Affine Mirković-Vilonen polytopes

Peter Tingley (MIT)

Includes joint work with Thomas Dunlap, Pierre Baumann and Joel Kamnitzer

Loyola U. Chicago, Jan 18, 2012

1Slides available at www-math.mit.edu/~ptingley/
1 Motivation and background
   - Lie groups and representations
   - Crystals and combinatorics
   - PBW bases and polytopes

2 Affine MV polytopes
   - $\hat{\mathfrak{sl}}_2$ and its root system
   - $\hat{\mathfrak{sl}}_2$ MV polytopes (combinatorial construction)
   - Other affine (symmetric) types
   - Sketch of quiver varieties proof

3 Ongoing work
Group theory, from my point of view, is about symmetry. One might consider symmetries of:
- Objects.
- Sets (permutation representations).
- Vector spaces ($\mathbb{C}^n$) (called linear representations, or just representations).

There are many applications, in math and beyond.
- Galois theory; non-solvable polynomial equations.
- Physics (both classical and more modern).
- Knot theory/low dimensional topology.
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Motivation and background

Lie groups and representations

Lie groups

This talk is concerned with various infinite groups:

- $SO(n)$
- $SP(n)$
- $SL(n)$

**Definition:**

$SL_n$ is the group of $n \times n$ matrices with determinant 1, under ordinary matrix multiplication.

For a good while, I will just talk about the example of $SL_3$.

$SL_3$ naturally acts by symmetries on a 3 dimensional vector space:

$$
\begin{bmatrix}
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  v_2 \\
  v_3
\end{bmatrix} \rightarrow
\begin{bmatrix}
  v_1 \\
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But there are actually many more linear representations of $SL_3$, and we want to understand them all.
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Other representations of $SL(3)$

There is a 9-dimensional representation of $SL(3)$; the tensor product of two copies of the 3-dimensional representation:

- If \{v_1, v_2, v_3\} is a basis for the three-dimensional representation $V$, $V \otimes V$ has a basis consisting of pairs $(v_i, v_j)$, written $v_i \otimes v_j$.

- $M \in SL(3)$ acts on $v \otimes w$ by $M \cdot (v \otimes w) = Mv \otimes Mw$, where e.g. $(2v_1 + v_3) \otimes (v_2 + 4v_3) = 2v_1 \otimes v_2 + v_3 \otimes v_2 + 8v_1 \otimes v_3 + 4v_3 \otimes v_3$.

This breaks up as the sum of two simpler representations:

- $\text{Sym}^2(V)$ is spanned by \{\(v_1 \otimes v_1, v_2 \otimes v_2, v_3 \otimes v_3, v_1 \otimes v_2 + v_2 \otimes v_1, v_1 \otimes v_3 + v_3 \otimes v_1, v_2 \otimes v_3 + v_3 \otimes v_2\}.

- $\wedge^2(V)$ is spanned by \{\(v_1 \otimes v_2 - v_2 \otimes v_1, v_1 \otimes v_3 - v_3 \otimes v_1, v_2 \otimes v_3 - v_3 \otimes v_2\}.

We are really interested in irreducible representations, i.e. ones that can't be broken up in this way.
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- We are really interested in **irreducible** representations, i.e. ones that can’t be broken up in this way.
From Lie groups to Lie algebras

People who study Lie groups often actually work with Lie algebras. This is a calculusy idea: you can understand a lot about a continuous group by understanding the tangent space near the identity element. The tangent space \( \mathfrak{sl}_3 \) to the identity in \( \text{SL}(3) \) is matrices with trace 0 (since \( \det(I + \epsilon M) = 1 \mod \epsilon^2 \) if and only if \( \text{tr}(M) = 0 \)).

Multiplication in the group gives rise to the bracket on the Lie algebra. For \( \mathfrak{sl}_3 \), this is given by \( [A, B] = AB - BA \).

A representation of a Lie algebra is a vector space \( V \) and an endomorphism of \( V \) for each element, such that \( [A, B]v = ABv - BAv \).

Every representation of a Lie group gives rise to a representation of the Lie algebra. This is almost an equivalence, and for \( \text{SL}(n) \) it is exactly an equivalence.
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- A representation of a Lie algebra is a vector space $V$ and an endomorphism of $V$ for each element, such that $[A, B]v = ABv - BAv$.
- Every representation of a Lie group gives rise to a representation of the Lie algebra. This is almost an equivalence, and for $SL(n)$ it is exactly an equivalence.
Example: $\mathfrak{sl}_3$

The standard generators of $\mathfrak{sl}_3$ are:

$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

$E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $F_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Any representation of $\mathfrak{sl}_3$ decomposes (as a vector space) into the direct sum of the simultaneous eigenspaces for the diagonal matrices (weight spaces). The standard generators move you from one weight space to another in a predictable way.
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Crystals (by example)

Motivation and background
Crystals and combinatorics

There are 6 one-dimensional weight spaces and 1 two-dimensional weight space. The generators $F_1$ and $F_2$ act between weight spaces. There are 4 distinguished one-dimensional spaces in the middle. If we use $U_q(sl_3)$ and 'rescale' the operators, then 'at $q = 0$', they match up. You get a colored directed graph.
Consider the adjoint representation of $\mathfrak{sl}_3$ (i.e. $\mathfrak{sl}_3$ acting on itself).
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Often the vertices of the crystal graph can be parametrized by combinatorial objects.
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There is a simple tensor product rule for crystals, which correctly predicts tensor product multiplicities for representations. For $\mathfrak{sl}_2$, crystals are just directed segments. Let's take the tensor product of two of these. For other types, just treat each $\mathfrak{sl}_2$ independently!

As an example, consider $V \otimes V$ for $\mathfrak{sl}_3$. 
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- For $\mathfrak{sl}_2$, crystals are just directed segments.
- Let's take the tensor product of two of these.
- For other types, just treat each $\mathfrak{sl}_2$ independently!
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Motivation and background
Crystals and combinatorics

Loyola U. Chicago, Jan 18, 2012
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The infinity crystal

Motivation and background

Crystals and combinatorics

There is a crystal $B(\lambda)$ for each dominant integral weight $\lambda$. 

$\{B(\lambda)\}$ forms a directed system. The limit of this system is $B(\infty)$. 

Peter Tingley (MIT)
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I am interested in realizing all this combinatorics explicitly.
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That means understanding $B(\infty)$, and all the embeddings $B(\lambda) \hookrightarrow B(\infty)$. 
Motivation and background

PBW bases and polytopes

PBW bases and parameterizations of $B(\infty)$

The roots of a Lie algebra $\mathfrak{g}$ are the non-zero weight spaces of $\mathfrak{g}$ acting on itself (these define a set of vector closed under reflection; a root system).

For each reduced expression $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$, Lusztig defines an order $\alpha_{i_1} = \beta_1 < \beta_2 < \cdots < \beta_N$ on positive roots, and elements $F_{\beta_1} \cdots F_{\beta_N}$ in $U_q(\mathfrak{g})$. 

$\{F(\ell_{\beta_1}) \cdots F(\ell_{\beta_N})\}$ is a crystal basis for $U_q(\mathfrak{g})$ (the PBW basis).

In particular, these monomials index $B(\infty)$.

There is one indexing of $B(\infty)$ for each expression for $w_0$.

We record each such monomial as a path in weight space, and this is the 1-skeleton of the MV polytope.

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Motivation and background

PBW bases and polytopes

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Motivation and background
PBW bases and polytopes

Characterization of MV polytopes

It is natural to ask which polytopes show up in this way. For rank two cases, this can be done using tropical Plücker relations. Equivalently, the conditions can be given in terms of the two diagonals:

• Both have slope at most the corresponding simple root.
• For one of the diagonals, the slope is equal to the simple root.

Remarkably, understanding rank 2 is enough! A polytope is MV exactly if all its rank 2 faces are MV polytopes of the right types.
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All edges are parallel to roots (here \( \alpha_1 \), \( \alpha_2 \), \( \alpha_1 + \alpha_2 \)).

Any path from bottom to top that passes through exactly one edge parallel to each root uniquely determines an MV polytope.

The crystal operators \( f_i \) just increase the length of the bottom edge in a well chosen path, and adjust the rest of the polytope accordingly.

\( B(\lambda) \subset B(\infty) \) is the set of polytopes contained in a given ambient polytope.

Tensor product multiplicities are given by counting MV polytopes subject to conditions on top and bottom edge lengths.
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- All edges are parallel to roots (here $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$).
- Any path from bottom to top that passes though exactly one edge parallel to each root uniquely determines an $MV$ polytope.
- The crystal operators $f_i$ just increase the length of the bottom edge in a well chosen path, and adjust the rest of the polytope accordingly.
- $B(\lambda) \subset B(\infty)$ is the set of polytopes contained in a given ambient polytope.
- Tensor product multiplicities are given by counting $MV$ polytopes subject to conditions on top and bottom edge lengths.
$\hat{\mathfrak{sl}}_2$ and its root system

The Kac-Moody algebra with Dynkin diagram $\begin{array}{cc}
0 & 1 \\
\end{array}$.

It can also be realized as $\mathfrak{sl}_2 \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$.

You should think of this as (an extension of) the loop group for $\text{SL}(2)$ (i.e. maps from $\mathbb{C}^*$ into $\text{SL}(2)$, under point-wise multiplication), something physicists care about.

The theory of crystals works just fine (for highest weight integrable representations), but until recently there were no MV polytopes.

The "root system" (i.e. non-zero weight spaces of $\hat{\mathfrak{sl}}_2$ acting on itself) is $\alpha_0, \alpha_1, \alpha_0 + \delta, \alpha_1 + \delta, \alpha_0 + 2\delta, \alpha_1 + 2\delta$. 

Peter Tingley (MIT)

Affine MV polytopes

Loyola U. Chicago, Jan 18, 2012

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\[ \hat{\mathfrak{sl}}_2 \text{ is the Kac-Moody algebra with Dynkin diagram} \]
\[ 0 \cdots 1 \]

\[ \hat{\mathfrak{sl}}_2 \text{ and its root system} \]

- \( \hat{\mathfrak{sl}}_2 \) and its root system
\( \widehat{sl}_2 \) and its root system

- \( \widehat{sl}_2 \) is the Kac-Moody algebra with Dynkin diagram
  \[
  0 \longrightarrow 1
  \]
- It can also be realized as \( \widehat{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d \).
\( \hat{\mathfrak{sl}}_2 \) and its root system

- \( \hat{\mathfrak{sl}}_2 \) is the Kac-Moody algebra with Dynkin diagram
  \[
  0 \longrightarrow 1
  \]

- It can also be realized as \( \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d \).
- You should think of this as (an extension of) the loop group for \( SL(2) \) (i.e. maps from \( \mathbb{C}^* \) into \( SL(2) \), under point-wise multiplication), something physicists care about.
\( \hat{sl}_2 \) and its root system

- \( \hat{sl}_2 \) is the Kac-Moody algebra with Dynkin diagram
  
  \[
  0 \quad \quad \quad \quad 1
  \]

- It can also be realized as \( \hat{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d \).

- You should think of this as (an extension of) the loop group for \( SL(2) \) (i.e. maps from \( \mathbb{C}^* \) into \( SL(2) \), under point-wise multiplication), something physicists care about.

- The theory of crystals works just fine (for highest weight integrable representations), but until recently there were no MV polytopes.
\(\hat{\mathfrak{sl}}_2\) and its root system

- \(\hat{\mathfrak{sl}}_2\) is the Kac-Moody algebra with Dynkin diagram
  \[
  \begin{array}{c}
  0 \\
  \end{array}
  \begin{array}{c}
  1 \\
  \end{array}
  \]
- It can also be realized as \(\mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d\).
- You should think of this as (an extension of) the loop group for \(SL(2)\) (i.e. maps from \(\mathbb{C}^*\) into \(SL(2)\), under point-wise multiplication), something physicists care about.
- The theory of crystals works just fine (for highest weight integrable representations), but until recently there were no MV polytopes.
- The “root system” (i.e. non-zero weight spaces of \(\hat{\mathfrak{sl}}_2\) acting on itself) is
  \[
  \begin{array}{c}
  \alpha_0 + 3\delta \\
  \alpha_0 + 2\delta \\
  \alpha_0 + \delta \\
  \alpha_0 \\
  \end{array}
  \begin{array}{c}
  k\delta \\
  \alpha_1 + 3\delta \\
  \alpha_1 + 2\delta \\
  \alpha_1 + \delta \\
  \alpha_1 \\
  \end{array}
  \]
Theorem (B-D-K-T)
There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize $B(\infty)$. They have all the properties we want.

- $(\mu_k - \mu_{k-1}, \omega_1) \leq 0$ and $(\mu_k - \mu_{k-1}, \omega_0) \leq 0$, with at least one of these being an equality.

- $(\mu_k - \mu_{k-1}, \omega_0) \geq 0$ and $(\mu_k - \mu_{k-1}, \omega_1) \geq 0$, with at least one of these being an equality.

- Either $\lambda = \lambda$, or $\lambda$ is obtained from $\lambda$ by adding or removing a single part of size the width of the polytope.

- $\lambda_1, \lambda_1$ are at most the width of the polytope.
Theorem (B-D-K-T)

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize $\mathfrak{sl}_2(\mathbb{C})$.

They have all the properties we want.

- $(\mu_k - \mu_{k-1}, \omega_1) \leq 0$ and $(\mu_k - \mu_{k-1}, \omega_0) \leq 0$, with at least one of these being an equality.

- $(\mu_k - \mu_{k-1}, \omega_0) \geq 0$ and $(\mu_k - \mu_{k-1}, \omega_1) \geq 0$, with at least one of these being an equality.

- Either $\lambda = \lambda$, or $\lambda$ is obtained from $\lambda$ by adding or removing a single part of size the width of the polytope.

- $\lambda_1$, $\lambda_2$ are at most the width of the polytope.
$\mathfrak{sl}_2$ MV polytopes

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize $B(\infty)$. They have all the properties we want.

- $\mu_k - \mu_{k-1}, \omega_1 \leq 0$ and $\mu_k - \mu_{k-1}, \omega_0 \leq 0$, with at least one of these being an equality.
- $\mu_k - \mu_{k-1}, \omega_0 \geq 0$ and $\mu_k - \mu_{k-1}, \omega_1 \geq 0$, with at least one of these being an equality.
- Either $\lambda = \lambda$, or $\lambda$ is obtained from $\lambda$ by adding or removing a single part of size the width of the polytope.
- $\lambda_1, \lambda_1$ are at most the width of the polytope.
\( \hat{\mathfrak{sl}_2} \) MV polytopes

Theorem (B-D-K-T)

There is a unique decorated polytope of this type for any choice of edge lengths on the right side.

These, along with natural crystal operators, realize \( B(\infty) \).

They have all the properties we want.

- \( (\mu_k - \mu_{k-1}, \omega_1) \leq 0 \) and \( (\mu_k - \mu_{k-1}, \omega_0) \leq 0 \), with at least one of these being an equality.

- \( (\mu_k - \mu_{k-1}, \omega_0) \geq 0 \) and \( (\mu_k - \mu_{k-1}, \omega_1) \geq 0 \), with at least one of these being an equality.

- Either \( \lambda = \lambda \), or \( \lambda \) is obtained from \( \lambda \) by adding or removing a single part of size the width of the polytope.

- \( \lambda_1, \lambda_1 \) are at most the width of the polytope.
Theorem (B-D-K-T)
There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize $B(\infty)$.

They have all the properties we want.

• $(\mu_k - \mu_{k-1}, \omega_1) \leq 0$ and $(\mu_k - \mu_{k-1}, \omega_0) \leq 0$, with at least one of these being an equality.

• $(\mu_k - \mu_{k-1}, \omega_0) \geq 0$ and $(\mu_k - \mu_{k-1}, \omega_1) \geq 0$, with at least one of these being an equality.

• Either $\lambda = \lambda'$, or $\lambda'$ is obtained from $\lambda$ by adding or removing a single part of size the width of the polytope.

• $\lambda_1, \lambda_1$ are at most the width of the polytope.
Theorem (B-D-K-T)

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize $B_\infty$. They have all the properties we want.

- $(\mu_k - \mu_{k-1}, \omega_1) \leq 0$ and $(\mu_k - \mu_{k-1}, \omega_0) \leq 0$, with at least one of these being an equality.
- $(\mu_k - \mu_{k-1}, \omega_0) \geq 0$ and $(\mu_k - \mu_{k-1}, \omega_1) \geq 0$, with at least one of these being an equality.
- Either $\lambda = \lambda$, or $\lambda$ is obtained from $\lambda$ by adding or removing a single part of size the width of the polytope.
- $\lambda_1, \lambda_1$ are at most the width of the polytope.
\( \hat{\mathfrak{sl}}_2 \) MV polytopes

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize \( B(\infty) \).

- \( \mu_k - \mu_{k-1}, \omega_1 \leq 0 \) and \( \mu_k - \mu_{k-1}, \omega_0 \leq 0 \), with at least one of these being an equality.
- \( \mu_k - \mu_{k-1}, \omega_0 \geq 0 \) and \( \mu_k - \mu_{k-1}, \omega_1 \geq 0 \) with at least one of these being an equality.
- Either \( \lambda = \lambda \), or \( \lambda \) is obtained from \( \lambda \) by adding or removing a single part of size the width of the polytope.
- \( \lambda_1, \lambda_1 \) are at most the width of the polytope.
Affine MV polytopes

\(\hat{\mathfrak{sl}}_2\) MV polytopes

Theorem (B-D-K-T)

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize \(B(\infty)\).

They have all the properties we want.

- \((\mu_k - \mu_{k-1}, \omega_1) \leq 0 \) and \((\mu_k - \mu_{k-1}, \omega_0) \leq 0\), with at least one of these being an equality.
- \((\mu_k - \mu_{k-1}, \omega_0) \geq 0 \) and \((\mu_k - \mu_{k-1}, \omega_1) \geq 0 \) with at least one of these being an equality.
- Either \(\lambda = \lambda\), or \(\lambda\) is obtained from \(\lambda\) by adding or removing a single part of size the width of the polytope.
- \(\lambda_1, \lambda_1\) are at most the width of the polytope.
\textbf{Theorem (B-D-K-T)}

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize $B(\infty)$.

They have all the properties we want.

- $(\mu_k - \mu_{k-1}, \omega_1) \leq 0$ and $(\mu_k - \mu_{k-1}, \omega_0) \leq 0$, with at least one of these being an equality.
- $(\mu_k - \mu_{k-1}, \omega_0) \geq 0$ and $(\mu_k - \mu_{k-1}, \omega_1) \geq 0$, with at least one of these being an equality.
- Either $\lambda = \lambda$, or $\lambda$ is obtained from $\lambda$ by adding or removing a single part of size the width of the polytope.
- $\lambda_0, \lambda_1$ are at most the width of the polytope.
\( \hat{\mathfrak{sl}_2} \) MV polytopes

Theorem (B-D-K-T)

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize \( B(\infty) \).

They have all the properties we want.

- \((\mu_k - \mu_{k-1}, \omega_1) \leq 0\) and \((\mu_k - \mu_{k-1}, \omega_0) \leq 0\), with at least one of these being an equality.
- \((\mu_k - \mu_{k-1}, \omega_0) \geq 0\) and \((\mu_k - \mu_{k-1}, \omega_1) \geq 0\), with at least one of these being an equality.
- Either \( \lambda = \lambda \), or \( \lambda \) is obtained from \( \lambda \) by adding or removing a single part of size the width of the polytope.
- \( \lambda_1, \lambda_1 \) are at most the width of the polytope.
\( \hat{\mathfrak{sl}}_2 \) MV polytopes

Theorem (B-D-K-T)

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize \( B(\infty) \). They have all the properties we want.

- \((\mu_k - \mu_{k-1}, \omega_1) \leq 0 \) and \((\mu_k - \mu_{k-1}, \omega_0) \leq 0\), with at least one of these being an equality.
- \((\mu_k - \mu_{k-1}, \omega_0) \geq 0 \) and \((\mu_k - \mu_{k-1}, \omega_1) \geq 0\), with at least one of these being an equality.
- Either \( \lambda = \lambda \), or \( \lambda \) is obtained from \( \lambda \) by adding or removing a single part of size the width of the polytope.
- \( \lambda_1, \lambda_1 \) are at most the width of the polytope.
Theorem (B-D-K-T)

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize \( \mathbf{B}(\infty) \).

They have all the properties we want.

\begin{itemize}
\item \( (\mu_k - \mu_{k-1}, \omega_1) \leq 0 \) and \( (\mu_k - \mu_{k-1}, \omega_0) \leq 0 \), with at least one of these being an equality.
\item \( (\mu_k - \mu_{k-1}, \omega_0) \geq 0 \) and \( (\mu_k - \mu_{k-1}, \omega_1) \geq 0 \) with at least one of these being an equality.
\item Either \( \lambda = \lambda \), or \( \lambda \) is obtained from \( \lambda \) by adding or removing a single part of size the width of the polytope.
\item \( \lambda_1, \lambda_1 \) are at most the width of the polytope.
\end{itemize}
Theorem (B-D-K-T)

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize $\mathfrak{sl}_2$. They have all the properties we want.

- $\mu_k - \mu_{k-1}, \omega_1 \leq 0$ and $\mu_k - \mu_{k-1}, \omega_0 \leq 0$, with at least one of these being an equality.
- $\mu_k - \mu_{k-1}, \omega_0 \geq 0$ and $\mu_k - \mu_{k-1}, \omega_1 \geq 0$, with at least one of these being an equality.
- Either $\lambda = \lambda$, or $\lambda$ is obtained from $\lambda$ by adding or removing a single part of size the width of the polytope.
- $\lambda_1, \lambda_{1-} \leq \text{width of the polytope}$. 

Peter Tingley (MIT)
\( \hat{\mathfrak{sl}}_2 \) MV polytopes

Theorem (B-D-K-T)

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize \( \hat{\mathfrak{b}}(\infty) \).

They have all the properties we want.

- \((\mu_k - \mu_{k-1}, \omega_1) \leq 0 \) and \((\mu_k - \mu_{k-1}, \omega_0) \leq 0\), with at least one of these being an equality.

- \((\mu_k - \mu_{k-1}, \omega_0) \geq 0 \) and \((\mu_k - \mu_{k-1}, \omega_1) \geq 0\), with at least one of these being an equality.

- Either \( \lambda = \lambda \), or \( \lambda \) is obtained from \( \lambda \) by adding or removing a single part of size the width of the polytope.

- \( \lambda_1, \lambda_1 \) are at most the width of the polytope.
Theorem (B-D-K-T)
There is a unique decorated polytope of this type
for any choice of edge lengths on the right side.
These, along with natural crystal operators,
realize $\mathfrak{sl}_2$. They have all the properties we want.

• $\mu_k - \mu_{k-1}, \omega_1 \leq 0$ and $\mu_k - \mu_{k-1}, \omega_0 \leq 0$, with at least one of these being an equality.

• $\mu_k - \mu_{k-1}, \omega_0 \geq 0$ and $\mu_k - \mu_{k-1}, \omega_1 \geq 0$, with at least one of these being an equality.

• Either $\lambda = \lambda$, or $\lambda$ is obtained from $\lambda$ by adding or removing a single part of size the width of the polytope.

• $\lambda_1, \lambda_1$ are at most the width of the polytope.
\( \hat{\mathfrak{sl}}_2 \) MV polytopes

Theorem (B-D-K-T)

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize \( B(\infty) \).

They have all the properties we want.

- \( (\mu_k - \mu_{k-1}, \omega_1) \leq 0 \) and \( (\mu_k - \mu_{k-1}, \omega_0) \leq 0 \), with at least one of these being an equality.

- \( (\mu_k - \mu_{k-1}, \omega_0) \geq 0 \) and \( (\mu_k - \mu_{k-1}, \omega_1) \geq 0 \) with at least one of these being an equality.

- Either \( \lambda = \lambda \), or \( \lambda \) is obtained from \( \lambda \) by adding or removing a single part of size the width of the polytope.

- \( \lambda_1, \lambda_1 \) are at most the width of the polytope.
Theorem (B-D-K-T)

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize $\mathfrak{sl}_2(\infty)$. They have all the properties we want.

- $(\mu_k - \mu_{k-1}, \omega_1) \leq 0$ and $(\mu_k - \mu_{k-1}, \omega_0) \leq 0$, with at least one of these being an equality.
- $(\mu_k - \mu_{k-1}, \omega_0) \geq 0$ and $(\mu_k - \mu_{k-1}, \omega_1) \geq 0$, with at least one of these being an equality.

- Either $\lambda = \lambda$, or $\lambda$ is obtained from $\lambda$ by adding or removing a single part of size the width of the polytope.

- $\lambda_1, \lambda_1$ are at most the width of the polytope.
\[ \hat{\mathfrak{sl}}_2 \text{ MV polytopes} \]

**Theorem (B-D-K-T)**

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize $\hat{\mathfrak{sl}}_2(\infty)$.

They have all the properties we want.

- \((\mu_k - \mu_{k-1}, \omega_1) \leq 0\) and \((\mu_k - \mu_{k-1}, \omega_0) \leq 0\), with at least one of these being an equality.
- \((\mu_k - \mu_{k-1}, \omega_0) \geq 0\) and \((\mu_k - \mu_{k-1}, \omega_1) \geq 0\), with at least one of these being an equality.
- Either $\lambda = \lambda$, or $\lambda$ is obtained from $\lambda$ by adding or removing a single part of size the width of the polytope.
- $\lambda_1, \lambda_1$ are at most the width of the polytope.
\( \text{\hat{sl}_2} \) MV polytopes

Theorem (B-D-K-T)
There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize \( B(\infty) \).

They have all the properties we want.

- \((\mu_k - \mu_{k-1}, \omega_1) \leq 0\) and \((\mu_k - \mu_{k-1}, \omega_0) \leq 0\), with at least one of these being an equality.

- \((\mu_k - \mu_{k-1}, \omega_0) \geq 0\) and \((\mu_k - \mu_{k-1}, \omega_1) \geq 0\), with at least one of these being an equality.

- Either \( \lambda = \lambda \), or \( \lambda \) is obtained from \( \lambda \) by adding or removing a single part of size the width of the polytope.

- \( \lambda_1, \lambda_1 \) are at most the width of the polytope.
\[ \mu_1 = \mu_2 = \cdots = \mu_1 = \mu_2 = \cdots \]

Theorem (B-D-K-T)

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize \( B(\infty) \). They have all the properties we want.

- \( (\mu_k - \mu_{k-1}, \omega_1) \leq 0 \) and \( (\mu_k - \mu_{k-1}, \omega_0) \leq 0 \), with at least one of these being an equality.
- \( (\mu_k - \mu_{k-1}, \omega_0) \geq 0 \) and \( (\mu_k - \mu_{k-1}, \omega_1) \geq 0 \), with at least one of these being an equality.
- Either \( \lambda = \lambda \), or \( \lambda \) is obtained from \( \lambda \) by adding or removing a single part of size the width of the polytope.
- \( \lambda_1, \lambda_1 \) are at most the width of the polytope.
Theorem (B-D-K-T)
There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize $B(\infty)$. They have all the properties we want.

- $(\mu_k - \mu_{k-1}, \omega_1) \leq 0$ and $(\mu_k - \mu_{k-1}, \omega_0) \leq 0$, with at least one of these being an equality.
- $(\mu_k - \mu_{k-1}, \omega_0) \geq 0$ and $(\mu_k - \mu_{k-1}, \omega_1) \geq 0$, with at least one of these being an equality.
- Either $\lambda = \lambda$, or $\lambda$ is obtained from $\lambda$ by adding or removing a single part of size the width of the polytope.
- $\lambda_1, \lambda_1$ are at most the width of the polytope.
Theorem (B-D-K-T)

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize $B(\infty)$.

They have all the properties we want:

- $(\mu_k - \mu_{k-1}, \omega_1) \leq 0$ and $(\mu_k - \mu_{k-1}, \omega_0) \leq 0$, with at least one of these being an equality.

- $(\mu_k - \mu_{k-1}, \omega_0) \geq 0$ and $(\mu_k - \mu_{k-1}, \omega_1) \geq 0$, with at least one of these being an equality.

- Either $\lambda = \lambda$, or $\lambda$ is obtained from $\lambda$ by adding or removing a single part of size the width of the polytope.

- $\lambda_1, \lambda_1$ are at most the width of the polytope.
\textbf{$\hat{\mathfrak{sl}}_2$ MV polytopes}

Theorem (B-D-K-T)

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize $B(\infty)$. They have all the properties we want.

- $\left( \mu_k - \mu_{k-1}, \omega_1 \right) \leq 0$ and $\left( \mu_k - \mu_{k-1}, \omega_0 \right) \leq 0$, with at least one of these being an equality.
- $\left( \mu_k - \mu_{k-1}, \omega_0 \right) \geq 0$ and $\left( \mu_k - \mu_{k-1}, \omega_1 \right) \geq 0$, with at least one of these being an equality.
- Either $\lambda = \lambda$, or $\lambda$ is obtained from $\lambda$ by adding or removing a single part of size the width of the polytope.
- $\lambda_1, \lambda_1$ are at most the width of the polytope.
Theorem (B-D-K-T)
There is a unique decorated polytope of this type
for any choice of edge lengths on the right side.
These, along with natural crystal operators,
realize $B(\infty)$.
They have all the properties we want.
• $(\mu_k - \mu_{k-1}, \omega_1) \leq 0$ and $(\mu_k - \mu_{k-1}, \omega_0) \leq 0$,
  with at least one of these being an equality.
• $(\mu_k - \mu_{k-1}, \omega_0) \geq 0$ and $(\mu_k - \mu_{k-1}, \omega_1) \geq 0$,
  with at least one of these being an equality.
• Either $\lambda = \lambda$, or $\lambda$ is obtained from $\lambda$ by adding or removing a single part
  of size the width of the polytope.
• $\lambda_1, \lambda_1$ are at most the width of the polytope.
Affine MV polytopes

\[ \hat{\mathfrak{sl}}_2 \text{ MV polytopes (combinatorial construction)} \]

**Theorem (B-D-K-T)**

There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize \( B(\infty) \).

They have all the properties we want.

- \((\overline{\mu}_k - \mu_{k-1}, \omega_1) \leq 0 \) and \((\mu_k - \overline{\mu}_{k-1}, \omega_0) \leq 0\), with at least one of these being an equality.
- \((\overline{\mu}_k - \mu_{k-1}, \omega_0) \geq 0 \) and \((\mu_k - \overline{\mu}_{k-1}, \omega_1) \geq 0\), with at least one of these being an equality.
- Either \( \lambda = \overline{\lambda} \), or \( \lambda \) is obtained from \( \overline{\lambda} \) by adding or removing a single part of size the width of the polytope.
- \( \lambda_1, \overline{\lambda}_1 \) are at most the width of the polytope.

\[ \alpha_0 \]

\[ 6\alpha_1 \]

\[ \overline{\mu}_1 \]

\[ 2\delta \]

\[ \alpha_0 + \delta \]

\[ \alpha_1 + \delta \]

\[ \mu_2 \]

\[ \mu_3 \]

\[ \mu_0 \]

\[ \delta \]

\[ 8\delta \]

\[ 2(\alpha_0 + 2\delta) \]

\[ 3\alpha_1 \]

\[ \overline{\mu}_1 \]

\[ \alpha_0 \]

\[ \alpha_1 \]

\[ \alpha_0 + \delta \]

\[ \alpha_1 + \delta \]

\[ \mu_2 \]

\[ \mu_3 \]

\[ \mu_0 \]

\[ \delta \]

\[ 8\delta \]

\[ 2(\alpha_0 + 2\delta) \]

\[ 3\alpha_1 \]

\[ \overline{\mu}_1 \]

\[ \alpha_0 \]

\[ \alpha_1 \]

\[ \overline{\mu}_1 \]

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Theorem (B-D-K-T)
There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize $B(\infty)$. They have all the properties we want.

- $(\mu_k - \mu_{k-1}, \omega_1) \leq 0$ and $(\mu_k - \mu_{k-1}, \omega_0) \leq 0$, with at least one of these being an equality.
- $(\mu_k - \mu_{k-1}, \omega_0) \geq 0$ and $(\mu_k - \mu_{k-1}, \omega_1) \geq 0$, with at least one of these being an equality.

- Either $\lambda = \lambda$, or $\lambda$ is obtained from $\lambda$ by adding or removing a single part of size the width of the polytope.
- $\lambda_1, \lambda_1$ are at most the width of the polytope.
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Theorem (B-D-K-T)
There is a unique decorated polytope of this type for any choice of edge lengths on the right side. These, along with natural crystal operators, realize $B(\infty)$. They have all the properties we want.
The crystal operators

- Increase the length of the bottom right edge by 1.
- Fix the left side in the unique possible way.
- The symmetry $\ast$ rotates by 180 degrees.
- The operators $f_\ast i$: $\ast f_i \ast$ act at the top.
- The operators $e_i$ shrink instead of expand.
- If there is an active $\alpha_0$ diagonal, $e_1 e_\ast 1(P) = e_\ast 1 e_1(P)$.
- Otherwise, $e_1(P) = e_\ast 1(P)$.
- This is essentially enough!
The crystal operators

- Increase the length of the bottom right edge by 1.
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- The operators $f^\ast i = \ast f^i \ast$ act at the top.
- The operators $e_i$ shrink instead of expand.
- If there is an active $\alpha_0$ diagonal, $e_1 e^\ast_1 (P) = e^\ast_1 e_1 (P)$.
- Otherwise, $e_1 (P) = e^\ast_1 (P)$.

This is essentially enough!
The crystal operators

To apply $f_1$:

- Increase the length of the bottom right edge by 1.
- Fix the left side in the unique possible way.
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• Otherwise, $e_1 (P) = e_\ast 1 (P)$.

This is essentially enough! (must also show there is an active $\alpha_0$ diagonal exactly when $\phi_1 ((e_\ast 1)_{\max} (P)) > \varepsilon_\ast 1 (P)$.)
The crystal operators

- The symmetry $\times$ rotates by 180 degrees.
The crystal operators

- The symmetry $\ast$ rotates by 180 degrees.
- The operators $f_i^\ast := \ast f_i \ast$ act at the top.
The crystal operators

- The symmetry $*$ rotates by 180 degrees.
- The operators $f_i^* := *f_i*$ act at the top.
- The operators $e_i$ shrink instead of expand.
The crystal operators

- The symmetry $\ast$ rotates by 180 degrees.
- The operators $f_i^* := \ast f_i \ast$ act at the top.
- The operators $e_i$ shrink instead of expand.
- If there is an active $\alpha_0$ diagonal,
  \[
e_1 e_1^*(P) = e_1^* e_1(P).
  \]
The crystal operators

- The symmetry $\ast$ rotates by 180 degrees.
- The operators $f_i^\ast := \ast f_i \ast$ act at the top.
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- If there is an active $\alpha_0$ diagonal,
  $$e_1 e_1^\ast (P) = e_1^\ast e_1 (P).$$
The crystal operators

- The symmetry $\ast$ rotates by 180 degrees.
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- Otherwise, $e_1(P) = e_1^*(P)$. 
The crystal operators

- The symmetry $\ast$ rotates by 180 degrees.
- The operators $f_i^* := \ast f_i \ast$ act at the top.
- The operators $e_i$ shrink instead of expand.
- If there is an active $\alpha_0$ diagonal,
  \[ e_1 e_1^*(P) = e_1^* e_1(P). \]
- Otherwise, $e_1(P) = e_1^*(P)$. 
The crystal operators

- The symmetry \( * \) rotates by 180 degrees.
- The operators \( f_i^* := *f_i * \) act at the top.
- The operators \( e_i \) shrink instead of expand.
- If there is an active \( \alpha_0 \) diagonal, 
  \[ e_1 e_1^*(P) = e_1^* e_1(P). \]
- Otherwise, \( e_1(P) = e_1^*(P) \).
- This is essentially enough!
The crystal operators

- The symmetry $\ast$ rotates by 180 degrees.
- The operators $f_i^*: = \ast f_i \ast$ act at the top.
- The operators $e_i$ shrink instead of expand.
- If there is an active $\alpha_0$ diagonal,
  $$e_1 e_1^*(P) = e_1^* e_1(P).$$
- Otherwise, $e_1(P) = e_1^*(P)$.
- This is essentially enough!

(must also show there is an active $\alpha_0$ diagonal exactly when $\varphi_1((e_1^*)^{\text{max}}(P)) > \varepsilon_1^*(P)$)
Definition

Let $g$ be an affine Kac-Moody algebra of rank $n+1$. A type $g$ MV polytope is a convex polytope whose edges are all parallel to roots, along with a partition associated to each vertical $n$-face, such that

For each edge $e$ which is a translate of $k\delta$, the sum of $|\lambda_{F}|$ over all faces $F$ incident to $e$ is $k$.

Each 2-face is an MV polytope of the correct type ($\text{sl}_2 \times \text{sl}_2$, $\text{sl}_3$, $B_2$, $G_2$ or $\hat{\text{sl}}_2$), where in the $\hat{\text{sl}}_2$ case one should only consider the decoration for the $n$-faces $F, F'$ which are incident to the 2-face, but do not contain it.

At least in symmetric types, these polytopes can be used to realize $B(\infty)$, and have all the properties we want.
Definition

Let \( \mathfrak{g} \) be an affine Kac-Moody algebra of rank \( n + 1 \). A type \( \mathfrak{g} \) MV polytope is a convex polytope whose edges are all parallel to roots, along with a partitions associated to each vertical \( n \)-face, such that

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Other (symmetric) affine types

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At least in symmetric types, these polytopes can be used to realize \( B(\infty) \), and have all the properties we want.
A picture of an $\hat{sl}_3$ MV polytope
Main theorem

Theorem (Baumann-Kamnitzer-T-)

Assume $g$ is of symmetric affine type. Then there is a bijection between type $g$ MV polytopes and $B(\infty)$ such that:

- The path through the polytope defined by any biconvex ordering on root directions, along with the partition associated to each vertical $n$-face containing the vertical edge in the path, uniquely defines the polytope.
- All candidate paths do correspond to polytopes.
- The action of the crystal operator $f_i$ stretches the bottom edge of an appropriate path.
- The other properties we want hold as well.

We conjecture this holds in non-symmetric types, but our proof fails as we use results from quiver varieties that only hold in symmetric type. An alternative approach would be to develop the theory of affine MV polytopes directly from the notion of affine PBW bases (which exist in all affine types do to work of Beck-Kac)...

we are working on this.

Peter Tingley (MIT)
Main theorem

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We conjecture this holds in non-symmetric types, but our proof fails as we use results from quiver varieties that only hold in symmetric type.

An alternative approach would be to develop the theory of affine MV polytopes directly from the notion of affine PBW bases (which exist in all affine types do to work of Beck-Kac)...we are working on this.
Proof (via quiver varieties)
I will briefly explain a much harder way to prove the realization I gave for \( \hat{sl}_2 \) but one that we know how to generalize.
Proof (via quiver varieties)

- I will briefly explain a much harder way to prove the realization I gave for $\hat{sl}_2$...but one that we know how to generalize.

- Our combinatorial data (diagonals, partitions...) has natural geometric/representation theoretic meaning.
The $\hat{\mathfrak{sl}}_2$ quiver varieties

We consider the double quiver $x_1$, $x_2$, $y_1$, $y_2$.

$\tilde{Q} = 0$

The $\hat{\mathfrak{sl}}_2$ preprojective algebra is the quotient of the path algebra by the relations $x_1 y_1 = x_2 y_2$ and $y_1 x_1 = y_2 x_2$.

$\Lambda(v)$ is the variety of actions of this algebra on a fixed vector space $V$ of dimension $v = (v_0, v_1)$.

A theorem of Kashiwara-Saito says that the irreducible components of all $\Lambda(v)$ parameterize $B(\infty)$.

Theorem (B-K-T) Find the component $Z_b$ corresponding to $b \in B(\infty)$. Fix $\pi \in Z_b$ generic. Then the undecorated polytope corresponding to $b$ is the convex hull of the dimension vectors of the subrepresentations of $(V, \pi)$.

In finite type, this construction gives exactly the MV polytope (follows from Baumann-Kamnitzer). The decoration comes from studying finer structure of $(V, \pi)$. 

Peter Tingley (MIT)
The $\hat{\mathfrak{sl}}_2$ quiver varieties

- We consider the double quiver

$$\tilde{Q} = \begin{array}{c}
0 \\
1
\end{array}$$
The $\widehat{sl}_2$ quiver varieties

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0 \\
1
\end{array}$$

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The \( \hat{\mathfrak{sl}_2} \) quiver varieties

- We consider the double quiver

\[
\tilde{Q} = \begin{array}{c}
\xrightarrow{x_1, x_2} \\
0 & 1 \\
\xleftarrow{y_1, y_2}
\end{array}
\]

- The \( \hat{\mathfrak{sl}_2} \) preprojective algebra is the quotient of the path algebra by the relations \( x_1y_1 = x_2y_2 \) and \( y_1x_1 = y_2x_2 \).

- \( \Lambda(\mathbf{v}) \) is the variety of actions of this algebra on a fixed vector space \( V \) of dimension \( \mathbf{v} = (v_0, v_1) \).

- A theorem of Kashiwara-Saito says that the irreducible components of all \( \Lambda(\mathbf{v}) \) parameterize \( B(\infty) \).
The $\hat{sl}_2$ quiver varieties

\[ \hat{Q} = \begin{array}{c}
0 \\
y_1, y_2
\end{array} \xrightarrow{x_1, x_2} 1 \]
The $\widehat{\mathfrak{sl}_2}$ quiver varieties

The $\widehat{\mathfrak{sl}_2}$ quiver varieties are given by the double quiver:

$$\tilde{Q} = \begin{array}{c} x_1, x_2 \\ 0 \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{y_1, y_2} \end{array} \end{array}$$

Theorem (B-K-T)

Find the component $Z_b$ corresponding to $b \in B(\infty)$. Fix $\pi \in Z_b$ generic. Then the undecorated polytope corresponding to $b$ is the convex hull of the dimension vectors of the subrepresentations of $(V, \pi)$. 

Peter Tingley (MIT)
The $\hat{\mathfrak{sl}}_2$ quiver varieties

The double quiver $\tilde{Q}$ is:

\[ \begin{array}{c}
\downarrow \hspace{1cm} \downarrow \\
0 & 1 \\
\uparrow & \uparrow \\
y_1, y_2 & x_1, x_2
\end{array} \]

Theorem (B-K-T)

*Find the component $Z_b$ corresponding to $b \in B(\infty)$. Fix $\pi \in Z_b$ generic. Then the undecorated polytope corresponding to $b$ is the convex hull of the dimension vectors of the subrepresentations of $(V, \pi)$.>*

- In finite type, this construction gives exactly the MV polytope (follows from Baumann-Kamnitzer).
The $\hat{\mathfrak{sl}}_2$ quiver varieties

$$Q = \begin{array}{c} 0 \quad 1 \\ \xleftarrow{y_1, y_2} \quad \xrightarrow{x_1, x_2} \end{array}$$

Theorem (B-K-T)

Find the component $Z_b$ corresponding to $b \in B(\infty)$. Fix $\pi \in Z_b$ generic. Then the undecorated polytope corresponding to $b$ is the convex hull of the dimension vectors of the subrepresentations of $(V, \pi)$.

- In finite type, this construction gives exactly the MV polytope (follows from Baumann-Kamnitzer).
- The decoration comes from studying finer structure of $(V, \pi)$. 
Filtrations

We consider various distinguished subrepresentations:

\[
\begin{array}{ccc}
1 & 1 & 1 \\
0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
AA & U & AA \\
U & AA & U \\
\end{array}
\]

There is a canonical filtration of \((V, \pi)\) with these as subquotients (a Harder-Narasimhan (HN) filtration), where at one step one sees a complicated extension of the second type of representation. The non vertical edges correspond to sub-quotients which are direct sums of a single indecomposable, and the edge lengths give the multiplicities. The (dual to) the decoration records the dimensions of the indecomposable summands at the complicated step.
Filtrations

- We consider various distinguished subrepresentations:

```
1 1 \cdots 1 1
0 0 \cdots 0 0
\cdots
```

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Filtrations

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\[
\begin{array}{cccccc}
1 & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c}
1 \\
a \left( \begin{array}{c} 1 \\ 0 \end{array} \right) b \\
\end{array}
\]

\[
\begin{array}{cccccc}
1 & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & 0 \\
\end{array}
\]

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Filtrations

- We consider various distinguished subrepresentations:

\[
\begin{array}{cccccc}
1 & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cc}
a & 1 \\
b & 0 \\
\end{array}
\]

\[
\begin{array}{cccccc}
1 & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & 0 \\
\end{array}
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1 & 1 & \cdots & 1 \\
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\end{array}
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Filtration for $\hat{\mathfrak{sl}}_2$ example
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Filtration for \( \hat{sl}_2 \) example
Filtration for $\hat{sl}_2$ example
Filtration for $\hat{\mathfrak{sl}}_2$ example
Filtration for $\hat{sl}_2$ example

6 summands of dim $(1, 1)$
Filtration for $\hat{\mathfrak{sl}}_2$ example

1 summand of dim $(2, 2)$
6 summands of dim $(1, 1)$
Filtration for $\hat{\mathfrak{sl}}_2$ example

- 1 summand of dim $(4, 4)$
- 1 summand of dim $(2, 2)$
- 6 summands of dim $(1, 1)$
Filtration for $\hat{sl}_2$ example

1 summand of dim $(4, 4)$

1 summand of dim $(2, 2)$

6 summands of dim $(1, 1)$
Filtration for $\hat{\mathfrak{sl}}_2$ example

- 1 summand of dim $(4, 4)$
- 1 summand of dim $(2, 2)$
- 6 summands of dim $(1, 1)$
Filtration for $\hat{sl}_2$ example

- 1 summand of dim (4, 4)
- 1 summand of dim (2, 2)
- 6 summands of dim (1, 1)
Filtration for $\hat{\mathfrak{sl}}_2$ example

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Ongoing work

Relationship with PBW bases?
I'm currently working on this with Dinakar Muthiah

Other types?
One may be able to prove things still work using PBW basis theory.

Relationship with double affine grassmannian?
This seems harder, although Dinakar Muthiah has partial results.
Ongoing work

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Thanks!