SOLUTIONS

5.1.4 \[ |a_n| = \left| \frac{a_n}{b_n} \right| \cdot |b_n| \to E \cdot N. \text{ Since } \frac{a_n}{b_n} \to L \text{ as } n \to \infty \text{ there exists } M > 0 \text{ s.t. } \left| \frac{a_n}{b_n} \right| \leq M \cdot N. \text{ Let } \varepsilon > 0 \text{ be arbitrary.}

Since \( b_n \to 0 \text{ as } n \to \infty \), \exists N \in \mathbb{N} \text{ s.t. } |b_n| < \frac{\varepsilon}{M} \text{ for } n \geq N.

Hence \( |a_n| < M \cdot \frac{E}{M} = \varepsilon \) \( \Rightarrow n \geq N \). We deduce that \( a_n \to 0 \text{ as } n \to \infty \).

5.2.1 We have \( \frac{\sqrt{n} - 1}{\sqrt{u+1}} \leq \frac{\sqrt{n} + \cos n}{\sqrt{u+1}} \leq \frac{\sqrt{n} + 1}{\sqrt{u+1}} \). Since \( \frac{\sqrt{n}}{\sqrt{u+1}} = \frac{\sqrt{\frac{1}{u+1}} \to 1} {\sqrt{1}} = 1 \text{ and } \frac{1}{\sqrt{u+1}} \to 0 \) we conclude by the Squeeze Theorem that \( \lim_{n \to \infty} \frac{\sqrt{n} + \cos n}{\sqrt{u+1}} = 1 \).

5.2.4 From the figure, we have:

\[ \frac{1}{u+1} + \frac{1}{u+2} + \ldots + \frac{1}{2n} \leq \int_{\frac{1}{n+1}}^{\frac{2}{n}} dx = \ln 2 - \ln n = \ln 2 \]

On the other hand:

\[ \frac{1}{u+1} + \frac{1}{u+2} + \ldots + \frac{1}{2n} \geq \int_{\frac{1}{n+1}}^{\frac{2}{n}} dx = \ln \frac{2n+1}{n+1} = \ln \left( 1 + \frac{n}{u+1} \right) \]

Since \( \ln \left( 1 + \frac{n}{u+1} \right) \to \ln (1 + 1) = \ln 2 \), the Squeeze Theorem implies that \( \lim_{n \to \infty} \left( \frac{1}{u+1} + \frac{1}{u+2} + \ldots + \frac{1}{2n} \right) = \ln 2 \).

5.3.4 (a) Since \( \{b_n\} \) converges, it is bounded above. Hence, \( \{a_n\} \) is bounded above. Since \( \{a_n\} \) is increasing and bounded above, it is convergent (by the Completeness Principle). \( \text{(5c)} \) p. 67.

(b) Take \( a_n = 1 - \frac{1}{n+2}, \ b_n = 1 - \frac{1}{(n+2)^2} \), \( M = 1 \). Then \( \{a_n\} \) is increasing, \( a_n < b_n \to M = 1 + \frac{\varepsilon}{u \in N} \), and \( \lim_{n \to \infty} a_n = M. \).
6.2.1 Let $a_n := \sin\left(\frac{n+1}{n} \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} + \frac{\pi}{2n}\right)$. The sequence $\{a_n\}$ is convergent (to $\sin\frac{\pi}{2} = 1$), thus the only cluster point is 1.

Let $b_n := \sin\left(\frac{n+1}{n} \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} + \frac{\pi}{2n}\right)$

Since $b_{2k} = \sin\left(k\pi + \frac{\pi}{4k}\right) = \sin(k\pi) \cos\frac{\pi}{4k} + \cos(k\pi) \sin\frac{\pi}{4k} = (-1)^k \sin\left(\frac{\pi}{4k}\right) \rightarrow 0$ as $k \rightarrow \infty$

$b_{4(k+1)} = \sin\left(\frac{\pi}{2} + 2k\pi + \frac{\pi}{2(4k+1)}\right) = \sin\left(\frac{\pi}{2} + 2k\pi\right) \cos\left(\frac{\pi}{8k+2}\right) + \cos\left(\frac{\pi}{2} + 2k\pi\right) \sin\left(\frac{\pi}{8k+2}\right) \rightarrow 1$ as $k \rightarrow \infty$

$b_{4(k+3)} = \sin\left(\frac{3\pi}{2} + 2k\pi + \frac{\pi}{8k+6}\right) = \sin\left(\frac{3\pi}{2} + 2k\pi\right) \cos\left(\frac{\pi}{8k+6}\right) + \cos\left(\frac{3\pi}{2} + 2k\pi\right) \sin\left(\frac{\pi}{8k+6}\right) 

we have that $-1, 0, 1$ are cluster points of $\{b_n\}$. The fact that these are the only cluster points can be proven similarly to the proof that $-1, 0, 1$ are the only cluster points of $\{\sin\left(\frac{\pi}{n}\right)\}$, done in class.

6.4.2 For $m > n > 1$ we have $|a_m - a_n| = \left| a_m - a_{m-1} - a_{m-2} + \ldots + a_{m+2} - a_{n+1} + a_{n+1} - a_n \right| \leq \left| a_{m+2} - a_{n+1} \right| + \left| a_{n+1} - a_n \right| < C(K^m + K^{m+1} + \ldots + K^{m-n+1})$

$= CK^m \left( 1 + K + \ldots + K^{m-n+1} \right) = CK^m \left( \frac{1 - K^{m-n+1}}{1 - K} \right)$

$= \frac{C}{1 - K} (K^m - K^n)$. Since $K \in (0, 1)$ we have $K^n \rightarrow 0$ as $n \rightarrow \infty$. Thus, $K^n$ is Cauchy, let $\varepsilon > 0$ be arbitrary. Then $K^m \nsim K^n$ for $m-n > 1$. In view of the above, we get $a_n \nsim a_m$ for $m-n > 1$. Thus, $a_n$ is Cauchy.
6-1 (a) We have \( |x_m - x_{m+1}| = \left| x_n - \frac{x_n + x_{n-1}}{2} \right| = \frac{1}{2} |x_n - x_{n-1}| \)

\[ = \frac{1}{2^2} |x_{n-1} - x_{n-2}| = \ldots = \frac{1}{2^m} |x_1 - x_0| = \frac{1}{2^n} |a - b| \]

Thus, for \( m > n \gg 1 \) we have

\[ \left| x_m - x_n \right| \leq \left| x_n - x_{n+1} \right| + \left| x_{n+1} - x_{n+2} \right| + \ldots + \left| x_{m-1} - x_m \right| \]

\[ \leq |a - b| \left( \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^{m-1}} \right) \]

\[ \leq \frac{|a - b|}{2^n} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{2^{m-n-1}} \right) \leq \frac{|a - b|}{2^{n-1}} \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k = 2 \]

Since \( \frac{|a - b|}{2^{n-1}} \to 0 \) as \( n \to \infty \), we deduce that \( \forall \varepsilon > 0 \)

\( \left| x_m - x_n \right| < \varepsilon \) for \( m > n \gg 1 \), which proves that \( \{x_m\} \) is a Cauchy sequence.

(b) From part (a), \( x_m - x_{m+1} = \frac{(-1)^m}{2^n} (x_0 - x_1) \). Thus,

\[ x_n = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \ldots + (x_1 - x_0) + x_0 \]

\[ = - (x_0 - x_1) \left[ \frac{(-1/2)^{n-1} + (-1/2)^{n-2} + \ldots + 1}{1 - (-1/2)} \right] + x_0 \]

\[ = (x_1 - x_0) \left[ \frac{1 - (-1/2)^n}{1 - (-1/2)} \right] + x_0 \to x_0 + \frac{2(x_1 - x_0)}{3} \text{ as } n \to \infty \]

Thus, \( \lim_{n \to \infty} x_n = \frac{2x_1 + x_0}{3} = \frac{2b + a}{3} \).
(a) We have, \( \forall u \in \mathbb{N}, a_{u+1} - a_u = \left( f(u)+f(u+1)+\cdots+f(u+r)+f(r) \right) - \int_{u}^{u+1} f(x) \, dx \) 
\[ = f(u) - \left( \int_{0}^{u+1} f(x) \, dx - \int_{0}^{u} f(x) \, dx \right) = f(u) - \int_{u}^{u+1} f(x) \, dx \]
Since \( f \) is decreasing, we have \( f(u+1) \leq f(x) \leq f(u) \), \( \forall x \in [u,u+1] \)
Thus, \( f(u+1) \leq \int_{u}^{u+1} f(x) \, dx \leq f(u) \). We obtain:
\[ 0 \leq a_{u+1} - a_u \leq f(u) - f(u+1). \]
For all \( m > n \) we have, similarly, \( a_m - a_{m-1} \leq f(m-1) - f(m) \)
\[ - f(m-1), \ldots, 0 \leq a_{m+1} - a_{m} \leq f(m) - f(m+1). \]
Adding all these inequalities term by term gives:
\[ 0 \leq a_m - a_n \leq f(n) - f(m), \quad m > n > 1. \]
Let \( \varepsilon > 0 \) be arbitrary.
Since \( f(n) \to 0 \) as \( n \to \infty \) we have \( f(n) - f(n) < \varepsilon \) for all \( m > n \).
Thus, we deduce that \( |a_m - a_n| < \varepsilon \) for all \( m > n \).
Which shows that \( \{a_n\} \) is a Cauchy sequence.

(b) If \( f(x) = e^{-x} \), we have
\[ a_n = 1 + e^{-1} + e^{-2} + \cdots + e^{-n+1} - \int_{0}^{n} e^{-x} \, dx = 1 + e^{-1} + e^{-2} + \cdots + e^{-n+1} + e^{-m} - \frac{1 - e^{-n}}{1 - e^{-1}} - 1 \]
Thus, \( a_n \to \frac{1}{1-e^{-1}} - 1 = \frac{e}{e-1} - 1 = \frac{1}{e-1} \) as \( n \to \infty \).