The goal for today is to characterize crystals. We would perhaps like to give a set of axioms on the data $B, e_i, f_i$ of a set $B$ along with candidates $e_i, f_i$ for the Kashiwara operators that would guarantee that we see the crystal of a representation $V$ or $\mathfrak{g}$. However, this is difficult, and we take a sort of half-way ground: A axiomatize the notion of a combinatorial crystal, which includes things which are not crystal graphs of representations. We then study various ways to prove that such a combinatorial crystal does in fact arise as the crystal of a representation. That is what we mean by recognition theorems.

For today, $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra.

1. Combinatorial crystals

**Definition 1.1.** An integrable combinatorial $\mathfrak{g}$-crystal is a set $B$ with operators $e_i, f_i$ with the properties:

1. **integrable:** For all $i$ and all $b \in B$, there is some $N > 0$ such that $e_i^N(b) = f_i^N(b) = 0$.
2. **weighted:** Define
   \[
   \varepsilon_i(b) = \max\{n \mid e_i^n b \neq 0\}, \quad \varphi_i(b) = \max\{n \mid f_i^n b \neq 0\}
   \]
   \[
   \varepsilon(b) = \sum_i \varphi_i(b) \omega_i, \quad \varphi(b) = \sum_i \varepsilon_i(b) \omega_i
   \]
   \[
   \text{wt}(b) = \varphi(b) - \varepsilon(b).
   \]
   Then $f_i$ has weight $-\alpha_i$ and $e_i$ has weight $\alpha_i$.
3. **Partial permutations:** $f_i b = b'$ if and only if $e_i b' = b$.

**Example 1.2.** This is not enough to characterize the crystals of integrable representations of $\mathfrak{g}$. For example

\[
\text{satisfies all the axioms, but is not the crystal of any representation.}
\]

**Remark 1.3.** $B(\infty)$ is not covered by this definition.

**Definition 1.4.** (see [K3, Section 7.2]) A combinatorial crystal is a tuple $(B, f_i, e_i, \text{wt}, \varepsilon_i, \varphi_i)$ where $\text{wt}: B \to P$ ($P$ is the weight lattice), $e_i, f_i: B \to B \cup \{0\}$, $\varepsilon_i, \varphi_i: B \to \mathbb{Z} \cup \{-\infty\}$ such that
(1) \( \varphi_i(b) - \varepsilon_i(b) = \langle \text{wt}(b), \alpha_i \rangle \) whenever the left hand side is finite.
(2) \( e_i \) increases \( \varphi_i \) by 1 and decreases \( \varepsilon_i \) by 1.
(3) \( e_i \) has weight \( \alpha_i \).
(4) \( f_i b = b' \) if and only if \( e_i b' = b \).
(5) If \( \varphi_i(b) = -\infty \), then \( e_i b = f_i b = 0 \). \( \square \)

**Remark 1.5.** An integrable combinatorial crystal is in particular a combinatorial crystal, where one defines \( \varphi_i(b) = \langle \varphi(b), \alpha_i^\vee \rangle \) and \( \varepsilon_i(b) = \langle \varepsilon(b), \alpha_i^\vee \rangle \).

**Remark 1.6.** The cases \( \varphi_i(b) = -\infty \) are included to allow certain combinatorial constructions that come in handy. They don’t ever really correspond to crystals of representations.

**Example 1.7.** \( B(\infty) \) is a combinatorial crystal where we define \( \text{wt}(1) = 0, \varepsilon_i(b) = \max \{ n \mid e_i^n b \neq 0 \} \), and then allow this to determine \( \varphi_i(b) \). \( \square \)

## 2. Recognizing integrable crystals

The follow theorems will be our tools for showing that later constructions (using quiver varieties etc.) actually give the crystals of integrable representations of \( \mathfrak{g} \).

**Theorem 2.1** (Kashiwara). There is a unique family of integrable crystals \( \{ B(\lambda) \} \) indexed by dominant weights \( \lambda \) which is closed under \( \otimes \) and taking connected components. These are the crystals for the highest weight representations \( V(\lambda) \).

**Theorem 2.2** (Kashiwara). Assume \( C(\lambda) \) is a graph with a unique source \( b_\lambda \) such that the length of the \( i \) root string leaving \( b_\lambda \) is \( \langle \alpha_i^\vee, \lambda \rangle \), and such that, for each \( i, j \in I \), the graph obtained by ignoring the other colors is an integrable rank 2 crystal for the corresponding rank 2 Kac–Moody algebra. Then \( C(\lambda) = B(\lambda) \).

**Remark 2.3.** For crystals of simply laced Kac-Moody algebras, Theorem 2.2 was enhanced by Stembridge [S] by giving a “local” characterization of integrable \( \mathfrak{sl}_3 \) crystals.

## 3. Recognizing \( B(\infty) \)

The previous section is only concerned with integrable crystals, so cannot recognize \( B(\infty) \). For that, we need to introduce more structure. Recall that we have an inner product \( (\cdot, \cdot) \) on \( U^-, L(\infty) \), and \( L(\infty)/q L(\infty) \). Working over \( \mathbb{Z} \), \( B(\infty) \cup -B(\infty) = \{ b \mid (b, b) = 1 \} \).

**Definition 3.1.** \( * \) is the anti-algebra involution on \( U^- \) which fixes each \( F_i \). \( \square \)

**Proposition 3.2** ([K2] Proposition 5.2.1]). For all \( u, v \in U^- \), we have \( (u, v) = (u, v) \). In particular, \( * \) preserves the set \( \{ (b, -b) \mid b \in B(\infty) \} \) (but not pointwise). By ignoring signs, we get an involution, also denoted \( * \), on \( B(\infty) \). Also, define \( e_i^* = * \circ e_i \circ * \), and define \( \varphi_i^*, \varepsilon_i^* \) in the obvious way.

**Remark 3.3.** It was shown by Lusztig that, at least in types ADE, the signs in the above are all +. This is conjectured to hold in other types as well.

**Definition 3.4** (see [K3] Section 7.5)]. Let \( B^{(i)} \) be the following crystal

\[
\cdots b^{(i)}(1) \xrightarrow{\hat{i}} b^{(i)}(0) \xrightarrow{\hat{i}} b^{(i)}(-1) \xrightarrow{\hat{i}} b^{(i)}(-2) \cdots
\]

where \( \text{wt}(b^{(i)}(k)) = k \alpha_i \), \( \varphi_i(b^{(i)}(k)) = k \), \( \varepsilon_i(b^{(i)}(k)) \), and for \( j \neq i \), \( \varphi_i(b^{(i)}(k)) = \varepsilon(b^{(i)}(k)) = -\infty \). Here the arrows show the action of \( f_i \). \( \square \)
Theorem 3.5. [K1] Theorem 2.2.1] For each i, there is a morphism of crystals
\[ \Phi_i : B(\infty) \rightarrow B(\infty) \otimes B^{(i)} \]
\[ u_0 \mapsto u_0 \otimes b^{(i)}(0). \]

Furthermore, \( \Psi \) satisfies
1. \( \Phi_i(b) = (e_i^*) \xi_i^*(b) \otimes b^{(i)}(-\xi_i^*(b)) \).
2. Image \( \Phi_i = \{ b \otimes b^i(k) \mid k \leq 0, \xi_i^*(b) = 0 \} \). \( \square \)

Furthermore \( B(\infty) \) can be characterized as the unique crystal for which the above holds. This is made precise as follows

Theorem 3.6 (KS, Proposition 3.2.3]). Let \( \{ B, e_i, f_i, \text{wt}, \xi_i, \varphi_i \} \) be a combinatorial crystal with an element \( b_+ \) such that \( \text{wt}(b) = 0, \xi_i(b) = 0 \) for all \( i \in I \), and \( \{ f_{i_1}, \ldots, f_{i_N}, b_+ : i_1, \ldots, i_N \in I \} = B \). Assume also that
1. \( \xi_i(b) \in \mathbb{Z} \) for every \( i \).
2. For every \( i \), there is a strict embedding (that is, an embedding of crystals) \( \Phi_i : B \rightarrow B \otimes B^{(i)} \).
3. \( \Phi_i(B) \subset B \otimes \{ b_i(-k) : k \geq 0 \} \).
4. For each \( b \neq b_+ \in B \), there some \( i \in I \) such that \( \Phi_i(b) = b' \otimes b^{(i)}(-k) \) with \( k > 0 \).

Then \( B \) is isomorphic to \( B(\infty) \).

The proof that the properties from Theorem 3.6 uniquely characterize the resulting crystal is quite simple. Essentially, one can see that these properties imply that \( B \) is isomorphic to the crystal generated by
\[ \cdots b^{(i_2)}(0) \otimes b^{(i_1)}(0) \in \cdots B^{(i_2)} \otimes B^{(i_1)}, \]
for appropriate \( \cdots i_2, i_1 \). It then follows from Theorem 3.5 that this unique crystal is in fact \( B(\infty) \).

The following is another wording of this result which makes the role of * more obvious.

Corollary 3.7. Let \( B \) be a highest weight combinatorial crystal with highest weight element \( b_+ \) such that, for all \( b \in B \) and all \( i \), \( \xi_i(b) \geq 0 \), and \( e_i(b_+) = 0 \). Fix an involution * on \( B \), and define \( e_i^* = *e_i* \) and \( \xi_i^*(b) = \xi_i(*b) \). Define \( \Phi_i : B \rightarrow B \otimes B^{(i)} \) by
\[ \Phi_i(b) = (e_i^*) \xi_i^*(b) \otimes b^{(i)}(-\xi_i^*(b)). \]

If \( \Phi_i \) is an embedding of crystals for all \( i \), then \( B = B(\infty) \) and * is Kashiwara’s involution.

REFERENCES


