Exam # 2  Solutions

Question 1 (20 points):

Question 2 (20 points):

Question 3 (20 points):

Question 4 (20 points):

Question 5 (20 points):

TOTAL SCORE:

Notes:
1. You have 60 minutes to complete the exam.
2. For full credit you must show your work completely. Simply writing down an answer without justifying it will receive very little partial credit.
3. NO TEXTBOOKS or NOTES are allowed while you take this exam.
1. (20 points) The quantity demanded of a certain product, \( q \), is given in terms of \( p \), the price, by

\[
q = 1000e^{-0.02p}
\]

(a) Write the revenue, \( R \), as a function of price.

(b) Find the rate of change of the revenue with respect to price.

(c) Find the revenue and the rate of change of the revenue with respect to price when the price is $10. Interpret your answers in economic terms.

\[
\begin{align*}
(\text{a}) \quad R &= pq = p \cdot 1000e^{-0.02p} = 1000pe^{-0.02p} \\
(\text{b}) \quad \frac{dR}{dp} &= \left(1000p \cdot e^{-0.02p}\right)' = \left(1000p\right)'e^{-0.02p} + 1000p \cdot (e^{-0.02p})' = \\
&= 1000e^{-0.02p} + 1000p \cdot (-0.02)e^{-0.02p} \\
&= 1000e^{-0.02p} - 20pe^{-0.02p} \\
&= (1000 - 20p)e^{-0.02p} \\
(\text{c}) \quad \text{The revenue when } p = 10 \text{ is } R(10) = 1000 \cdot 10 \cdot e^{-0.02 \cdot 10} = 10000e^{-0.2} \approx $8187.30 \\
\text{When the price of the product is $10 the revenue is $8187.30.}
\]

The rate of change of the revenue with respect to price when the price is $10 is

\[
\frac{dR}{dp}(10) = (1000 - 20 \cdot 10)e^{-0.02 \cdot 10} = 800e^{-0.2} \approx $654.98
\]

If the price of the item increases by $1 to $11, the revenue will increase by approximately $654.98.
2. (20 points) The quantity, \( Q \) mg, of nicotine in the body \( t \) minutes after a cigarette is smoked is given by \( Q(t) = f(t) \).
(a) Interpret the statements \( f(20) = 0.36 \) and \( f'(20) = -0.002 \) in terms of nicotine. What are the units of the numbers 20, 0.36, and \(-0.002\)?
(b) Use the information given in part (a) to estimate \( f(21) \) and \( f(30) \).

\[
\begin{align*}
\text{(a)} & \quad f(20) = 0.36 \text{ means that 20 minutes after the cigarette is smoked there are 0.36 mg of nicotine in the body.} \\
f'(20) = -0.002 \text{ means that 20 minutes after the cigarette is smoked the nicotine in the body is decreasing at a rate of 0.002 mg/minute.}
\end{align*}
\]

The units of 20 are minutes.

The units of 0.36 are mg.

The units of \(-0.002\) are mg/minute.

\[
\begin{align*}
\text{(b)} & \quad f(21) = f(20) + f'(20)(21-20) = 0.36 - 0.002 = 0.358 \text{ mg} \\
f(30) = f(20) + f'(20)(30-20) = 0.36 - 0.02 = 0.34 \text{ mg}
\end{align*}
\]

21 minutes after the cigarette was smoked there will be 0.358 mg of nicotine in the body.

30 minutes after the cigarette was smoked there will be 0.34 mg of nicotine in the body.
3. (20 points) At exactly two of the labeled points in the figure, the derivative $f'$ is 0; the second derivative $f''$ is not zero at any of the labeled points. In the table, give the signs of $f, f', f''$ at each marked point.

<table>
<thead>
<tr>
<th>Point</th>
<th>$f$</th>
<th>$f'$</th>
<th>$f''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$-$</td>
<td>$0$</td>
<td>$+$</td>
</tr>
<tr>
<td>$B$</td>
<td>$+$</td>
<td>$0$</td>
<td>$-$</td>
</tr>
<tr>
<td>$C$</td>
<td>$+$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$D$</td>
<td>$-$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

- Since $A, D$ are below the $x$-axis $\Rightarrow f$ is negative at $A$ and $D$
- Since $B, C$ are above the $x$-axis $\Rightarrow f$ is positive at $B$ and $C$
- Since the graph of $f$ is neither increasing nor decreasing at $A, B$ $\Rightarrow f'$ is zero at $A$ and $B$
- Since the graph of $f$ is decreasing at $C$ $\Rightarrow f'$ is negative at $C$
- Since the graph of $f$ is increasing at $D$ $\Rightarrow f'$ is positive at $D$
- Since the graph of $f$ is concave up at $A, D$ $\Rightarrow f''$ is positive at $A$ and $D$
- Since the graph of $f$ is concave down at $B, C$ $\Rightarrow f''$ is negative at $B$ and $C$
4. (20 points) An industrial production process costs $C(q)$ million dollars to produce $q$ million units; these units then sell for $R(q)$ million dollars. If $C(2.1) = 5.1$, $R(2.1) = 6.9$, $MC(2.1) = 0.6$, and $MR(2.1) = 0.7$, calculate

(a) The profit earned by producing 2.1 million units.

(b) The approximate change in revenue if production increases from 2.1 to 2.14 million units.

(c) The approximate change in revenue if production decreases from 2.1 to 2.05 million units.

(d) The approximate change in profit in parts (b) and (c).

\[
\begin{align*}
(a) & \quad \overline{\tau}(2.1) = R(2.1) - C(2.1) = 6.9 - 5.1 = 1.8 \\
(b) & \quad R(2.14) = R(2.1) + R'(2.1)(2.14 - 2.1) = 0.7 \cdot 0.04 = 0.028 \\
& \quad \Rightarrow R(2.14) - R(2.1) = R'(2.1)(2.14 - 2.1) = 0.028 \\
& \quad \Rightarrow \text{The approximate change in revenue if production increases from 2.1 to 2.14 million units is $0.028$ million dollars.} \\
(c) & \quad R(2.05) = R(2.1) + R'(2.1)(2.05 - 2.1) = -0.035 \\
& \quad \Rightarrow R(2.05) - R(2.1) = R'(2.1)(2.05 - 2.1) = -0.035 \\
& \quad \Rightarrow \text{The approximate change in revenue if production decreases from 2.1 to 2.05 million units is $-0.035$ million dollars.} \\
(d) & \quad C(2.14) = C(2.1) + MC(2.1)(2.14 - 2.1) = 5.1 + 0.6 \cdot 0.04 = 5.124 \\
& \quad \Rightarrow \overline{\tau}(2.14) = R(2.14) - C(2.14) = 1.804 \\
& \quad \Rightarrow \text{The change in profit if production increases from 2.1 to 2.14 million units is $1.804 - 1.8 = 0.004$ million dollars.} \\
& \quad C(2.05) = C(2.1) + MC(2.1)(2.05 - 2.1) = 5.13 \text{ million} \\
& \quad \Rightarrow \overline{\tau}(2.05) = R(2.05) - C(2.05) = 6.865 - 5.13 = 1.735 \text{ million} \\
& \quad \Rightarrow \text{The change in profit if production decreases from 2.1 to 2.05 million units is $1.735 - 1.8 = -0.065$ million dollars.}
\end{align*}
\]
5. (20 points) Suppose $1000$ is deposited in a bank account that pays 8% annual interest, compounded continuously.
(a) Find a formula $f(t)$ for the balance $t$ years after the initial deposit.
(b) Find $f(10)$ and $f'(10)$ and explain what your answers mean in terms of money.

(a) The balance after $t$ years is $f(t) = 1000e^{0.08t}$

(b) $f(10) = 1000e^{0.08 \cdot 10} = 1000e^{0.8} \approx 2225.54$

The balance after 10 years is $2225.54$. 

To find $f'(10)$, we compute $f'(t)$:

$f'(t) = 1000(e^{0.08t})' = 1000 \cdot 0.08e^{0.08t} = 80e^{0.08t}$

$\Rightarrow f'(10) = 80e^{0.08 \cdot 10} = 80e^{0.8} \approx 178.04$

After 10 years, the balance is growing at a rate of about $178$ per year.