Some combinatorics of $\hat{sl}_n$ crystals
(different models for $\hat{sl}_n$ crystals and how they are related)

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\textsuperscript{1}Notes "Explicit crystal maps between cylindric plane partitions, multi-partitions and multi-segments" are available at www-math.mit.edu/~ptingley/
Outline

1 Motivation
   - Crystals, Characters and Combinatorics
   - What does “understand" mean anyway?
   - Two examples

2 Some structures I understand
   - The multi-partition realization of $B(\Lambda)$
   - Understanding the infinity crystal
   - Relationship with the Kyoto path model

3 A structure I only partly understand
   - Fayers’ crystals
   - Relationship with monomial crystals (partly conjectural)
The adjoint representation of $\mathfrak{sl}_3$

There are 6 one-dimensional weight spaces and 1 two-dimensional weight space. The generators $F_1$ and $F_2$ act between weight spaces. There are 4 distinguished one-dimensional spaces in the middle. If we use $U_q(\mathfrak{sl}_3)$ and 'rescale' the operators, then 'at $q=0$', they match up. You get a colored directed graph. We will only work with highest weight crystals, and ignore the functions $\text{wt}, \epsilon, \phi$: $B \rightarrow P$ usually included in the definition. These are recoverable from the graph (up to global shifting by a null weight in the affine case).
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\[ \text{2 dim} \]

\[ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \]

$\text{Peter Tingley (MIT)}$
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Then the combinatorics gives information about representation theory, and vise-versa.
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\( \hat{\mathfrak{sl}}_n \) crystals

\( \hat{\mathfrak{sl}}_n \) crystals for \( n \geq 3 \) is the Kac-Moody algebra with Dynkin diagram \( \bullet \overset{u}{\rightarrow} \overset{u}{\rightarrow} \overset{u}{\rightarrow} \overset{u}{\rightarrow} \overset{u}{\rightarrow} \overset{u}{\rightarrow} \cdots \).

\( \hat{\mathfrak{sl}}_n \) is (almost) generated by \( \{ E_i, F_i \} \) for \( 0 \leq i \leq n-1 \) subject to the relations that for each pair \( 0 \leq i < j \leq n-1 \), \( \{ E_i, F_i, E_j, F_j \} \) generate a copy of \( \mathfrak{sl}_3 \) if \( |i-j| = 1 \) mod \( n \), \( \mathfrak{sl}_2 \times \mathfrak{sl}_2 \) otherwise.

Fix \( n \geq 3 \). An (infinite) \( n \)-colored directed graph is an \( \hat{\mathfrak{sl}}_n \) crystal if, for each pair of colors \( c_i \) and \( c_j \), the graph consisting of all edges of those 2 colors is an \( \hat{\mathfrak{sl}}_3 \) crystal graph if \( |i-j| = 1 \) mod \( n \), \( \hat{\mathfrak{sl}}_2 \times \mathfrak{sl}_2 \) crystal graph otherwise.
\( \hat{\mathfrak{sl}}_n \) crystals

- \( \hat{\mathfrak{sl}}_n \) (for \( n \geq 3 \)) is the Kac-Moody algebra with dynkin diagram

\[ \text{Diagram} \]

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  ![Dynkin diagram](attachment:image.png)

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The infinity crystal

There is a crystal $B^\lambda$ for each dominant weight $\lambda$. 

$\{B^\lambda\}$ forms a directed system. 

The limit of this system is $B^\infty$. 

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- $\{B_\lambda\}$ forms a directed system.
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$B_{\omega_1+2\omega_2} \cup B_{\omega_1+\omega_2}$
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The infinity crystal
Motivation

Crystals, Characters and Combinatorics

The infinity crystal

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In order to understand a model I want to:
Explicitly (non-recursively) describe the vertex set.
If it is a family of models for all highest weights, I want to explicitly describe the embeddings $\mathcal{B}(\Lambda) \hookrightarrow \mathcal{B}(\Lambda')$.
Explicitly describe the limit $\mathcal{B}(\infty)$.
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In order to understand the relationship between two models for $B(\Lambda)$ I want:

* An explicitly description of the unique bijection commuting with the crystal operators
* This description should be "better" than using the crystal operators to get to the highest weight element, then using the crystal operators on the other side to go back down.

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The Misra-Miwa-Hayashi realization of $B_{\Lambda_0}$ for $\hat{\mathfrak{sl}}_3$
The Misra-Miwa-Hayashi realization of $B_{\Lambda_0}$ for \( \hat{\mathfrak{sl}}_3 \)

- We define crystal operators on partitions.
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- $F_{\tilde{2}}$ adds a $\tilde{2}$ colored box.
The Misra-Miwa-Hayashi realization of $B_{\Lambda_0}$ for $\hat{\mathfrak{sl}}_3$

\[
\begin{array}{cccccccc}
\bar{2} & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\
1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\
1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
\end{array}
\]

- $F_{\bar{2}}$ adds a $\bar{2}$ colored box.
\[ F_2 \text{ adds a } \bar{2} \text{ colored box.} \]
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Motivation

Two examples

The Misra-Miwa-Hayashi realization of $B_{\Lambda_0}$ for $\hat{\mathfrak{sl}}_3$

- $F_{\overline{2}}$ adds a $\overline{2}$ colored box.
The Misra-Miwa-Hayashi realization of $B_{\Lambda_0}$ for $\widehat{\mathfrak{sl}_3}$

- $F_2$ adds a $\bar{2}$ colored box.
- $E_2$ would send this partition to 0.
The Misra-Miwa-Hayashi realization of $B_{\Lambda_0}$ for $\hat{\mathfrak{sl}}_3$
The Misra-Miwa-Hayashi realization of $B_{\Lambda_0}$ for $\hat{\mathfrak{sl}}_3$

- Every connected is a copy of $B_{\Lambda_0}$. 

In particular, the subcrystal generated by the empty partition is a model for $B_{\Lambda_0}$. This is not enough to "understand" the model. The subcrystal generated by the empty partition consists exactly of the 3-regular partitions are a single copy of $B_{\Lambda_0}$ (no 3 rows of same length). Now we can say we understand the model.
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- Now we can say we understand the model.
Nakajima’s monomial crystal
Consider monomials on variables $Y_{i,k}^\pm, i \in \mathbb{Z}/n\mathbb{Z}, k \in \mathbb{Z}$.
Nakajima’s monomial crystal

Consider monomials on variables $Y^{-1}_{\bar{1},k}, i \in \mathbb{Z}/n\mathbb{Z}, k \in \mathbb{Z}$

\[
Y^{-2}_{1,15}Y_{2,14}Y_{1,13}Y_{0,10}Y_{9,9}Y_{3,9}Y_{7,7}Y_{5,7}Y_{0,4}Y_{1,1}
\]
Motivation

Nakajima’s monomial crystal

\[ Y_{\bar{1},15} Y_{\bar{2},14} Y_{\bar{1},13} Y_{0,10} Y_{\bar{1},9} Y_{3,9} Y_{\bar{1},7} Y_{\bar{3},7} Y_{\bar{1},5} Y_{0,4} Y_{\bar{1},1} \]

- Consider monomials on variables \( Y_{\bar{i},k}^{\pm 1} \), \( i \in \mathbb{Z}/n\mathbb{Z} \), \( k \in \mathbb{Z} \) (here \( n = 4 \)).
Nakajima’s monomial crystal

Consider monomials on variables $Y_{i,k}^{\pm 1}$, $i \in \mathbb{Z}/n\mathbb{Z}$, $k \in \mathbb{Z}$ (here $n = 4$).

Define operators $E_i$ and $F_i$ on this set. We show $E\bar{1}$, $F\bar{1}$. 

$$Y_{\bar{1},15} Y_{\bar{2},14} Y_{\bar{1},13} Y_{\bar{0},10} Y_{\bar{1},9} Y_{3,9} Y_{\bar{1},7} Y_{3,7} Y_{\bar{1},5} Y_{\bar{0},4} Y_{\bar{1},1}$$
Nakajima’s monomial crystal

\[ Y_{\overline{1},15} Y_{\overline{2},14} Y_{\overline{1},13} Y_{\overline{0},10} Y_{\overline{1},9} Y_{3,9} Y_{\overline{1},7} Y_{3,7} Y_{\overline{1},5} Y_{0,4} Y_{\overline{1},1} \]

- Consider monomials on variables \( Y_{i,k}^{\pm 1} \), \( i \in \mathbb{Z}/n\mathbb{Z}, k \in \mathbb{Z} \) (here \( n = 4 \)).
- Define operators \( E_{\overline{i}} \) and \( F_{\overline{i}} \) on this set. We show \( E_{\overline{1}}, F_{\overline{1}} \).
- Put a “(" for every \( Y_{\overline{1},k} \) and a “")" for every \( Y_{\overline{1},k}^{-1} \), ordered left to right by decreasing \( k \).
Nakajima’s monomial crystal

\[
\begin{align*}
Y_{\bar{1},15}Y_{\bar{2},14}Y_{\bar{1},13}Y_{\bar{0},10}Y_{\bar{1},9}Y_{\bar{3},9}Y_{\bar{1},7}Y_{\bar{3},7}Y_{\bar{1},5}Y_{\bar{0},4}Y_{\bar{1},1}
\end{align*}
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\[
\left( \begin{array}{c}
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Put a "(" for every $Y_{\bar{1},k}$ and a "")" for every $Y_{\bar{1},k}^{-1}$, ordered left to right by decreasing $k$. 

\[
\begin{pmatrix}
Y_{\bar{1},15} & Y_{\bar{1},13} & Y_{\bar{0},10} & Y_{\bar{1},9} & Y_{\bar{3},9} & Y_{\bar{1},7} & Y_{\bar{3},7} & Y_{\bar{1},5} & Y_{\bar{0},4} & Y_{\bar{1},1}
\end{pmatrix}
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Put a “(" for every $Y_{\bar{1},k}$ and a “")" for every $Y_{\bar{1},k}^{-1}$, ordered left to right by decreasing $k$.

$F_{\bar{\bar{1}}}$ multiplies $m$ by $A_{\bar{1},k+1}^{-1} := Y_{\bar{1},k}^{-1} Y_{\bar{1},k+2}^{-1} Y_{0,k+1}^{-1} Y_{2,k+1}$, where the first uncanceled “(" corresponds to a $Y_{\bar{1},k}$.
Motivation

Nakajima’s monomial crystal

\[
\begin{align*}
Y_{\bar{1},15} & Y_{2,14} Y_{\bar{1},13} Y_{0,10} Y_{\bar{1},9} Y_{3,9} Y_{\bar{1},7} Y_{3,7} Y_{\bar{1},5} Y_{0,4} Y_{\bar{1},1} \\
( & ) & ( & ) & ( & )
\end{align*}
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- Put a "(" for every \( Y_{\bar{1},k} \) and a ")" for every \( Y_{\bar{1},k}^{-1} \), ordered left to right by decreasing \( k \).
- \( F_{\bar{1}} \) multiplies \( m \) by \( A_{\bar{1},k+1}^{-1} := Y_{\bar{1},k}^{-1} Y_{\bar{1},k+2}^{-1} Y_{0,k+1}^{-1} Y_{2,k+1}^{-1} \), where the first uncanceled "(" corresponds to a \( Y_{\bar{1},k} \). Or sends \( m \) to 0 if there is no uncanceled "")".
Nakajima’s monomial crystal

\[
\begin{pmatrix}
Y^{-1}_{1,15} & Y_{2,14}^{-1} & Y_{1,13}^{-2} & Y_{0,10} & Y_{1,9}^{-1} & Y_{3,9} & Y_{1,7}^{-1} & Y_{3,7}^{-1} & Y_{1,5}^{-1} & Y_{0,4}^{-1} & Y_{1,1}^{-1} \\
\end{pmatrix}
\]

- Consider monomials on variables \(Y_{i,k}^{\pm 1}, i \in \mathbb{Z}/n\mathbb{Z}, k \in \mathbb{Z}\) (here \(n = 4\)).
- Define operators \(E_{\bar{i}}\) and \(F_{\bar{i}}\) on this set. We show \(E_{\bar{1}}, F_{\bar{1}}\).
- Put a "(" for every \(Y_{1,k}\) and a "")" for every \(Y_{1,k}^{-1}\), ordered left to right by decreasing \(k\).
- \(F_{\bar{1}}\) multiplies \(m\) by \(A_{\bar{1},k+1}^{-1} := Y_{1,k}^{-1}Y_{1,k+2}^{-1}Y_{0,k+1}^{-1}Y_{2,k+1}^{-1}\), where the first uncanceled "(" corresponds to a \(Y_{1,k}\). Or sends \(m\) to 0 if there is no uncanceled "")".
- \(F_{\bar{1}}\) multiplies \(m\) by \(A_{\bar{1},k-1}^{-1} := Y_{1,k-2}^{-1}Y_{1,k}^{-1}Y_{0,k-1}^{-1}Y_{2,k-1}^{-1}\), where the first uncanceled "")" corresponds to a \(Y_{1,k}^{-1}\).
Nakajima’s monomial crystal

\[
\begin{pmatrix}
Y_{\bar{1},15} & Y_{2,14} & Y_{\bar{1},13} & Y_{0,10} & Y_{\bar{1},9} & Y_{3,9} & Y_{\bar{1},7} & Y_{3,7} & Y_{\bar{1},5} & Y_{0,4} & Y_{\bar{1},1}
\end{pmatrix}
\]
Nakajima’s monomial crystal

\[
\begin{array}{c}
(\quad )
\end{array}
\quad
\begin{array}{c}
(\quad )
\end{array}
\quad
\begin{array}{c}
(\quad )
\end{array}
\quad
\begin{array}{c}
(\quad )
\end{array}
\quad
\begin{array}{c}
(\quad )
\end{array}

\begin{array}{c}
Y_{1,15} Y_{2,14} Y_{-2} Y_{0,10} Y_{9} Y_{3,9} Y_{1,7} Y_{-1} Y_{3,7} Y_{5,1} Y_{-1} Y_{0,4} Y_{1,1}
\end{array}

\begin{array}{c}
F_{1}
\end{array}
\]
Nakajima’s monomial crystal

\[
\begin{array}{c}
\left( \begin{array}{c}
Y_{\bar{1},15}
\end{array} \right) \ast \left( \begin{array}{c}
Y_{\bar{2},14}
\end{array} \right) \left( \begin{array}{c}
Y_{0,10}
\end{array} \right) \left( \begin{array}{c}
Y_{\bar{9},3,9}
\end{array} \right) \left( \begin{array}{c}
Y_{\bar{7},3,7}
\end{array} \right) \left( \begin{array}{c}
Y_{\bar{5},0,4}
\end{array} \right) \left( \begin{array}{c}
Y_{\bar{1},1}
\end{array} \right)
\end{array}
\]

\[
F_{\bar{1}}
\]
Nakajima’s monomial crystal

\[
\begin{pmatrix}
    Y^{-1}_{\bar{1},15} & Y^{-2}_{2,14} & Y^{-2}_{\bar{1},13} & Y_{0,10} & Y_{\bar{1},9} & Y_{3,9} & Y_{\bar{3},7} & Y_{\bar{1},5} & Y_{0,4} & Y_{\bar{1},1}
\end{pmatrix}
\]

\[
F_{\bar{1}}
\]

\[
\begin{pmatrix}
    A^{-1}_{\bar{1},10} & Y_{\bar{1},15} & Y^{-2}_{2,14} & Y_{\bar{1},13} & Y_{0,10} & Y_{\bar{1},9} & Y_{3,9} & Y_{\bar{1},7} & Y_{3,7} & Y_{\bar{1},5} & Y_{0,4} & Y_{\bar{1},1}
\end{pmatrix}
\]
Nakajima’s monomial crystal

\[
\begin{array}{c}
\begin{pmatrix}
Y_{1,15} & Y_{2,14} & Y_{\bar{1},13} & Y_{\bar{0},10} & Y_{\bar{1},9} & Y_{3,9} & Y_{\bar{1},7} & Y_{3,7} & Y_{\bar{1},5} & Y_{\bar{0},4} & Y_{\bar{1},1}
\end{pmatrix}
\end{array}
\]

\[
\begin{array}{c}
\begin{pmatrix}
A^{-1}_{1,10} & Y_{\bar{1},15} & Y_{2,14} & Y_{\bar{1},13} & Y_{\bar{0},10} & Y_{\bar{1},9} & Y_{3,9} & Y_{\bar{1},7} & Y_{3,7} & Y_{\bar{1},5} & Y_{\bar{0},4} & Y_{\bar{1},1}
\end{pmatrix}
\end{array}
\]

\[
F_{\bar{1}}
\]

\[
\begin{array}{c}
| | | |
\end{array}
\]
Nakajima’s monomial crystal

\[(\quad) \quad * \quad (\quad) \quad (\quad) \quad (\quad) \quad (\quad) \quad \]

\[
\begin{align*}
Y_{1,15} & Y_{2,14} Y_{0,10} Y_{1,9} Y_{3,9} Y_{1,7} Y_{3,7} Y_{1,5} Y_{0,4} Y_{1,1} \\
\end{align*}
\]

\[
\begin{align*}
F_{1} \\
A_{1,10}^{-1} Y_{1,15} Y_{2,14} Y_{0,10} Y_{1,9} Y_{3,9} Y_{1,7} Y_{3,7} Y_{1,5} Y_{0,4} Y_{1,1} \\
\end{align*}
\]

\[
\begin{align*}
Y_{1,9} Y_{1,11} Y_{0,10} Y_{2,10} Y_{1,15} Y_{2,14} Y_{0,10} Y_{1,9} Y_{3,9} Y_{1,7} Y_{3,7} Y_{1,5} Y_{0,4} Y_{1,1} \\
\end{align*}
\]
Nakajima’s monomial crystal

\[
(\quad ) \quad \quad (\quad )
\]

\[
Y_{\bar{1},15} Y_{\bar{2},14} Y_{\bar{0},10} Y_{\bar{1},9} Y_{\bar{3},9} Y_{\bar{1},7} Y_{\bar{3},7} Y_{\bar{1},5} Y_{0,4} Y_{\bar{1},1}
\]

\[
F_{\bar{1}}
\]

\[
A_{\bar{1},10}^{-1} Y_{\bar{1},15} Y_{\bar{2},14} Y_{\bar{0},10} Y_{\bar{1},9} Y_{\bar{3},9} Y_{\bar{3},7} Y_{\bar{1},5} Y_{0,4} Y_{\bar{1},1}
\]

\[
Y_{\bar{1},9} Y_{\bar{1},11} Y_{0,10} Y_{\bar{2},10} Y_{\bar{1},15} Y_{\bar{2},14} Y_{\bar{0},10} Y_{\bar{1},9} Y_{\bar{3},9} Y_{\bar{1},7} Y_{\bar{3},7} Y_{\bar{1},5} Y_{0,4} Y_{\bar{1},1}
\]
Motivation

Two examples

Nakajima’s monomial crystal

\[
\begin{array}{c}
\begin{array}{c}
( ) \quad ( ) \quad \ast \quad ( ) \quad ( ) \\
Y_{\bar{1},15}Y_{\bar{2},14}Y_{\bar{2},14}Y_{\bar{2},14}Y_{\bar{1},13}Y_{0,10}Y_{\bar{1},9}Y_{3,9}Y_{\bar{3},7}Y_{\bar{3},7}Y_{\bar{1},5}Y_{0,4}Y_{\bar{1},1}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
F_{\bar{1}}
\end{array}
\]

\[
\begin{array}{c}
A_{\bar{1},10}^{-1}Y_{\bar{1},15}Y_{\bar{2},14}Y_{\bar{2},14}Y_{\bar{2},14}Y_{\bar{1},13}Y_{0,10}Y_{\bar{1},9}Y_{3,9}Y_{\bar{3},7}Y_{\bar{3},7}Y_{\bar{1},5}Y_{0,4}Y_{\bar{1},1}
\end{array}
\]

\[
\begin{array}{c}
Y_{\bar{1},9}Y_{\bar{1},11}Y_{0,10}Y_{\bar{2},10}Y_{\bar{1},15}Y_{\bar{2},14}Y_{\bar{2},14}Y_{\bar{1},13}Y_{0,10}Y_{\bar{1},9}Y_{3,9}Y_{\bar{3},7}Y_{\bar{3},7}Y_{\bar{1},5}Y_{0,4}Y_{\bar{1},1}
\end{array}
\]

\[
\begin{array}{c}
Y_{\bar{1},15}Y_{\bar{2},14}Y_{\bar{2},14}Y_{\bar{2},14}Y_{\bar{1},13}Y_{\bar{1},11}Y_{0,10}Y_{\bar{2},10}Y_{\bar{3},9}Y_{\bar{1},7}Y_{\bar{3},7}Y_{\bar{1},5}Y_{0,4}Y_{\bar{1},1}
\end{array}
\]
Nakajima’s monomial crystal

\[
\begin{pmatrix}
Y_{\bar{1},15} & Y_{\bar{2},14} & Y_{\bar{0},10} & Y_{\bar{1},9} & Y_{\bar{3},9} & Y_{\bar{1},7} & Y_{\bar{3},7} & Y_{\bar{1},5} & Y_{\bar{0},4} & Y_{\bar{1},1}
\end{pmatrix}
\]

- The component generated by a dominant monomial is a highest weight crystal
Nakajima’s monomial crystal

\[
(\phantom{\text{\textbullet}}) \quad (\phantom{\text{\textbullet}}) \quad (\phantom{\text{\textbullet}}) \quad (\phantom{\text{\textbullet}}) \quad (\phantom{\text{\textbullet}})
\]

\[
Y_{\bar{1},15} Y_{\bar{2},14} Y_{\bar{1},13} Y_{0,10} Y_{\bar{1},9} Y_{3,9} Y_{\bar{1},7} Y_{3,7} Y_{\bar{1},5} Y_{0,4} Y_{\bar{1},1}
\]

- The component generated by a dominant monomial is a highest weight crystal (provided \( n \) is even, and some parity conditions hold).
Nakajima’s monomial crystal

The component generated by a dominant monomial is a highest weight crystal (provided \( n \) is even, and some parity conditions hold). CAUTION: other components are not all crystals.
Nakajima’s monomial crystal

\[
\begin{pmatrix}
Y_{\bar{1},15} & Y_{2,14} & Y_{\bar{1},13} & Y_{0,10} & Y_{\bar{1},9} & Y_{\bar{3},9} & Y_{\bar{1},7} & Y_{\bar{3},7} & Y_{\bar{1},5} & Y_{0,4} & Y_{\bar{1},1}
\end{pmatrix}
\]

- The component generated by a dominant monomial is a highest weight crystal (provided \( n \) is even, and some parity conditions hold).

CAUTION: other components are not all crystals. Hernandez and Nakajima have described some other components that are crystals.
Nakajima’s monomial crystal

The component generated by a dominant monomial is a highest weight crystal (provided \(n\) is even, and some parity conditions hold).
CAUTION: other components are not all crystals
Hernandez and Nakajima have described some other components that are crystals.

In particular, the component generated by \(Y_{0,0}\) is a copy of \(B(\Lambda_0)\).
Nakajima’s monomial crystal

\[
\left(\begin{array}{c}
Y_{\bar{1},15}^1 \quad Y_{\bar{1},13}^{-2} \quad Y_{\bar{0},10}^{-1} \quad Y_{\bar{1},9} \quad Y_{\bar{3},9} \quad Y_{\bar{1},7} \quad Y_{\bar{3},7}^{-1} \quad Y_{\bar{1},5}^{-1} \quad Y_{\bar{0},4}^{-1} \quad Y_{\bar{1},1}^{-1}
\end{array}\right)
\]

- The component generated by a dominant monomial is a highest weight crystal (provided \( n \) is even, and some parity conditions hold). CAUTION: other components are not all crystals. Hernandez and Nakajima have described some other components that are crystals.
- In particular, the component generated by \( Y_{\bar{0},0} \) is a copy of \( B(\Lambda_0) \). This holds for ALL \( n \geq 3 \).
Nakajima’s monomial crystal

\[
\begin{pmatrix}
Y_{\bar{1},15} & Y_{\bar{2},14} & Y_{\bar{1},13} & Y_{\bar{0},10} & Y_{\bar{1},9} & Y_{\bar{3},9} & Y_{\bar{1},7} & Y_{\bar{3},7} & Y_{\bar{1},5} & Y_{\bar{0},4} & Y_{\bar{1},1}
\end{pmatrix}
\]

- The component generated by a dominant monomial is a highest weight crystal (provided \( n \) is even, and some parity conditions hold). CAUTION: other components are not all crystals. Hernandez and Nakajima have described some other components that are crystals.

- In particular, the component generated by \( Y_{0,0} \) is a copy of \( B(\Lambda_0) \). This holds for ALL \( n \geq 3 \).

- I do not understand this crystal, since I do not know a good rule for checking if a given monomial is in \( B(\Lambda_0) \).
Nakajima’s monomial crystal

\[
\begin{pmatrix}
Y_{\bar{1},15} & Y_{\bar{2},14} & Y_{\bar{1},13} & Y_{0,10} & Y_{\bar{3},9} & Y_{\bar{1},7} & Y_{\bar{3},7} & Y_{\bar{1},5} & Y_{0,4} & Y_{\bar{1},1}
\end{pmatrix}
\]

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- I also do not know an explicit isomorphism with the Misra-Miwa model.
Nakajima’s monomial crystal

\[
\begin{array}{ccccccc}
Y_{\bar{1},15} & Y_{2,14} & Y_{\bar{1},13} & Y_{0,10} & Y_{3,9} & Y_{\bar{1},7} & Y_{\bar{3},7} & Y_{\bar{1},5} & Y_{0,4} & Y_{\bar{1},1}
\end{array}
\]

- The component generated by a dominant monomial is a highest weight crystal (provided \( n \) is even, and some parity conditions hold). CAUTION: other components are not all crystals. Hernandez and Nakajima have described some other components that are crystals.

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- I do know an explicit isomorphism with modification of the Misra-Miwa model due to Fayers.
Nakajima’s monomial crystal

\[
\begin{pmatrix}
Y_{1,15} & Y_{2,14} & Y_{-1,13} & Y_{0,10} & Y_{9,9} & Y_{7,9} & Y_{-1,7} & Y_{3,7} & Y_{-1,5} & Y_{0,4} & Y_{1,1}
\end{pmatrix}
\]

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- I also do not know an explicit isomorphism with the Misra-Miwa model.
- I do know an explicit isomorphism with modification of the Misra-Miwa model due to Fayers. I’ll mention this at the end.
The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)
The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

- For $\ell = 4$, use 4-tuples of charged partitions.
The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

- For $\ell = 4$, use 4-tuples of charged partitions.
For \( \ell = 4 \), use 4-tuples of charged partitions. The charge is a residue mod \( n = 3 \), which is the color of the vertex of the partition.
The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

For $\ell = 4$, use 4-tuples of charged partitions. The charge is a residue mod $n = 3$, which is the color of the vertex of the partition. We choose the multi-charge $(\bar{0}, \bar{1}, \bar{1}, \bar{2})$. 
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\[ \lambda^{(0)} \quad \lambda^{(1)} \quad \lambda^{(2)} \quad \lambda^{(3)} \]
The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

For $\ell = 4$, use 4-tuples of charged partitions. The charge is a residue mod $n = 3$, which is the color of the vertex of the partition. We choose the multi-charge ($\bar{0}, \bar{1}, \bar{1}, \bar{2}$). These must satisfy a “shifted containment" condition.
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The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

- $\lambda^{(0)}$
- $\lambda^{(1)}$
- $\lambda^{(2)}$
- $\lambda^{(3)}$

- Again $F_0$ will add a box colored $\bar{0}$. 

The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

- Again $F_0$ will add a box colored $\overline{0}$.
- Again places you can add a box are labeled "(" and places you can remove a box are labeled ")".
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- Again $F_0$ will add a box colored $\bar{0}$.
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Some structures I understand

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- $\lambda^{(0)}$
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- Again $F_0$ will add a box colored $\bar{0}$.
- Again places you can add a box are labeled "(" and places you can remove a box are labeled ")".
The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

Some structures I understand

- Again $F_0$ will add a box colored $0$.
- Again places you can add a box are labeled "(" and places you can remove a box are labeled ")".

\[ \lambda^{(0)} \]
\[ \lambda^{(1)} \]
\[ \lambda^{(2)} \]
\[ \lambda^{(3)} \]
Some structures I understand

The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

- Again $F_0$ will add a box colored $0$.
- Again places you can add a box are labeled "(" and places you can remove a box are labeled ")".
The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

- Again $F_0$ will add a box colored $\bar{0}$.
- Again places you can add a box are labeled "(" and places you can remove a box are labeled ")".
- The brackets are reordered appropriately, and $F_0$ adds the box corresponding to the first uncanceled "(".
The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

- Again $F_0$ will add a box colored $\bar{0}$.
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$$
\lambda^{(0)} = 3 \quad 0 \quad -3
$$

$$
\lambda^{(1)} = 3 \quad 0 \quad -3
$$

$$
\lambda^{(2)} = 3 \quad 0 \quad -3 \quad 6
$$

$$
\lambda^{(3)} = 3 \quad 0 \quad -3 \quad 6
$$
The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

- Again $F_0$ will add a box colored $\bar{0}$.
- Again places you can add a box are labeled "(" and places you can remove a box are labeled ")".
- The brackets are reordered appropriately, and $F_0$ adds the box corresponding to the first uncanceled "(".
The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

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Some structures I understand

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Some structures I understand

The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)
Every connected component is a copy of $B_\Lambda$. 
The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

$\Lambda = \Lambda_0 + 2\Lambda_1 + \Lambda_2$

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Some structures I understand
The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

$\Lambda = \Lambda_0 + 2\Lambda_1 + \Lambda_2$

- Every connected component is a copy of $B_\Lambda$.
- The "3-regular" multi-partitions form a single copy of $B_\Lambda$. 
The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

Every connected component is a copy of $B_\Lambda$.

The “3-regular” multi-partitions form a single copy of $B_\Lambda$.

3-regular means no three differently colored rows have the same length.
The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

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The multi-partition realization of $B(\Lambda)$ (JMMO, FLOTW)

\[ \Lambda = \Lambda_0 + 2\Lambda_1 + \Lambda_2 \]

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Every connected component is a copy of $B_\Lambda$.
The "3-regular" multi-partitions form a single copy of $B_\Lambda$.
3-regular means no three differently colored rows have the same length.
Our example is not 3-regular.
I would say we do understand this model for $B(\Lambda)$.
Understanding embeddings and $B(\infty)$
An $\ell$-tuple of partitions satisfying the shifted containment conditions fits together into a three dimensional picture called a “cylindric partition".
An $\ell$-tuple of partitions satisfying the shifted containment conditions fits together into a three dimensional picture called a "cylindric partition".

Consider $n = 3$, $\ell = 2$, and multi-charge $(0, 1)$. 
Understanding embeddings and $B(\infty)$
Some structures I understand
Understanding the infinity crystal

Understanding embeddings and $B(\infty)$
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Understanding the infinity crystal

Understanding embeddings and $B(\infty)$
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Some structures I understand

Understanding the infinity crystal
Understanding embeddings and $B(\infty)$
Understanding embeddings and $B(\infty)$

A “multi-segment” $\ell$-tuple of partitions satisfying the shifted containment conditions fits together into a three dimensional picture called a “cylindric partition”. Consider $n = 3$, $\ell = 2$, and multi-charge $(-\bar{0}, -\bar{1})$.

A cylindric partition is in $B(\Lambda)$ if and only if it does not have three differently colored piles of the same height.

To understand the embedding $B(\Lambda_0 + \Lambda_1) \hookrightarrow B(2\Lambda_1 + \Lambda_1)$, consider the “dual” $n$-tuple of partitions. Just shift the cylindric partition so that this dual $n$-tuple does not change.

For $B(\infty)$, just record the vertical piles, not the arrangement into an $\ell$-tuple of partitions.
Some structures I understand

Understanding the infinity crystal

Understanding embeddings and $B(\infty)$
Understanding embeddings and $B(\infty)$

A "multi-segment" $\ell$-tuple of partitions satisfying the shifted containment conditions fits together into a three dimensional picture called a "cylindric partition". Consider $n=3$, $\ell=2$, and multi-charge $(\bar{0}, \bar{1})$. A cylindric partition is in $B(\Lambda)$ if and only if it does not have three differently colored piles of the same height. To understand the embedding $B(\Lambda_0 + \Lambda_1) \hookrightarrow B(2\Lambda_1 + \Lambda_1)$, consider the "dual" $n$-tuple of partitions. Just shift the cylindric partition so that this dual $n$-tuple does not change. For $B(\infty)$, just record the vertical piles, not the arrangement into an $\ell$-tuple of partitions.
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Understanding embeddings and \( B(\infty) \)
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Understanding embeddings and $B(\infty)$

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Understanding embeddings and $B(\infty)$
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Understanding embeddings and $B(\infty)$

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Some structures I understand

Understanding the infinity crystal

Understanding embeddings and $B(\infty)$

A "multi-segment" $\ell$-tuple of partitions satisfying the shifted containment conditions fits together into a three dimensional picture called a "cylindric partition".

Consider $n=3$, $\ell=2$, and multi-charge $\bar{\bar{0}}, \bar{\bar{1}}$.

A cylindric partition is in $B(\Lambda)$ if and only if it does not have three differently colored piles of the same height.

To understand the embedding $B(\Lambda_0+\Lambda_1) \hookrightarrow B(2\Lambda_1+\Lambda_1)$, consider the "dual" $n$-tuple of partitions.

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Understanding embeddings and $B(\infty)$

A "multi-segment" $\ell$-tuple of partitions satisfying the shifted containment conditions fits together into a three dimensional picture called a "cylindric partition". Consider $n = 3$, $\ell = 2$, and multicharge $(\bar{0}, \bar{1})$. A cylindric partition is in $B(\Lambda)$ if and only if it does not have three differently colored piles of the same height. To understand the embedding $B(\Lambda_0 + \Lambda_1) \hookrightarrow B(2\Lambda_1 + \Lambda_1)$, consider the "dual" $n$-tuple of partitions. Just shift the cylindric partition so that this dual $n$-tuple does not change. For $B(\infty)$, just record the vertical piles, not the arrangement into an $\ell$-tuple of partitions.
Understanding embeddings and $B(\infty)$
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Understanding the infinity crystal

Understanding embeddings and $B(\infty)$

A "multi-segment"
Relation to the Kyoto path model
Relation to the Kyoto path model
Relation to the Kyoto path model
Relation to the Kyoto path model
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Relation to the Kyoto path model

[Diagram showing a structure related to the Kyoto path model]
Relation to the Kyoto path model
Relation to the Kyoto path model
Relation to the Kyoto path model

\[
\begin{array}{c|c|c|c|c|c|c}
\hline
& 1 & 2 & 1 & 1 & 0 & 1 \\
\hline
\end{array}
\]
Relation to the Kyoto path model

![Diagram of crystal structures]

Some structures I understand

Relationship with the Kyoto path model
Relation to the Kyoto path model
Relation to the Kyoto path model
Relation to the Kyoto path model

Some structures I understand

Relationship with the Kyoto path model

Peter Tingley (MIT)
Relation to the Kyoto path model
Relation to the Kyoto path model

\[ \cdots 1 \bar{2} \quad \otimes \quad \bar{0} \quad 1 \quad \otimes \quad \bar{0} \quad 2 \quad \otimes \quad \bar{1} \quad \bar{2} \quad \otimes \quad \bar{1} \quad \bar{2} \quad \otimes \quad 1 \quad 1 \quad \otimes \quad \bar{0} \quad 1 \]

Peter Tingley (MIT)
The “horizontal" crystal
The “horizontal" crystal

Define new operators $E_i$ and $F_i$ on the set of partitions.
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Define new operators $E_{\bar{i}}$ and $F_{\bar{i}}$ on the set of partitions.

for $\bar{i} = \bar{2}$, construct a string of brackets as before, but ordered lexicographically by height, then right to left.
The “horizontal" crystal

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Define new operators $E_{\bar{i}}$ and $F_{\bar{i}}$ on the set of partitions.

for $\bar{i} = \bar{2}$, construct a string of brackets as before, but ordered lexicographically by height, then right to left.
Define new operators $E_{\tilde{i}}$ and $F_{\tilde{i}}$ on the set of partitions.

for $\tilde{i} = \tilde{2}$, construct a string of brackets as before, but ordered lexicographically by height, then right to left.
The "horizontal" crystal

- Define new operators $E_{\bar{i}}$ and $F_{\bar{i}}$ on the set of partitions.
- for $\bar{i} = \bar{2}$, construct a string of brackets as before, but ordered lexicographically by height, then right to left.
Define new operators $E_{\vec{i}}$ and $F_{\vec{i}}$ on the set of partitions.

for $\vec{i} = \vec{2}$, construct a string of brackets as before, but ordered lexicographically by height, then right to left.
The “horizontal" crystal

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The “horizontal” crystal

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for $\bar{i} = \bar{2}$, construct a string of brackets as before, but ordered lexicographically by height, then right to left.
The “horizontal” crystal

- Define new operators $E_{\bar{\imath}}$ and $F_{\bar{\imath}}$ on the set of partitions.
- for $\bar{\imath} = \bar{2}$, construct a string of brackets as before, but ordered lexicographically by height, then right to left.
The “horizontal” crystal

- Define new operators $E_{\vec{i}}$ and $F_{\vec{i}}$ on the set of partitions.
- for $\vec{i} = \vec{2}$, construct a string of brackets as before, but ordered lexicographically by height, then right to left.
The “horizontal” crystal

- Define new operators $E_{\bar{i}}$ and $F_{\bar{i}}$ on the set of partitions.
- for $\bar{i} = \bar{2}$, construct a string of brackets as before, but ordered lexicographically by height, then right to left.
Define new operators $E_\vec{i}$ and $F_\vec{i}$ on the set of partitions.

for $\vec{i} = \vec{2}$, construct a string of brackets as before, but ordered lexicographically by height, then right to left.

$F_\vec{2}$ adds the box corresponding to the first uncanceled $\vec{\circ}$.
The “horizontal” crystal

- Define new operators $E_{\bar{i}}$ and $F_{\bar{i}}$ on the set of partitions.
- For $\bar{i} = \bar{2}$, construct a string of brackets as before, but ordered lexicographically by height, then right to left.
- $F_{\bar{2}}$ adds the box corresponding to the first uncanceled $\circ$. 
The "horizontal" crystal

- Define new operators $E_{\bar{i}}$ and $F_{\bar{i}}$ on the set of partitions.
- for $i = 2$, construct a string of brackets as before, but ordered lexicographically by height, then right to left.
- $F_{\overline{2}}$ adds the box corresponding to the first uncanceled $\circ$. 
The “horizontal" crystal

Define new operators $E_{ar{i}}$ and $F_{ar{i}}$ on the set of partitions. For $\bar{i} = \bar{2}$, construct a string of brackets as before, but ordered lexicographically by height, then right to left. $F_{\bar{2}}$ adds the box corresponding to the first uncanceled $\dag$. The component generated by the empty partition is a copy of $B(\Lambda_0)$. CAUTION: other components are not all crystals. A partition is in $B(\Lambda_0)$ if and only if there are no illegal hooks. I cautiously say this is "understood". One can actually read the boxes according to ANY slope. The same result is true, although definition of "illegal hook" is a bit more complicated.

Peter Tingley (MIT)
The “horizontal" crystal

- The component generated by the empty partition is a copy of $B(\Lambda_0)$. 
The “horizontal" crystal

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The “horizontal" crystal

One can actually read the boxes according to ANY slope
The “horizontal" crystal

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The “horizontal" crystal

- One can actually read the boxes according to ANY slope
A structure I only partly understand

Fayers’ crystals

The “horizontal” crystal

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The "horizontal" crystal

- One can actually read the boxes according to ANY slope
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Horizontal to monomial
Horizontal to monomial

- There is a natural isomorphism between $B(\Lambda_0)$ realized using the horizontal crystal and $B(\Lambda_0)$ realized using the monomial crystal.
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Each “inner" corner corresponds to a $Y$ and each “outer" corner to a $Y^{-1}$.
A structure I only partly understand

Relationship with monomial crystals (partly conjectural)

Horizontal to monomial

There is a natural isomorphism between $B(\Lambda_0)$ realized using the horizontal crystal and $B(\Lambda_0)$ realized using the monomial crystal. Each "inner" corner corresponds to a $Y$ and each "outer" corner to a $Y^{-1}$.

Some other slopes correspond to known models. The Misra-Miwa crystal (at many slopes, although for some only the highest component works). A recent crystal due to Chris Berg.

Peter Tingley (MIT)
Korea, September 2009
A structure I only partly understand

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There is a natural isomorphism between $B(\Lambda_0)$ realized using the
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Peter Tingley (MIT)
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Each "inner" corner corresponds to a $Y$ and each "outer" corner to a $Y_{-1}$.

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The Misra-Miwa crystal (at many slopes, although for some only the highest component works).

A recent crystal due to Chris Berg.

Peter Tingley (MIT)

$\tilde{sl}_n$ crystals

Korea, September 2009
Horizontal to monomial

There is a natural isomorphism between $B(\Lambda_0)$ realized using the horizontal crystal and $B(\Lambda_0)$ realized using the monomial crystal. Each "inner" corner corresponds to a $Y_{\lambda, \mu}$ and each "outer" corner to a $Y_{\lambda', \mu'}$. Some other slopes correspond to known models. The Misra-Miwa crystal (at many slopes, although for some only the highest component works). A recent crystal due to Chris Berg.

$Y_{\lambda, \mu}$
Horizontal to monomial

There is a natural isomorphism between $\mathcal{B}(\Lambda_0)$ realized using the horizontal crystal and $\mathcal{B}(\Lambda_0)$ realized using the monomial crystal. Each "inner" corner corresponds to a $Y$ and each "outer" corner to a $Y^{-1}$. Some other slopes correspond to known models. The Misra-Miwa crystal (at many slopes, although for some only the highest component works). A recent crystal due to Chris Berg.
A structure I only partly understand

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$Y_{\bar{3}, 11} Y^{-1}_{\bar{2}, 12}$
A structure I only partly understand

Relationship with monomial crystals (partly conjectural)

Horizontal to monomial

\[ Y_{\frac{3}{2}, 11} Y_{-1}^{-1} \]

Peter Tingley (MIT)  
Korea, September 2009  
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Horizontal to monomial

\[
Y_{\bar{3},11} Y_{\bar{2},12}^{-1}
\]
Horizontal to monomial

$Y_{\tilde{3},11} Y_{\tilde{2},12} Y_{\tilde{2},8}$
A structure I only partly understand

Horizontal to monomial crystals (partly conjectural)

There is a natural isomorphism between $B(\Lambda_0)$ realized using the horizontal crystal and $B(\Lambda_0)$ realized using the monomial crystal. Each "inner" corner corresponds to a $Y$ and each "outer" corner to a $Y^{-1}$.

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Peter Tingley (MIT)
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Korea, September 2009
A structure I only partly understand
Relationship with monomial crystals (partly conjectural)

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$Y_{3,11}Y_{2,12}^{-1}Y_{2,8}Y_{1,9}^{-1}$
A structure I only partly understand

Relationship with monomial crystals (partly conjectural)

Horizontal to monomial

\[
Y_{\bar{3},11} Y_{2,12}^{-1} Y_{\bar{2},8}^{-1} Y_{\bar{1},9}^{-1}
\]
A structure I only partly understand

Relationship with monomial crystals (partly conjectural)

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A structure I only partly understand

Relationship with monomial crystals (partly conjectural)

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$Y_{3,11} Y_{2,12}^{-1} Y_{2,8} \ Y_{1,9}^{-1} \ Y_{2,6}$
A structure I only partly understand

Relationship with monomial crystals (partly conjectural)

Horizontal to monomial

There is a natural isomorphism between $B(\Lambda_0)$ realized using the horizontal crystal and $B(\Lambda_0)$ realized using the monomial crystal.

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Some other slopes correspond to known models. The Misra-Miwa crystal (at many slopes, although for some only the highest component works). A recent crystal due to Chris Berg.

Peter Tingley (MIT)
A structure I only partly understand

Relationship with monomial crystals (partly conjectural)

Horizontal to monomial

\[ Y_{3,11} Y_{2,12}^{-1} Y_{2,8} Y_{1,9}^{-1} Y_{2,6} \]
Horizontal to monomial

\[ Y_{\overline{3},11} Y_{\overline{2},12} Y_{\overline{2},8} Y_{\overline{1},9} Y_{\overline{2},6} Y_{\overline{1},7} \]
A structure I only partly understand

Relationship with monomial crystals (partly conjectural)

Horizontal to monomial

There is a natural isomorphism between $B(\Lambda^0)$ realized using the horizontal crystal and $B(\Lambda^0)$ realized using the monomial crystal.

Each “inner” corner corresponds to a $Y$ and each “outer” corner to a $Y^{-1}$.

Some other slopes correspond to known models.

The Misra-Miwa crystal (at many slopes, although for some only the highest component works).

A recent crystal due to Chris Berg.

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The Misra-Miwa crystal (at many slopes, although for some only the highest component works).

A recent crystal due to Chris Berg.

$Y_{\bar{3},11}^{-1} Y_{\bar{2},12} Y_{\bar{2},8} Y_{\bar{1},9}^{-1} Y_{\bar{2},6} Y_{\bar{1},7}^{-1}$
Horizontal to monomial

\[ Y_{\frac{3}{1},11} Y_{\frac{2}{12}} Y_{\frac{2}{8}} Y_{\frac{1}{9}} Y_{\frac{2}{6}} Y_{\frac{1}{7}} \]
A structure I only partly understand

Relationship with monomial crystals (partly conjectural)

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There is a natural isomorphism between $B(\Lambda_0)$ realized using the horizontal crystal and $B(\Lambda_0)$ realized using the monomial crystal. Each "inner" corner corresponds to a $Y$ and each "outer" corner to a $Y^{-1}$. Some other slopes correspond to known models. The Misra-Miwa crystal (at many slopes, although for some only the highest component works). A recent crystal due to Chris Berg.

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$\hat{s}l_n$ crystals

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Relationship with monomial crystals (partly conjectural)

Horizontal to monomial

\[
\begin{align*}
Y &_{\bar{3}, 11} \quad Y_{2, 12}^{-1} \quad Y_{\bar{2}, 8}^{-1} \quad Y_{2, 6} \quad Y_{\bar{1}, 7}^{-1} \quad Y_{3, 5} \quad Y_{2, 6}^{-1} \quad Y_{\bar{1}, 5}
\end{align*}
\]
Horizontally to monomial

\[
Y_{\frac{3}{11}} \ Y^{-1}_{\frac{2}{12}} \ Y_{\frac{2}{8}} \ Y^{-1}_{\frac{1}{9}} \ Y_{\frac{2}{6}} \ Y^{-1}_{\frac{1}{7}} \ Y_{\frac{3}{5}} \ Y^{-1}_{\frac{2}{6}} \ Y_{\frac{1}{5}} \ Y^{-1}_{\frac{3}{11}}
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Questions

The monomial crystals come out of deep structures (q-characters, quiver varieties). Can these be used to give an algebraic/geometric explanation for Fayers' other crystal structures?

- Positive evidence: Kim has shown that the Misra-Miwa crystal is naturally isomorphic to a known modification of the monomial crystal.

The monomial crystals work for higher levels. There are also (multi) partition models at higher levels. Do Fayers' crystals generalize beyond level 1?

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