8.3.1 We have, after replacing $x$ by $-x$ in the given power series:

$$-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \ldots - \frac{x^m}{m} \ldots = \ln(1-x)$$

Thus, using Theorem 8.3 we find:

$$\ln \left( \frac{1+x}{1-x} \right) = \ln (1+x) - \ln (1-x) = 2x + 2 \frac{x^3}{3} + \ldots + 2 \frac{x^{2k+1}}{2k+1} + \ldots$$

9.3.5 We proceed by contradiction. If $f$ is not constant, then there exist $x_1 < x_2$ such that $f(x_1) < f(x_2)$. Since $f$ is periodic (of, say, period $T > 0$) then

$$f(x_1) = f(x_1 + T) = f(x_1 + 2T) = \ldots = f(x_1 + nT)$$

Let $m = \lceil \frac{x_2 - x_1}{T} \rceil + 1$. We have $x_1 + nT > x_2$ and thus, since $f$ is increasing, $f(x_2) \leq f(x_1 + nT)$, we deduce that $f(x_1) < f(x_2) \leq f(x_1 + nT) = f(x_1)$, a contradiction.

8-1 If $|x| < 1$ then, since $|\sin(n)| x^n \leq |x|^n$ and the series $\sum |x|^n$ converges it follows by the comparison theorem that the series $\sum \sin(n) x^n$ is (absolutely) convergent. On the other hand if $x = 1$, the series becomes $\sum \sin(n)$ which diverges by the divergence test (as shown in class), hence $\lim_{n \to \infty} \sin(n)$ does not exist, while if $x = -1$, the same reasoning shows that $\sum (-1)^n \sin(n)$ diverges. Thus, the radius of convergence is 1.
We need to show that 

$$A := \{ x \geq 0 \mid f(x) = x^2 \} = \{ x \geq 0 \mid f(x) = g(x) \} =: B$$

If \( x \in A \), then \( f(x) = x = f(g(x)) \). Since \( f \) is strictly increasing, it is injective, so \( x = g(x) \).

Since \( x = f(x) \) we deduce that \( f(x) = g(x) \), so \( x \in B \).

This shows that \( A \subseteq B \). Next, we show that \( B \subseteq A \). Let \( x \in B \). Thus, \( f(x) = g(x) \).

We claim that \( f(x) = x \). Indeed, if not, then either \( x < f(x) \) or \( x > f(x) \) must hold.

If we are in Case I, since \( x = f(g(x)) \) we get \( f(g(x)) < f(x) \) which gives \( g(x) < x \) (since \( f \) is strictly increasing).

Hence, \( f(x) < x \) which is a contradiction.

Case II is handled analogously. Where do we use it-2?

We have \( f(\cos(x + \frac{2\pi}{2})) = f(\cos x) \) for all \( x \in \mathbb{R} \).

Thus, \( f \circ \cos \) is periodic and \( 2\pi \) is a period.

(a) Let \( g(x) = \sin x \). \( g : \mathbb{R} \to \mathbb{R} \) and it is periodic with \( 2\pi \) as a period. If we assume that there exists \( f : \mathbb{R} \to \mathbb{R} \) such that \( g(x) = f(\cos x) \) for all \( x \in \mathbb{R} \), then we would have \( f(0) = f(\cos \frac{\pi}{2}) = g(\frac{\pi}{2}) = \sin \frac{\pi}{2} = 1 \) and, or the other hand, \( f(0) = f(\cos \frac{3\pi}{2}) = g(\frac{3\pi}{2}) = \sin \frac{3\pi}{2} = -1 \).

This is a contradiction.