THE LITTELMAN PATH MODEL

TAEDONG YUN (LIVE TEXED BY STEVEN SAM)

1. Littlewood–Richardson rule

Set $g = \mathfrak{sl}_{n+1}$. Write $\lambda = \sum \lambda_i \varepsilon_i$ ($\lambda_1 \geq \lambda_2 \geq \cdots$) and $\mu = \sum \mu_i \varepsilon_i$ with $\mu_1 \geq \mu_2 \geq \cdots$. Then we have a decomposition

$$B(\lambda) \otimes B(\mu) = \bigoplus_{(j_1, \ldots, j_N) \in \mu} B(\lambda[j_1, \ldots, j_N]),$$

where $\lambda[j_1, \ldots, j_r]$ is obtained by adding a box at the $j_r$th row to $\lambda[j_1, \ldots, j_{r-1}]$. This term is 0 if the result is not a Young tableau.

2. Path model

Now let $g$ be any Kac–Moody algebra. Let $P$ be the weight lattice, $P_\mathbb{R} = P \otimes \mathbb{R}$.

Definition 2.1. A path is a piecewise linear continuous map $\pi: [0, 1] \to P_\mathbb{R}$. We say that $\pi_1 = \pi_2$ if there exists a surjective nondecreasing continuous function $p: [0, 1] \to [0, 1]$ such that $\pi_1 = \pi_2 \circ p$.

Define

$$\Pi = \{\text{paths } \pi \mid \pi(0) = 0, \ \pi(1) \in P\}.$$

The weight of a path $\pi$ is $\text{wt}(\pi) = \pi(1)$.

Given a simple root $\alpha_i$, let $s_i$ be the corresponding simple reflection. Let

$$h = \min(\mathbb{Z} \cap \{\langle \pi(t), \alpha_i^\vee \rangle \mid t \in [0, 1]\}).$$

If $h \geq 0$, define $\bar{e}_i \pi = 0$. If $h < 0$, let

$$t_1 = \min \{t \mid \langle \pi(t), \alpha_i^\vee \rangle = h\}$$
$$t_0 = \max \{t < t_1 \mid \langle \pi(t), \alpha_i^\vee \rangle = h + 1\}.$$

Define

$$\bar{e}_i \pi = \begin{cases} \pi(t) & t \leq t_0 \\ \pi(t_0) + s_i(\pi(t) - \pi(t_0)) & t_0 \leq t \leq t_1 \\ \pi(t) + \alpha_i & t_1 \leq t \end{cases}.$$

Define the path $\pi^\vee$ by $\pi^\vee(t) = \pi(1-t) - \pi(1)$ and set $\bar{f}_i \pi = (\bar{e}_i(\pi^\vee))^\vee$.

Theorem 2.2. $(\Pi, \bar{e}, \bar{f}, \text{wt})$ is a (combinatorial) crystal.

Recall the definition of the dominant weights

$$P^+ = \{\lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i\},$$

and the dominant chamber

$$P^+_\mathbb{R} = \{\lambda \in P_\mathbb{R} \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i\}.$$

Define $\Pi^+$ to be the set of paths that lie entirely in $P^+_\mathbb{R}$. For $\pi \in \Pi^+$, define

$$B_\pi = \{\bar{f}_{i_1} \cdots \bar{f}_{i_r} \pi \mid i_1, \ldots, i_r \in I\}.$$
Theorem 2.3. (1) For $\pi, \pi' \in \Pi^+$, $B_\pi \cong B_{\pi'}$ if and only if $\pi(1) = \pi'(1)$.
(2) For $\lambda \in P^+$, define $\pi_\lambda : [0, 1] \to P_\mathbb{R}$ by $t \mapsto t\lambda$. Then $B(\lambda) \cong B_{\pi_\lambda}$.

Example 2.4 (Adjoint representation of $\mathfrak{sl}_3$). Let $\mathfrak{g} = \mathfrak{sl}_3$ and let $\alpha_1, \alpha_2$ be the simple roots. The lowest weight is $-\alpha_1 - \alpha_2$, so let $\pi_1$ be the path $t \mapsto t(-\alpha_1 - \alpha_2)$.

$\bar{e}_1\pi_1 = \pi_2$ is the path $t \mapsto -t\alpha_2$. $\bar{e}_2\pi_2 = \pi_3$ is the path $t \mapsto -t\alpha_2$ for $0 \leq t \leq 1/2$ and $t \mapsto -(1-t)\alpha_2$ for $1/2 \leq t \leq 1$. Similarly,

$\bar{e}_2\pi_3 = \pi_4 : t \mapsto t\alpha_2,$
$\bar{e}_2\pi_1 = \pi_5 : t \mapsto -t\alpha_1,$
$\bar{e}_1\pi_5 = \pi_6 : t \mapsto -t\alpha_1$ for $0 \leq t \leq 1/2$, $(t-1)\alpha_1$ for $1/2 \leq t \leq 1$,
$\bar{e}_1\pi_6 = \pi_7 : t \mapsto t\alpha_1,$
$\bar{e}_2\pi_7 = \bar{e}_1\pi_4 = \pi_8 : t \mapsto t(\alpha_1 + \alpha_2)$.

Thus we get the crystal for the adjoint representation of $\mathfrak{sl}_3$ as in Figure 1.

3. Generalized Littlewood–Richardson rule

Given $\pi_1, \pi_2 \in \Pi$, define concatenation $\pi_1 \ast \pi_2$ by

$$(\pi_1 \ast \pi_2)(t) = \begin{cases} 
\pi_1(2t) & 0 \leq t \leq 1/2 \\
\pi_1(1) + \pi_2(2t-1) & 1/2 \leq t \leq 1 
\end{cases}.$$

Theorem 3.1. The map $\Pi \otimes \Pi \to \Pi$ given by $\pi_1 \otimes \pi_2 \mapsto \pi_1 \ast \pi_2$ is a morphism of crystals.

Corollary 3.2. Given $\pi_1, \pi_2 \in \Pi^+$, $B_{\pi_1} \otimes B_{\pi_2} = \bigoplus_{\pi} B_{\pi}$ where the sum is over all paths $\pi \in \Pi^+$ such that $\pi = \pi_1 \ast \eta$ for some $\eta \in B_{\pi_2}$.

Example 3.3. We compute $B(\lambda) \otimes B(\lambda)$ for $\lambda = \alpha_1 + \alpha_2$ for $\mathfrak{g} = \mathfrak{sl}_3$ (i.e. $V(\lambda)$ is the adjoint representation.) Let $\pi(t) = (\alpha_1 + \alpha_2)t$. We see that $\pi \ast \eta \in \Pi^+$ for $\eta$ of weights $\{\alpha_1 + \alpha_2, \alpha_1, \alpha_2, 0, 0, -\alpha_1 - \alpha_2\}$, so the decomposition is

$B(\lambda) \otimes B(\lambda) = B(2\alpha_1 + 2\alpha_2) \oplus B(2\alpha_1 + \alpha_2) \oplus B(\alpha_1 + 2\alpha_2) \oplus B(\alpha_1 + \alpha_2) \oplus B(\alpha_1 + \alpha_2) \oplus B(0).$
4. Connection to Young tableaux model

Given a semistandard Young tableau $T$, let $w_T = i_1 \cdots i_s$ be the word obtained by reading from bottom to top (in French notation) starting from rightmost column and then moving to the left. This gives a path $\pi_T = \pi_{\varepsilon_1} \ast \cdots \ast \pi_{\varepsilon_s}$ where $\varepsilon_1 = \omega_1$, $\varepsilon_2 = \omega_2 - \omega_1$, $\ldots$, $\varepsilon_{n-1} = \omega_{n-1} - \omega_{n-2}$, $\varepsilon_n = -\omega_{n-1}$.

Then the crystal operator on paths coincides with the crystal operator on Young tableaux.