Please do not start working until instructed to do so.

You have 50 minutes.

You must show your work to receive full credit.

Calculators are OK (but don’t expect them to be useful).

You may use one double-sided 8.5 by 11 sheet of handwritten (by you) notes.

Problem 1. _______

Problem 2. _______

Problem 3. _______

Problem 4. _______

Problem 5. _______

Total. _______
Problem 1. (20 points) Calculate the following limits:

a. (4 points) \( \lim_{n \to \infty} \frac{n^2 - 7n^3 + 15n}{4 + 9n + 2n^3} \)

Solution:
\[
\lim_{n \to \infty} \frac{n^2 - 7n^3 + 15n}{4 + 9n + 2n^3} = \lim_{n \to \infty} \frac{n^2 - 7 + 15n^{-2}}{4n^{-3} + 9n^{-2} + 2} = \frac{-7}{2}
\]

b. (4 points) \( \lim_{n \to \infty} \frac{\cos(n\pi)}{n + 6} \)

Solution:
\[
\lim_{n \to \infty} \frac{\cos(n\pi)}{n + 6} = 1
\]
because \( |\cos(n\pi)| \leq 1 \).

c. (4 points) \( \lim_{n \to \infty} \left( 1 + \frac{2}{n} \right)^{7n} \)

Solution:
\[
\lim_{n \to \infty} \left( 1 + \frac{2}{n} \right)^{7n} = \lim_{n \to \infty} \left( \left( 1 + \frac{1}{(n/2)} \right)^{n/2} \right)^{14} = e^{14}
\]

d. (4 points) \( \lim_{n \to \infty} \sqrt{n^2 + n} - \sqrt{n^2 + 98} \)

Solution:
\[
\lim_{n \to \infty} \sqrt{n^2 + n} - \sqrt{n^2 + 98} = \lim_{n \to \infty} \frac{n^2 + n - n^2 - 98}{\sqrt{n^2 + n} + \sqrt{n^2 + 98}} = \lim_{n \to \infty} \frac{n - 98}{\sqrt{n^2 + n} + \sqrt{n^2 + 98}}
\]
\[
= \lim_{n \to \infty} \frac{n - 98}{\sqrt{1 + 1/n} + \sqrt{1 + 98/n}} = \frac{1}{2}
\]

e. (4 points) \( \lim_{n \to \infty} \left( 1 - \frac{1}{n + 3} \right)^{2n} \)

Solution:
\[
\lim_{n \to \infty} \left( 1 - \frac{1}{n + 3} \right)^{2n} = \lim_{n \to \infty} \left( \left( 1 - \frac{1}{n + 3} \right)^{n+3} \left( 1 - \frac{1}{n + 3} \right)^{-3} \right)^{2} = e^{-2}
\]
Problem 2. (10 points) Use the definition of the limit of a sequence to establish that
\[
\lim_{n \to \infty} \frac{3n + 1}{2n + 5} = \frac{3}{2}
\]

Solution: For any \(\varepsilon > 0\) need to find \(N\) such that for all \(n > N\),
\[
\left| \frac{3n + 1}{2n + 5} - \frac{3}{2} \right| < \varepsilon.
\]
Do some algebra:
\[
\left| \frac{3n + 1}{2n + 5} - \frac{3}{2} \right| = \frac{13}{4n} < \frac{13}{4\varepsilon}
\]
Need \(\frac{13}{4\varepsilon} < n\) so \(\frac{13}{4\varepsilon} < n\). Hence, let \(N\) be any natural number greater than \(\frac{13}{4\varepsilon}\).

Problem 3. (10 points) Let \(S = \{ \frac{2}{n} + \frac{3}{m} \mid n \in \mathbb{N}, m \in \mathbb{Z}, m \neq 0 \}\). Find inf \(S\) and sup \(S\).

Solution:
\[
\inf S = -3, \quad \sup S = 5
\]
Brief justification:
\[
\inf \left\{ \frac{2}{n} \mid n \in \mathbb{N} \right\} = 0, \quad \sup \left\{ \frac{2}{n} \mid n \in \mathbb{N} \right\} = 2
\]
\[
\inf \left\{ \frac{3}{m} \mid m \in \mathbb{Z}, m \neq 0 \right\} = -3, \quad \sup \left\{ \frac{3}{m} \mid m \in \mathbb{Z}, m \neq 0 \right\} = 3
\]
Problem 4. (15 points)  Prove that the following sequence is convergent and find its limit:

\[ x_1 = 4, \quad x_{n+1} = 4 - \frac{3}{x_n} \text{ for } n = 1, 2, 3, \ldots \]

Solution: Write down the first few terms:

\[ x_1 = 4, \quad x_2 = 4 - \frac{3}{4} = \frac{13}{4}, \quad x_3 = 3 \frac{1}{13}, \quad x_4 = 3 \frac{1}{40}, \ldots \]

The sequence seems to be decreasing and bounded below by 3. Let’s try to prove this.

First, decreasing: clearly, \( x_1 > x_2 \). Then:

\[ x_1 > x_2, \quad \frac{3}{x_1} < \frac{3}{x_2}, \quad -\frac{3}{x_1} > -\frac{3}{x_2}, \quad 4 - \frac{3}{x_1} > 4 - \frac{3}{x_2} \]

and thus \( x_2 > x_3 \). The same argument shows that \( x_n > x_{n+1} \) implies \( x_{n+1} > x_{n+2} \). Hence the sequence is decreasing.

Second, bounded below by 3. Clearly, \( x_1 > 3 \). Now suppose that \( x_n > 3 \). Then

\[ x_n > 3, \quad \frac{3}{x_n} < 1, \quad -\frac{3}{x_n} > -1, \quad 4 - \frac{3}{x_n} > 4 - 1 \]

and so \( x_{n+1} > 3 \). Induction concludes that \( x_n > 3 \) for all \( n \in \mathbb{N} \).

Because the sequence is decreasing and bounded below, it has a limit. Let this limit be \( L \). Because

\[ x_{n+1} = 4 - \frac{3}{x_n} \text{ for } n = 1, 2, 3, \ldots \]

we have

\[ \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} 4 - \frac{3}{x_n} \]

and thus

\[ L = 4 - \frac{3}{L} \]

Note that \( L \neq 0 \) (why?). Then \( L^2 = 4L - 3, L^2 - 4L + 3 = 0, (L - 3)(L - 1) = 0 \) so \( L = 3 \) or \( L = 1 \). But \( L \geq 3 \), so it must be that \( L = 3 \).
Problem 5. (15 points total)

a. (5 points) Give an example of an increasing sequence that does not have a limit.

Solution: $x_n = n$.

b. (10 points) Prove that every nonincreasing and bounded below sequence is convergent.

Solution: see the textbook.