Solutions

(10.1.9) (a) If $f$ is bounded, there exists $M > 0$ such that $|f(y)| \leq M$ for all $y \in \mathbb{R}$. Thus $|f(g(x))| \leq M$ for all $x \in \mathbb{R}$. Hence, $f \circ g$ is bounded.

(b) False. Let $f(x) = \sin x$ and define $g: \mathbb{R} \to \mathbb{R}$ by $g(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Then, we have

$$g(f(x)) = \begin{cases} \frac{1}{\sin x} & \text{if } x \notin \{k\pi | k \in \mathbb{Z}\} \\ 0 & \text{if } x \in \{k\pi | k \in \mathbb{Z}\} \end{cases},$$

which is unbounded, since $g(f(\frac{1}{n})) = \frac{1}{\sin(\frac{1}{n})} \to \infty$ as $n \to \infty$

(10.3.2) Since $f$ is bounded for $x > 1$, there exist $M > 0$ and $b \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \geq b$.

We claim that, in fact, the inequality $|f(x)| \leq M$ holds for all $x \in \mathbb{R}$. Indeed, let $x < b$ arbitrary. If $T > 0$ is the period of $f$, let $m \in \mathbb{N}$ be such that $x + mT \geq b$ (one can take, for example, $m = \lceil \frac{b-x}{T} \rceil + 1$). Then, $|f(x)| = |f(x + mT)| \leq M$.

The above shows that $f: \mathbb{R} \to \mathbb{R}$ is bounded.
Let $\varepsilon > 0$ be arbitrary, and choose $\delta = \frac{-1 + \sqrt{1 + 4\varepsilon}}{4}$. If $x$ is such that $0 < |x| < \delta$, we have

$$\left| \frac{1-x}{1+x^2} - 1 \right| = \left| \frac{-x-x^2}{1+x^2} \right| \leq \frac{1}{1+x^2} |1-x| < \delta \delta^2$$

Triangle Ineq.

We have $\delta + \delta^2 = \frac{-1 + \sqrt{1 + 4\varepsilon}}{4} + \frac{\sqrt{1 + 4\varepsilon} - 1}{16} = \frac{-4 + 4\sqrt{1 + 4\varepsilon} + 1 + 4\varepsilon + 1 - 2\sqrt{1 + 4\varepsilon} + 4\varepsilon}{16} = \frac{-2 + 2\sqrt{1 + 4\varepsilon} + 4\varepsilon}{16} = \frac{\sqrt{1 + 4\varepsilon} - 1 + 2\varepsilon}{8} < \varepsilon$.

Thus, $\left| \frac{1-x}{x^2+1} - 1 \right| < \varepsilon$ whenever $0 < |x| < \delta$,

that is, $\lim_{x \to 0} \frac{1-x}{x^2+1} = 1$.

We have $|f(x)| = \sqrt{x} \left| \cos \frac{1}{x} \right| \leq \sqrt{x}$.

Since $\lim_{x \to 0} \sqrt{x} = 0$, Squeeze Theorem implies that $\lim_{x \to 0} f(x) = 0$.

Hence, if we define $f(0) = 0$, $\lim_{x \to 0} f(x) = f(0)$, which means that $f$ is continuous at 0.
(a) We have \( \frac{d}{dx} = \int_0^x \frac{\cos t}{1+t^2} \, dt = -\int_0^x \frac{\cos(-t)}{1+(-t)^2} (-dy) \)

\[ = -\int_0^x \frac{\cos y}{1+y^2} \, dy = -f(x) \forall x \in \mathbb{R}. \text{ Thus, the function is odd.} \]

(b) Neither. If \( f \) was monotone on \( \mathbb{R} \), its natural domain, then since it's differentiable (by the Fundamental Theorem of Calculus), \( f' \) would need to be either \( \geq 0 \) or \( \leq 0 \) throughout \( \mathbb{R} \). However,

\[ f'(x) = \frac{d}{dx} \left( \int_0^x \frac{\cos t}{1+t^2} \, dt \right) = \frac{\cos x}{1+x^2} \]

changes sign on \( \mathbb{R} \).

(c) Since \( f \) is odd it suffices to study what happens when \( x > 0 \). The graph of \( g \) looks (roughly) as follows:

From the graph it appears that \( f \) has a maximum at \( \frac{\pi}{2} \), and \( f(\frac{\pi}{2}) = A \approx \frac{\pi}{4} \), where we have approximated \( A \) by the area of the triangle with vertices \( O, M, \) and \( N. \) [Diagram]
We have \( f(2) = f(1 + 1) = f(1) + f(1) = 2f(1) \), \( f(3) = f(2) + f(1) = 3f(1) \), etc. It can be shown by induction that \( f(n) = nf(1) \) for all \( n \in \mathbb{N} \). Next, note that \( f(0) = f(0 + 0) = f(0) + f(0) \), so \( f(0) = 0 \), and that \( f(x + (-x)) = f(x) + f(-x) \), giving \( f(-x) = -f(x) \) for all \( x \in \mathbb{R} \). We deduce that \( f(n) = nf(1) \) for all \( n \in \mathbb{Z} \).

By induction, one shows that \( f\left(\frac{1}{n}\right) = \frac{1}{n} f(1) \) for all \( n \in \mathbb{N} \) and using the fact that \( f \) is odd, we get \( f\left(\frac{1}{n}\right) = \frac{1}{n} f(1) \) for all \( n \in \mathbb{N} \). Now, if \( m \in \mathbb{N} \) we have

\[
 f\left(\frac{m}{n}\right) = f\left(\frac{1}{n} + \cdots + \frac{1}{n}\right) = \underbrace{f\left(\frac{1}{n}\right) + \cdots + f\left(\frac{1}{n}\right)}_{m \text{ times}} = mf\left(\frac{1}{n}\right).
\]

Thus, \( f\left(\frac{m}{n}\right) = \frac{m}{n} f(1) \) for all \( n \in \mathbb{N} \), and this clearly also holds for all negative integers \( m \) as well (why?). We have shown that \( f(x) = xf(1) \) for all rational numbers \( x \in \mathbb{Q} \).

(b) Let \( x \in \mathbb{R} \setminus \mathbb{Q} \). Using the density of \( \mathbb{Q} \) in \( \mathbb{R} \) we can construct a sequence \( \{r_n\} \subset \mathbb{Q} \) s.t. \( r_n \to x \) as \( n \to \infty \).

We have \( f(x) = \lim_{n \to \infty} f(r_n) = \lim_{n \to \infty} (r_n f(1)) = x \cdot f(1) \).

Since \( f \) is continuous on \( \mathbb{R} \). Since \( x \in \mathbb{R} \setminus \mathbb{Q} \) was arbitrary, we deduce that, overall, \( f(x) = x \cdot f(1) \) for all \( x \in \mathbb{R} \).

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Since \( f(cx) = f(x) \) for all \( x \in \mathbb{R} \) we deduce that \( f(x) = f\left(\frac{c}{c}x\right) = f\left(\frac{c}{c}\right)x = x \cdot f(1) \) for all \( x \in \mathbb{R} \).

If \( c \in (0, 1) \), \( f(x) = \lim_{u \to 0} f(x) = \lim_{u \to 0} f\left(\frac{1}{u}x\right) = f(0) \) since \( \frac{1}{u}x \to 0 \) as \( u \to 0 \).

If \( c > 1 \), \( f(x) = \lim_{u \to 0} f(x) = \lim_{u \to 0} f\left(\frac{x}{u}\right) = f(0) \) since \( \frac{x}{u} \to 0 \) as \( u \to \infty \).

The above shows that \( f(x) = f(0) \) for all \( x \in \mathbb{R} \).

(recall that \( x \) is fixed)