that \( u_n(x) \to u_0(x) \) \( \mathcal{L}^N \)-a.e. \( x \in Q_{\nu} \). Thus,
\[
\lim_{n \to \infty} \int_{Q_{\nu}} \left( W \left( \frac{x}{\epsilon_n}, u_n(x) \right) + \epsilon_n^2 |\nabla u_n(x)|^2 \right) \, dx = 0. \quad (4.34)
\]
By (A3) we have
\[
|u_n(x) - u_0(x)|^9 \leq C \left( W \left( \frac{x}{\epsilon_n}, u_n(x) \right) + 1 \right),
\]
and we deduce that
\[
C \mathcal{L}^N(Q_{\nu}) - \limsup_{n \to \infty} \|u_n - u_0\|_{L^9(Q_{\nu}; \mathbb{R}^d)}^2
\]
\[
= \liminf_{n \to \infty} \int_{Q_{\nu}} \left( CW \left( \frac{x}{\epsilon_n}, u_n(x) \right) + C - |u_n(x) - u_0(x)|^9 \right) \, dx
\]
\[
\geq \int_{Q_{\nu}} \liminf_{n \to \infty} \left( CW \left( \frac{x}{\epsilon_n}, u_n(x) \right) + C - |u_n(x) - u_0(x)|^9 \right) \, dx \geq C \mathcal{L}^N(Q_{\nu}),
\]
where we have used (4.34), and Fatou’s lemma. Therefore,
\[
\limsup_{n \to \infty} \int_{Q_{\nu}} |u_n - u_0|^9 \, dx = 0, \quad (4.35)
\]
and we conclude that \( u_n \to u_0 \) in \( L^9(Q_{\nu}; \mathbb{R}^d) \) as \( n \to \infty \).

For simplicity, assume in what follows that \( \nu = e_N \) and denote \( Q_{\nu} \) by \( Q \). Note that
\[
v_n(x) = \begin{cases} 
  b & \text{if } x_N > \epsilon_n, \\
  a & \text{if } x_N < -\epsilon_n,
\end{cases}
\]
and
\[
\|\nabla v_n\|_{\infty} = O(1/\epsilon_n), \quad \text{supp}\nabla v_n \subset \{ x \in Q : |x_N| < \epsilon_n \}, \quad \text{and } v_n \to u_0 \text{ in } L^9(Q; \mathbb{R}^d). \quad (4.36)
\]
For each \( k \in \mathbb{N} \) define
\[
L_k := \left\{ x \in Q : \text{dist}(x, \partial Q) \leq \frac{1}{k} \right\}.
\]
Consider \( n \) sufficiently large, and divide \( L_k \) into \( M_{k,n} \) layers \( L_{k,n}^{(i)} (i = 1, ..., M_{k,n}) \) of width \( \epsilon_n \|u_n - v_n\|_{L^2(Q; \mathbb{R}^d)}^{1/2} \), so that \( M_{k,n} \epsilon_n \|u_n - v_n\|_{L^2(Q; \mathbb{R}^d)} = O(1/k) \). Since
\[
\sum_{i=1}^{M_{k,n}} \int_{L_{k,n}^{(i)}} \left( 1 + |u_n|^9 + |v_n|^9 + \epsilon_n^2 |\nabla u_n|^2 + \frac{|u_n - v_n|^2}{\|u_n - v_n\|_{L^2(Q; \mathbb{R}^d)}} \right) \, dx
\]
\[
= \int_{L_k} \left( 1 + |u_n|^9 + |v_n|^9 + \epsilon_n^2 |\nabla u_n|^2 + \frac{|u_n - v_n|^2}{\|u_n - v_n\|_{L^2(Q; \mathbb{R}^d)}} \right) \, dx,
\]
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there exists \( i = i(k, n) \in \{1, ..., M_{k,n} \} \) such that
\[
\int_{L_{k,n}^{(i)}} \left( 1 + |u_n|^q + |v_n|^q + \epsilon_n^2 |\nabla u_n|^2 + \frac{|u_n - v_n|^2}{\|u_n - v_n\|_{L^2(Q;\mathbb{R}^d)}} \right) dx \\
\leq \frac{1}{M_{k,n}} \int_{L_k} \left( 1 + |u_n|^q + |v_n|^q + \epsilon_n^2 |\nabla u_n|^2 + \frac{|u_n - v_n|^2}{\|u_n - v_n\|_{L^2(Q;\mathbb{R}^d)}} \right) dx. \tag{4.37}
\]
Choose cut-off functions \( \varphi_{k,n} \in C_c^\infty(Q; [0,1]) \) such that \( \varphi_{k,n} = 0 \) on \( \bigcup_{j=i+1}^{M_{k,n}} L_{k,n}^{(j)} =: A_{k,n}, \varphi_{k,n} = 1 \) on \( (Q \setminus L_k) \cup \bigcup_{j=1}^{i-1} L_{k,n}^{(j)} =: B_{k,n} \), and define
\[
w_{k,n} := \varphi_{k,n} u_n + (1 - \varphi_{k,n}) v_n.
\]
We have
\[
\lim_{k \to \infty} \lim_{n \to \infty} \|w_{k,n} - u_0\|_{L^1(Q;\mathbb{R}^d)} = 0.
\]
Also
\[
\begin{align*}
\limsup_{k \to \infty} \limsup_{n \to \infty} & \int_Q \left( \frac{1}{\epsilon_n} W^i \left( \frac{x}{\epsilon_n}, w_{k,n}(x) \right) + \epsilon_n |\nabla w_{k,n}(x)|^2 \right) dx \\
\leq & \limsup_{k \to \infty} \limsup_{n \to \infty} \int_{A_{k,n}} \left( \frac{1}{\epsilon_n} W^i \left( \frac{x}{\epsilon_n}, v_n(x) \right) + \epsilon_n |\nabla v_n(x)|^2 \right) dx \\
& + \limsup_{k \to \infty} \limsup_{n \to \infty} \int_{L_{k,n}^{(i)}} \left( \frac{1}{\epsilon_n} W^i \left( \frac{x}{\epsilon_n}, w_{k,n}(x) \right) + \epsilon_n |\nabla w_{k,n}(x)|^2 \right) dx \\
& + \lim_{n \to \infty} \int_Q \left( \frac{1}{\epsilon_n} W^i \left( \frac{x}{\epsilon_n}, u_n(x) \right) + \epsilon_n |\nabla u_n(x)|^2 \right) dx. \tag{4.38}
\end{align*}
\]
By (A3) and (4.36) we have
\[
\begin{align*}
\limsup_{k \to \infty} & \limsup_{n \to \infty} \int_{A_{k,n}} \left( \frac{1}{\epsilon_n} W^i \left( \frac{x}{\epsilon_n}, v_n(x) \right) + \epsilon_n |\nabla v_n(x)|^2 \right) dx \\
& \leq \limsup_{k \to \infty} \limsup_{n \to \infty} \int_{L_k \cap \{x \in Q \mid x_n < \epsilon_n \}} \frac{C}{\epsilon_n} \left( 1 + |v_n|^q + \epsilon_n^2 |\nabla v_n|^2 \right) dx = 0,
\end{align*}
\]
and
\[
\begin{align*}
\limsup_{k \to \infty} & \limsup_{n \to \infty} \int_{L_{k,n}^{(i)}} \left( \frac{1}{\epsilon_n} W^i \left( \frac{x}{\epsilon_n}, w_{k,n}(x) \right) + \epsilon_n |\nabla w_{k,n}(x)|^2 \right) dx \\
& \leq \limsup_{k \to \infty} \limsup_{n \to \infty} \frac{C}{\epsilon_n M_{k,n}} \int_{L_k} \left( 1 + |u_n|^q + |v_n|^q + \epsilon_n^2 |\nabla u_n|^2 + \frac{|u_n - v_n|^2}{\|u_n - v_n\|_{L^2(Q;\mathbb{R}^d)}} \right) dx.
\end{align*}
\]
\[ \limsup_{k \to \infty} \limsup_{n \to \infty} C_k \|u_n - v_n\|_{L^2(Q; \mathbb{R}^d)}^{1/2} \left( \int_Q \left( 1 + |u_n|^q + |v_n|^q + \varepsilon_n^2 |\nabla u_n|^2 \right) \, dx + \|u_n - v_n\|_{L^2(Q; \mathbb{R}^d)} \right) = 0, \]

where we have used (4.35) and (4.37). Thus, (4.38) becomes

\[ \limsup_{k \to \infty} \limsup_{n \to \infty} \int_Q \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, w_{k,n}(x) \right) + \varepsilon_n |\nabla w_{k,n}(x)|^2 \right) \, dx \leq \limsup_{n \to \infty} \int_Q \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n |\nabla u_n(x)|^2 \right) \, dx. \]

Using a diagonalization process (see Lemma 2.5) we extract a subsequence \( \{k(n)\} \) of \( \{k\} \) such that, upon letting \( w_n := w_{k(n),n} \), we have \( w_n = v_n \) on \( \partial Q \),

\[ \lim_{n \to \infty} \|w_n - u_0\|_{L^1(\Omega; \mathbb{R}^d)} = 0, \]

and

\[ \limsup_{n \to \infty} \int_Q \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n |\nabla u_n(x)|^2 \right) \, dx \leq \liminf_{n \to \infty} \int_Q \left( \frac{1}{\varepsilon_n} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n |\nabla u_n(x)|^2 \right) \, dx. \]

**Proof of Proposition 4.4.** Let \( A(\Omega) \) be the family of all open subsets of \( \Omega \), and let \( C \) be the family of all open cubes in \( \Omega \) with faces parallel to the axes, centered at points \( x \in \Omega \cap Q^N \) and with rational edgelength. Denote by \( \mathcal{R} \) the countable subfamily of \( A(\Omega) \) obtained by taking all finite unions of elements of \( C \), i.e.,

\[ \mathcal{R} := \left\{ \bigcup_{i=1}^{k} C_i : k \in \mathbb{N}, \, C_i \in C \right\}. \]

Let \( \varepsilon_n \to 0^+ \). Since \( L^1(\Omega; \mathbb{R}^d) \) is a separable metric space, using Kuratowski’s Compactness Theorem (see, e.g., [17]), a diagonalization argument, and in the spirit of \( \Gamma \)-convergence (see Proposition 7.9 in [12]), we can assert the existence of a subsequence \( \{\varepsilon_n^R\} \) of \( \{\varepsilon_n\} \) such that, if

\[ W_{\{\delta_n\}}(u; A) := \inf \left\{ \liminf_{n \to \infty} \int_{\Omega} \left( \frac{1}{\delta_n} W \left( \frac{x}{\delta_n}, v_n(x) \right) + \delta_n |\nabla v_n|^2 \right) \, dx : v_n \to u \text{ in } L^1(A; \mathbb{R}^d), \, v_n \in H^1(A; \mathbb{R}^d) \right\}, \]

for \( A \in A(\Omega) \) and \( \delta_n \to 0^+ \), then for every \( u \in L^1(\Omega; \mathbb{R}^d) \) and \( C \in \mathcal{R} \), there exists a sequence \( \{u_{C_{\varepsilon_n}}^C\} \subset H^1(C; \mathbb{R}^d) \) such that

\[ u_{C_{\varepsilon_n}}^C \to u \text{ in } L^1(C; \mathbb{R}^d) . \]
and 
\[ W_{\{\varepsilon_n^\pm\}}(u; C) = \lim_{n \to \infty} \int_{\mathcal{C}} \left( \frac{1}{\varepsilon_n} W\left( \frac{x}{\varepsilon_n}, u_{\varepsilon_n}(x) \right) + \varepsilon_n \left| \nabla u_{\varepsilon_n}(x) \right|^2 \right) dx. \]

We will first prove that
\[ W_{\{\varepsilon_n^\pm\}}(u; \cdot) \] is a finite nonnegative Radon measure, absolutely continuous with respect to \( H^{N-1} | \partial^* A_0. \) (4.39)

For each \( k \in \mathbb{N}, \) let \( \{v_n^k\} \subset H^1(\Omega; \mathbb{R}^d) \) be such that \( \lim_{n \to \infty} \|v_n^k - u\|_{L^1(\Omega; \mathbb{R}^d)} = 0, \) and
\[ W_{\{\varepsilon_n^\pm\}}(u; \Omega) \leq \liminf_{n \to \infty} \int_{\Omega} \left( \frac{1}{\varepsilon_n} W\left( \frac{x}{\varepsilon_n}, v_n^k(x) \right) + \varepsilon_n \left| \nabla v_n^k(x) \right|^2 \right) dx \leq W_{\{\varepsilon_n^\pm\}}(u; \Omega) + \frac{1}{k}. \]

Extract an increasing subsequence \( \{n(j, k)\} \) of \( \{n\} \) such that
\[ \liminf_{n \to \infty} \int_{\Omega} \left( \frac{1}{\varepsilon_n} W\left( \frac{x}{\varepsilon_n}, v_n^k(x) \right) + \varepsilon_n \left| \nabla v_n^k(x) \right|^2 \right) dx = \lim_{j \to \infty} \int_{\Omega} \left( \frac{1}{\varepsilon_n(j, k)} W\left( \frac{x}{\varepsilon_n(j, k)}, v_n(j, k)^k(x) \right) + \varepsilon_n(j, k) \left| \nabla v_n(j, k)^k(x) \right|^2 \right) dx = W_{\{\varepsilon_n^\pm\}}(u; \Omega). \]

A diagonalization process allows us to extract a subsequence \( \{j(k)\} \) of \( \{j\}, \) such that, upon denoting \( n_k := n(j(k), k) \) and \( v_k := v_n(j(k), k), \) we have
\[ \lim_{k \to \infty} \|v_k - u\|_{L^1(\Omega; \mathbb{R}^d)} = 0, \]
and
\[ \lim_{k \to \infty} \int_{\Omega} \left( \frac{1}{\varepsilon_n} W\left( \frac{x}{\varepsilon_n}, v_k(x) \right) + \varepsilon_n \left| \nabla v_k(x) \right|^2 \right) dx = W_{\{\varepsilon_n^\pm\}}(u; \Omega). \]

The sequence of measures \( \{\mu_k\}, \) where \( \mu_k := \left( \frac{1}{\varepsilon_n} W\left( \frac{x}{\varepsilon_n}, v_k(x) \right) + \varepsilon_n \left| \nabla v_k(x) \right|^2 \right) \mathcal{L}^N | \Omega, \) is bounded in \( \mathcal{M}(\Omega). \) Thus, there exists a nonnegative Radon measure \( \mu \) such that, up to a subsequence (not relabelled), \( \mu_k \rightharpoonup^* \mu \) in \( \mathcal{M}(\Omega). \) We want to show that \( W_{\{\varepsilon_n^\pm\}}(u; A) = \mu(A) \) for all \( A \in \mathcal{A}(\Omega). \) To this end, and in view of Lemma 7.3 in [13] (see also [23]), it suffices to show that for any \( A, B, C \in \mathcal{A}(\Omega), W_{\{\varepsilon_n^\pm\}}(u; \cdot) : \mathcal{A}(\Omega) \to [0, \infty) \) satisfies

(i) if \( \overline{C} \subset B \subset A, \) then \( W_{\{\varepsilon_n^\pm\}}(u; A) \leq W_{\{\varepsilon_n^\pm\}}(u; A \setminus \overline{C}) + W_{\{\varepsilon_n^\pm\}}(u; B), \)

(ii) for any \( \varepsilon > 0, \) there exists \( C_\varepsilon \in \mathcal{A}(\Omega) \) with \( \overline{C_\varepsilon} \subset A \) and \( W_{\{\varepsilon_n^\pm\}}(u; A \setminus \overline{C_\varepsilon}) \leq \varepsilon, \)

(iii) \( W_{\{\varepsilon_n^\pm\}}(u; \Omega) \geq \mu(\mathbb{R}^N), \)

(iv) \( W_{\{\varepsilon_n^\pm\}}(u; A) \leq \mu(\overline{A}). \)
We will first prove (i). To this aim, let \( A, B, C \in \mathcal{A}(\Omega) \) be such that \( \overline{C} \subset B \subset A \). For \( \delta > 0 \), let \( B^\delta \) and \( D^\delta \) be two elements of \( \mathcal{R} \) such that \( B^\delta \subset B \), \( D^\delta \subset A \setminus \overline{C} \), and

\[
\mathcal{H}^{N-1} \left( (A \setminus (B^\delta \cup D^\delta)) \cap \partial^* A_0 \right) < \delta. \tag{4.40}
\]

Let \( \{ \frac{u^B_R}{\varepsilon_n^R} \} \) and \( \{ \frac{u^D_R}{\varepsilon_n^R} \} \) be sequences in \( H^1(B^\delta; \mathbb{R}^d) \) and \( H^1(D^\delta; \mathbb{R}^d) \), respectively, such that \( u^B_R \to u \) in \( L^1(B^\delta; \mathbb{R}^d) \), \( u^D_R \to u \) in \( L^1(D^\delta; \mathbb{R}^d) \),

\[
\lim_{n \to \infty} \int_{B^\delta} \left( \frac{1}{\varepsilon_n^R} W \left( \frac{x}{\varepsilon_n^R}, u^B_R(x) \right) + \varepsilon_n^R |\nabla u^B_R(x)|^2 \right) dx = W_1(\varepsilon_n^R)(u; B^\delta) < +\infty, \tag{4.41}
\]

and

\[
\lim_{n \to \infty} \int_{D^\delta} \left( \frac{1}{\varepsilon_n^R} W \left( \frac{x}{\varepsilon_n^R}, u^D_R(x) \right) + \varepsilon_n^R |\nabla u^D_R(x)|^2 \right) dx = W_1(\varepsilon_n^R)(u; D^\delta) < +\infty. \tag{4.42}
\]

Let \( \rho : \mathbb{R}^N \to [0, +\infty) \) be a symmetric mollifier, and define \( \rho \frac{\chi_R}{\tau_R^N} (x) := \frac{1}{(2\pi)^{\frac{N}{2}}} \rho \left( \frac{x}{\tau_R^N} \right) \). We may assume, without loss of generality, that

\[
u^B_R = \rho \frac{\chi_R}{\tau_R^N} \ast u \text{ on } \partial B^\delta, \quad u^B_R \to u \text{ in } L^2(B^\delta; \mathbb{R}^d) \text{ and } L^N - \text{a.e. } x \in B^\delta.
\]

The idea of the proof is along the lines of the proof of Lemma 4.5 (see also the proof of Proposition 3.3), where we replace \( Q \) by \( B^\delta \), and \( v_n \) by \( \rho \frac{\chi_R}{\tau_R^N} \ast u \) (with \( \rho \frac{\chi_R}{\tau_R^N} \) as defined above). Note that in this case \( \text{supp } \nabla (\rho \frac{\chi_R}{\tau_R^N} \ast u) \subset \{ x : \text{dist}(x, \partial^* A_0) < \varepsilon_n^R \} \), and for each \( k \in \mathbb{N} \), the layer \( L_k \) in the proof of Lemma 4.5 should be taken to be \( L_k := \{ x \in B^\delta : \text{dist}(x, \partial B^\delta) \leq \frac{1}{k} \} \).

Similarly, we may assume that

\[
u^D_R = \rho \frac{\chi_R}{\tau_R^N} \ast u \text{ on } \partial D^\delta, \quad u^D_R \to u \text{ in } L^2(D^\delta; \mathbb{R}^d) \text{ and } L^N - \text{a.e. } x \in D^\delta.
\]

Extend \( u^B_R \) and \( u^D_R \) as \( \rho \frac{\chi_R}{\tau_R^N} \ast u \) outside \( B^\delta \) and \( D^\delta \), respectively. Note that, in view of (4.35),

\[
\lim_{n \to \infty} \left\| \frac{u^B_R}{\varepsilon_n^R} - u \right\|_{L^2(A; \mathbb{R}^d)} = \lim_{n \to \infty} \left\| \frac{u^D_R}{\varepsilon_n^R} - u \right\|_{L^2(A; \mathbb{R}^d)} = 0. \tag{4.43}
\]

Write \( B \setminus \overline{C} \) as a union of \( M \) layers \( L_n^{(i)} \) \( (i = 1, \ldots, M) \) of width \( \varepsilon_n^R \) \( \left| \frac{u^B_R}{\varepsilon_n^R} - \frac{u^D_R}{\varepsilon_n^R} \right|_{L^2(A; \mathbb{R}^d)} = O(1). \tag{4.44}
\]

We have

\[
\sum_{i=1}^{M} \int_{L_n^{(i)}} \left( 1 + \left| \frac{u^B_R}{\varepsilon_n^R} \right|^q + \left| \frac{u^D_R}{\varepsilon_n^R} \right|^q + (\varepsilon_n^R)^2 |\nabla u^B_R|^2 + (\varepsilon_n^R)^2 |\nabla u^D_R|^2 \right) \left( \frac{|u^B_R - u^D_R|^2}{\varepsilon_n^R} \right) dx
\]

\[
= \int_{B \setminus \overline{C}} \left( 1 + \left| \frac{u^B_R}{\varepsilon_n^R} \right|^q + \left| \frac{u^D_R}{\varepsilon_n^R} \right|^q + (\varepsilon_n^R)^2 |\nabla u^B_R|^2 + (\varepsilon_n^R)^2 |\nabla u^D_R|^2 \right) \left( \frac{|u^B_R - u^D_R|^2}{\varepsilon_n^R} \right) dx,
\]

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and thus there exists $i_0 \in \{1, \cdots, M_n\}$ such that

$$
\int_{L_n^{(i_0)}} \left( 1 + |u_{\varepsilon_n}^{B_i}|^q + |u_{\varepsilon_n}^{D_i}|^q + (\varepsilon_n^R)^2 |\nabla u_{\varepsilon_n}^{B_i}|^2 + (\varepsilon_n^R)^2 |\nabla u_{\varepsilon_n}^{D_i}|^2 + \frac{|u_{\varepsilon_n}^{B_i} - u_{\varepsilon_n}^{D_i}|^2}{\|u_{\varepsilon_n}^{B_i} - u_{\varepsilon_n}^{D_i}\|_{L^2(\mathbb{R}^d)}} \right) dx \leq \frac{1}{M_n} \int_{B \setminus C} \left( 1 + |u_{\varepsilon_n}^{B_i}|^q + |u_{\varepsilon_n}^{D_i}|^q + (\varepsilon_n^R)^2 |\nabla u_{\varepsilon_n}^{B_i}|^2 + (\varepsilon_n^R)^2 |\nabla u_{\varepsilon_n}^{D_i}|^2 + \frac{|u_{\varepsilon_n}^{B_i} - u_{\varepsilon_n}^{D_i}|^2}{\|u_{\varepsilon_n}^{B_i} - u_{\varepsilon_n}^{D_i}\|_{L^2(\mathbb{R}^d)}} \right) dx.
$$

(4.45)

We remark that by (4.41), (4.42), (4.43), and (A3),

$$
\sup_{n \in \mathbb{N}} \int_{B \setminus C} \left( 1 + |u_{\varepsilon_n}^{B_i}|^q + |u_{\varepsilon_n}^{D_i}|^q + (\varepsilon_n^R)^2 |\nabla u_{\varepsilon_n}^{B_i}|^2 + (\varepsilon_n^R)^2 |\nabla u_{\varepsilon_n}^{D_i}|^2 + \frac{|u_{\varepsilon_n}^{B_i} - u_{\varepsilon_n}^{D_i}|^2}{\|u_{\varepsilon_n}^{B_i} - u_{\varepsilon_n}^{D_i}\|_{L^2(\mathbb{R}^d)}} \right) dx =: \epsilon_0 < +\infty.
$$

(4.46)

Consider cut-off functions $\varphi_n \in C^\infty_c (\Omega; [0, 1])$ such that

$$
\varphi_n(x) = 0 \text{ if } x \in \left( \bigcup_{j=i_0+1}^{M_n} L_n^{(i)} \right) \cup (A \setminus \overline{B}),
$$

$$
\varphi_n(x) = 1 \text{ if } x \in \left( \bigcup_{j=1}^{i_0-1} L_n^{(i)} \right) \cup C,
$$

and

$$
\| \nabla \varphi_n \|_{\infty} = O \left( \frac{1}{\varepsilon_n^R \|u_{\varepsilon_n}^{B_i} - u_{\varepsilon_n}^{D_i}\|_{L^2(\mathbb{R}^d)}^{1/2}} \right).
$$

Define

$$
u_n := \varphi_n u_{\varepsilon_n}^{D_i} + (1 - \varphi_n) u_{\varepsilon_n}^{B_i} + \chi_{(A \setminus (B^i \cup D^i))} \left( \frac{\rho_{\varepsilon_n}^D}{\varepsilon_n} * u \right).
$$

We have that $u_n \rightharpoonup u$ in $L^1(A; \mathbb{R}^d)$ as $n \to \infty$, and in view of (4.41), (4.42),

$$
W(\varepsilon_n)(u; A) \leq \liminf_{n \to \infty} \int_A \left( \frac{1}{\varepsilon_n^R} W \left( \frac{x}{\varepsilon_n}, u_n(x) \right) + \varepsilon_n^R |\nabla u_n(x)|^2 \right) dx
$$

$$
\leq \liminf_{n \to \infty} \left\{ \int_{A \setminus (B^i \cup D^i)} \left( \frac{1}{\varepsilon_n^R} W \left( \frac{x}{\varepsilon_n}, \rho_{\varepsilon_n}^D * u \right) + \varepsilon_n^R |\nabla \left( \rho_{\varepsilon_n}^D * u \right)|^2 \right) dx + \int_{B^i} \left( \frac{1}{\varepsilon_n^R} W \left( \frac{x}{\varepsilon_n}, u_{\varepsilon_n}^{B_i} \right) + \varepsilon_n^R |\nabla u_{\varepsilon_n}^{B_i}|^2 \right) dx + \int_{D^i} \left( \frac{1}{\varepsilon_n^R} W \left( \frac{x}{\varepsilon_n}, u_{\varepsilon_n}^{D_i} \right) + \varepsilon_n^R |\nabla u_{\varepsilon_n}^{D_i}|^2 \right) dx + \int_{L_n^{(i_0)}} \frac{1}{\varepsilon_n^R} W \left( \frac{x}{\varepsilon_n}, u_n \right) + \varepsilon_n^R |\nabla u_n|^2 \right\} dx \right\}
$$

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Indeed, it suffices to remark that for all $n > 0$:

$$W_{\{n\}}(u; \partial^* A_0) + W_{\{n\}}(u; D^\delta)$$

Next, we note that (ii) follows by the inner regularity of the Radon measure $\mathcal{H}^{N-1}(A \setminus (B^\delta \cup D^\delta)) \cap \partial^* A_0$:

$$\leq \mathcal{H}^{N-1}(A \setminus (B^\delta \cup D^\delta)) \cap \partial^* A_0 + W_{\{n\}}(u; D^\delta) + W_{\{n\}}(u; B^\delta)$$

By (4.44), (4.45), and the growth conditions in (A3), we obtain that:

$$\leq \frac{C}{\varepsilon_n} \int_{L^{(n)}} \left(1 + |u_{n}^{B_0}|^q + |u_{n}^{B_0}|^q + (\varepsilon_n R)^2 |\nabla u_{n}^{B_0}|^2 + (\varepsilon_n R)^2 |\nabla u_{n}^{B_0}|^2 + (\varepsilon_n R)^2 |\nabla \varphi_n|_\infty^2 |u_{n}^{B_0} - u_{n}^{B_0}|^2 \right) dx$$

where we have used (4.46). Thus,

$$\limsup_{n \to \infty} \int_{L^{(n)}} \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, u_n\right) + \varepsilon_n R |\nabla u_n|^2 \right) dx = 0,$$

and we deduce from (4.40) and (4.47) that:

$$W_{\{n\}}(u; A) \leq \delta + W_{\{n\}}(u; B^\delta) + W_{\{n\}}(u; D^\delta) \leq \delta + W_{\{n\}}(u; B) + W_{\{n\}}(u; A \setminus \overline{C}).$$

Letting $\delta \to 0^+$, we obtain that (i) holds.

Next, we note that (ii) follows by the inner regularity of the Radon measure $\mathcal{H}^{N-1}(A \setminus (B^\delta \cup D^\delta)) \cap \partial^* A_0$. Indeed, it suffices to remark that for all $A \in \mathcal{A}(\Omega)$, by the growth condition in (A3), and since $A_0$ is polyhedral:

$$W_{\{n\}}(u; A) \leq \liminf_{n \to \infty} \int_A \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, \left(\rho_{\frac{1}{\varepsilon_n}} \ast u\right)(x)\right) + \varepsilon_n R \left|\nabla \left(\rho_{\frac{1}{\varepsilon_n}} \ast u\right)(x)\right|^2 \right) dx$$

$$\leq \liminf_{n \to \infty} \int_{x \in A : \text{dist}(x, \partial^* A_0) \leq \varepsilon_n} \left(1 + \left|\rho_{\frac{1}{\varepsilon_n}} \ast u\right|^q + (\varepsilon_n R)^2 \left|\nabla \left(\rho_{\frac{1}{\varepsilon_n}} \ast u\right)\right|^2 \right) dx$$

$$\leq C \liminf_{n \to \infty} \mathcal{L}^N(\{x \in A : \text{dist}(x, \partial^* A_0) \leq \varepsilon_n\}) = C \mathcal{H}^{N-1}(A \cap \partial^* A_0).$$
Property (iii) follows immediately from
\[
\mu(\mathbb{R}^N) \leq \liminf_{k \to \infty} \mu_k(\mathbb{R}^N) = \lim_{k \to \infty} \int_{\Omega} \left( \frac{1}{\varepsilon_n} W\left( \frac{x}{\varepsilon_n}, v_k(x) \right) + \varepsilon |\nabla v_k(x)|^2 \right) dx = \mathcal{W}_{\{\varepsilon_n\}}(u; \Omega).
\]

Finally, since the sequence \( \{v_k\} \subset H^1(\mathbb{R}^d) \) is admissible for the definition of \( \mathcal{W}_{\{\varepsilon_n\}}(u; A) \), we obtain that
\[
\mathcal{W}_{\{\varepsilon_n\}}(u; A) \leq \liminf_{k \to \infty} \int_A \left( \frac{1}{\varepsilon_n} W\left( \frac{x}{\varepsilon_n}, v_k(x) \right) + \varepsilon |\nabla v_k(x)|^2 \right) dx
= \liminf_{k \to \infty} \mu_k(A) \leq \mu(A),
\]
thus asserting (iv). Since \( \mathcal{W}_{\{\varepsilon_n\}}(u; \cdot) \) satisfies (i)-(iv), we conclude that (4.39) holds.

We claim that
\[
\frac{d\mathcal{W}_{\{\varepsilon_n\}}(u; \cdot)}{d\mathcal{H}^{N-1}|_{\partial^* A_0}}(x_0) \leq K_1(\nu(x_0)) \text{ for } \mathcal{H}^{N-1} \text{-a.e. } x_0 \in \Omega \cap \partial^* A_0.
\]

Assume, without loss of generality, that \( x_0 \in \Omega \cap \partial^* A_0 \) is such that \( \nu(x_0) = e_N \) and
\[
\frac{d\mathcal{W}_{\{\varepsilon_n\}}(u; \cdot)}{d\mathcal{H}^{N-1}|_{\partial^* A_0}}(x_0) = \lim_{\varepsilon \to 0} \frac{\mathcal{W}_{\{\varepsilon_n\}}(u; Q(x_0, \varepsilon))}{\varepsilon^{N-1}},
\]
and denote \( Q(x_0) \) by \( Q \), \( \rho_{T, \nu(x_0)} \) by \( \rho_T \), and \( K_1(\nu(x_0)) \) by \( K_1 \). In view of Lemma 4.1, let \( \{T_k\} \subseteq \mathbb{N} \), with \( T_k \to \infty \), and \( \{u_k\} \subset H^1(T_k Q; \mathbb{R}^d) \) be such that \( u_k = \rho_{T_k} * u_0 \) on \( T_k Q \), and
\[
\lim_{k \to \infty} \int_{T_k Q} \left( W(y, u_k(y)) + |\nabla u_k(y)|^2 \right) dy = K_1.
\]

Changing variables, we obtain that
\[
K_1 = \lim_{k \to \infty} \int_Q T_k W(T_k x, v_k(x)) + \frac{1}{T_k} |\nabla v_k(x)|^2 \right) dx,
\]
where \( v_k(x) := u_k(T_k x), \ x \in Q \). For \( x \in \mathbb{R}^d \setminus [-\frac{1}{2}, \frac{1}{2}]^d \), extend \( v_k(\cdot, x) \) by periodicity outside \( Q' \), and define
\[
v_{n,k}(x) := \begin{cases} u_0(x) & \text{if } |x| > \frac{\varepsilon n}{T_k} \\ v_k \left( \frac{x \varepsilon n}{\varepsilon n T_k} \right) & \text{if } |x| \leq \frac{\varepsilon n}{2T_k}. \end{cases}
\]

For \( \varepsilon > 0 \), we have
\[
\int_Q \left( \frac{\varepsilon}{\varepsilon n} W\left( \frac{x}{\varepsilon n}, v_{n,k}(x) \right) + \frac{\varepsilon}{\varepsilon n} |\nabla v_{n,k}(x)|^2 \right) dx
= \int_{x \in Q : |x| \leq \frac{\varepsilon n}{2T_k}} \left( \frac{\varepsilon}{\varepsilon n} W\left( \frac{x}{\varepsilon n}, v_k \left( \frac{x \varepsilon n}{\varepsilon n T_k} \right) \right) + \frac{\varepsilon}{\varepsilon n T_k^2} \left| \nabla v_k \left( \frac{x \varepsilon n}{\varepsilon n T_k} \right) \right|^2 \right) dx
\]
\[
\int_{\frac{e}{e_nT_k}}^{\frac{e}{e_nT_k}} \int_{Q^c} \left( \frac{e}{e_nT_k} W \left( \left( \frac{e x'}{e_nT_k}, \frac{e x_N}{e_nT_k} \right), v_k \left( \frac{e x'}{e_nT_k}, \frac{e x_N}{e_nT_k} \right) \right) + \frac{e}{e_nT_k} \left| \nabla v_k \left( \frac{e x'}{e_nT_k}, \frac{e x_N}{e_nT_k} \right) \right|^2 \right) \, dx' \, dx_N
\]

\[
= \int_{-\frac{e}{e_nT_k}}^{\frac{e}{e_nT_k}} \int_{Q'} \left( \frac{e}{e_nT_k} W \left( \left( \frac{e T_k x'}{e_nT_k}, T_k y_N \right), v_k \left( \frac{e T_k x'}{e_nT_k}, y_N \right) \right) + \frac{1}{T_k} \left| \nabla v_k \left( \frac{e x'}{e_nT_k}, y_N \right) \right|^2 \right) \, dx' \, dy_N.
\]

Thus, by the Riemann-Lebesgue Lemma (recall that \(T_k \in \mathbb{N}\), and thus \(W(T_k, z)\) is \(Q^c\)-periodic) and the Dominated Convergence Theorem, together with (4.50), we obtain

\[
\lim_{k \to \infty} \lim_{n \to \infty} \lim_{\varepsilon \to 0} \int_Q \left( \frac{e}{e_n} W \left( \left( \frac{e x}{e_nT_k}, v_{n,k}(x) \right) \right) + \frac{e}{e_n} \left| \nabla v_{n,k}(x) \right|^2 \right) \, dx = K_1.
\]

(4.51)

Let \(m_n \in \mathbb{Z}^N\) and \(s_n \in [0, 1]^N\) be such that \(\frac{m_n}{n} = m_n + s_n\), and let \(y_{n,\varepsilon} := -\frac{y_n}{\varepsilon}\). Note that for any \(\varepsilon > 0\), \(\lim_{n \to \infty} x_{n,\varepsilon} = 0\). Define \(u_{n,\varepsilon,k} \in H^1(Q(x_0, \varepsilon); \mathbb{R}^d)\) by \(u_{n,\varepsilon,k}(x) := v_{n,k}(\frac{x-x_0}{\varepsilon} - x_{n,\varepsilon})\).

We claim that for any \(k \in \mathbb{N}\) and \(\varepsilon > 0\), we have

\[
\lim_{n \to \infty} \|u_{n,\varepsilon,k} - u\|_{L^1(Q(x_0, \varepsilon); \mathbb{R}^d)} = 0.
\]

(4.52)

Indeed, changing variables,

\[
\int_{Q(x_0, \varepsilon)} \left| u_{n,\varepsilon,k}(x) - u(x) \right| \, dx = \int_{Q(0, \varepsilon) - \varepsilon x_{n,\varepsilon}} \left| v_{n,k} \left( \frac{z}{\varepsilon} \right) - u(x_0 + z + \varepsilon x_{n,\varepsilon}) \right| \, dz
\]

\[
= \int_{(Q(0, \varepsilon) - \varepsilon x_{n,\varepsilon}) \cap \{z: |z| \leq \frac{e}{e_nT_k}\}} \left| v_{n,k} \left( \frac{z}{\varepsilon} \right) - u(x_0 + z + \varepsilon x_{n,\varepsilon}) \right| \, dz
\]

\[
+ \int_{(Q(0, \varepsilon) - \varepsilon x_{n,\varepsilon}) \cap \{z: |z| > \frac{e}{e_nT_k}\}} \left| u_0 \left( \frac{z}{\varepsilon} \right) - u(x_0 + z + \varepsilon x_{n,\varepsilon}) \right| \, dz
\]

\[
\leq \int_{(Q(0, \varepsilon) - \varepsilon x_{n,\varepsilon}) \cap \{z: |z| \leq \frac{e}{e_nT_k}\}} C \left( 1 + \left| v_{k} \left( \frac{z}{e_nT_k} \right) \right| \right) \, dz + \int_{(Q(0, \varepsilon) - \varepsilon x_{n,\varepsilon}) \cap \{z: 0 < |z| < \delta(x_{n,\varepsilon}) \}} |b - a| \, dz.
\]

(4.53)

Since \(\lim_{n \to \infty} x_{n,\varepsilon} = 0\), we obtain that

\[
\lim_{n \to \infty} \int_{(Q(0, \varepsilon) - \varepsilon x_{n,\varepsilon}) \cap \{z: 0 < |z| < \delta(x_{n,\varepsilon}) \}} |b - a| \, dz = 0
\]

and

\[
\lim_{n \to \infty} \int_{(Q(0, \varepsilon) - \varepsilon x_{n,\varepsilon}) \cap \{z: |z| \leq \frac{e}{e_nT_k}\}} C \left( 1 + \left| v_{k} \left( \frac{z}{e_nT_k} \right) \right| \right) \, dz
\]

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We claim that

Thus,

Consequently,

Thus, in view of (4.53), we deduce that (4.52) holds.

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Thus, in view of (4.53), we deduce that (4.52) holds.

Changing variables, we have by (A1),

Thus,

We claim that

\[
\lim_{k \to \infty} \limsup_{\epsilon \to 0^+} \limsup_{n \to \infty} \int_{(-x_n,+Q) \setminus Q} \left( \frac{\epsilon}{\epsilon_n^R} W \left( \frac{\epsilon x}{\epsilon_n}, v_{n,k}(x) \right) + \frac{\epsilon R}{\epsilon} |\nabla v_{n,k}(x)|^2 \right) dx = 0. \tag{4.57}
\]
After changing variables, we have

\[
\int_{(-x_n, \varepsilon) \setminus Q} \left( \frac{\varepsilon}{\varepsilon_n^2} W \left( \frac{\varepsilon x}{\varepsilon_n} v_n^{(c)}(x) \right) + \frac{\varepsilon R}{\varepsilon} |\nabla v_n^{(c)}(x)|^2 \right) dx
\]

\[
= \int_{(-x_n, \varepsilon) \setminus Q} \left( \frac{\varepsilon}{\varepsilon_n^2} W \left( \frac{\varepsilon x}{\varepsilon_n} v_n^{(c)}(x) \right) + \frac{\varepsilon n}{\varepsilon} |\nabla v_n^{(c)}(x)|^2 \right) dx
\]

\[
= \int_{(-x_n, \varepsilon) \setminus Q} \left( T_k W \left( \left( \frac{T_k x}{\varepsilon} \right), y_n T_k \right), v_k \left( \frac{\varepsilon x'}{\varepsilon k}, y_n \right) \right) + \frac{1}{T_k} |\nabla v_k \left( \frac{\varepsilon x'}{\varepsilon k}, y_n \right)|^2 \right) dy/\Omega.
\]

For each \( \varepsilon > 0 \), take \( n(\varepsilon) \in \mathbb{N} \) such that \( |x_{\varepsilon,n}| < \varepsilon \) for all \( n \geq n(\varepsilon) \). In particular, we have \( (-x_n, \varepsilon) \setminus Q \subset (1 + \varepsilon)Q \setminus Q \) and, in view of the Riemann-Lebesgue Lemma,

\[
\limsup_{n \to \infty} \int_{(-x_n, \varepsilon) \setminus Q} \left( \frac{\varepsilon}{\varepsilon_n^2} W \left( \frac{\varepsilon x}{\varepsilon_n} v_n^{(c)}(x) \right) + \frac{\varepsilon n}{\varepsilon} |\nabla v_n^{(c)}(x)|^2 \right) dx
\]

\[
\leq O(\varepsilon) \int_{Q} \left( T_k W \left( T_k y, v_k(y) \right) + \frac{1}{T_k} |\nabla v_k|^2 \right) dy,
\]

thus asserting (4.57). Taking into account (4.49), (4.54), (4.55), (4.56), and (4.57), we obtain that

\[
\frac{dW_{1/n}}{d\mathcal{H}^{N-1}} \left| \partial^* A_0 \left( x_0 \right) \right| \leq \limsup_{k \to \infty} \limsup_{\varepsilon \to 0^+} \int_{Q} \left( \frac{\varepsilon}{\varepsilon_n^2} W \left( \frac{\varepsilon x}{\varepsilon_n} v_n^{(c)}(x) \right) + \frac{\varepsilon n}{\varepsilon} |\nabla v_n^{(c)}(x)|^2 \right) dx
\]

\[
+ \limsup_{n \to \infty} \int_{(-x_n, \varepsilon) \setminus Q} \left( \frac{\varepsilon}{\varepsilon_n^2} W \left( \frac{\varepsilon x}{\varepsilon_n} v_n^{(c)}(x) \right) + \frac{\varepsilon n}{\varepsilon} |\nabla v_n^{(c)}(x)|^2 \right) dx
\]

\[
= \limsup_{k \to \infty} \limsup_{n \to \infty} \int_{Q} \left( \frac{\varepsilon}{\varepsilon_n^2} W \left( \frac{\varepsilon x}{\varepsilon_n} v_n^{(c)}(x) \right) + \frac{\varepsilon n}{\varepsilon} |\nabla v_n^{(c)}(x)|^2 \right) dx = K_1,
\]

where the last equality follows by (4.51). Thus, (4.48) holds. In view of (4.39) and (4.48), we have

\[
W_{1/n}(u; \Omega) \leq \int_{\Omega} \frac{dW_{1/n}}{d\mathcal{H}^{N-1}} \left| \partial^* A_0 \left( x_0 \right) \right| d\mathcal{H}^{N-1} \left| \partial^* A_0 \left( x \right) \right| = \int_{\Omega} \frac{dW_{1/n}}{d\mathcal{H}^{N-1}} \left| \partial^* A_0 \left( x \right) \right| d\mathcal{H}^{N-1} \left| \partial^* A_0 \left( x \right) \right|
\]

\[
\leq \int_{\Omega} K_1(\nu(x)) d\mathcal{H}^{N-1}(x).
\]

By Proposition 4.3, we deduce that, in fact,

\[
W_{1/n}(u; \Omega) = \int_{\Omega} K_1(\nu(x)) d\mathcal{H}^{N-1}(x),
\]

and the conclusion follows by a diagonalization argument.
References


