Problem 1. (4 pts) Five men and a monkey ... well, OK, not. Convert 291 to base 4.

Solution:

\[ 291 = 4 \cdot 72 + 3, \quad 72 = 4 \cdot 18 + 0, \quad 18 = 4 \cdot 4 + 2, \quad 4 = 4 \cdot 1 + 0 \]

so

\[ 291 = 4 \cdot 72 + 3 = 4(4 \cdot 18) + 3 = 4^4 + 2 \cdot 4^2 + 3 \cdot 4^0 = (10203)_4 \]

Problem 2. (6 pts) Find at least three different nonnegative integer pairs \((x, y)\) that solve the equation

\[ 26x - 18y = 20 \]

Solution: Extended Euclidean Algorithm shows that \(26(-2) + 18(3) = 2\), so \(26(-2) - 18(-3) = 2\), so \(26(-20) - 18(-30) = 20\). EEA also shows that \(26(9) + 18(-13) = 0\), so \(26(9) - 18(13) = 0\), so \(26(9k) - 18(13k) = 0\) for any integer \(k\). Then

\[ 26(-20 + 9k) - 18(-30 + 13k) = 20, \]

so all solution pairs are given by \(x = -20 + 9k\), \(y = -30 + 13k\). Both \(x\) and \(y\) are positive when \(k \geq 3\). Some solution pairs, corresponding to \(k = 3, 4, 5\), are then:

\[ x = 7, y = 9, \quad x = 16, y = 22, \quad x = 25, y = 35. \]
Problem 3. (4 pts) Can 2431 be written as a sum of two positive integers, one of which is divisible by 42 and the other is divisible by 91? (Say yes or no, and justify your answer.)

Solution: the question is really this: are there positive integer solutions $x$ and $y$ to

$$42x + 91y = 2431.$$  

There are integer solutions to this equation if and only if gcd(42, 91) divides 2431. But gcd(42, 91) = 7 and 2431/7 = 347.285... , so 7 does not divide 2431, so the answer is NO.

Problem 4. (6 pts) Prove that gcd($a,c$) = gcd($b,c$) = 1 if and only if gcd($ab,c$) = 1.

Solution:
First, we prove that gcd($a,c$) = gcd($b,c$) = 1 implies that gcd($ab,c$) = 1. Note that 1|ab and 1|c. It is then enough to show that there exist integer $x$ and $y$ such that $abx + cy = 1$. Because gcd($a,c$) = gcd($b,c$) = 1, there exist integers $k,l,m,n$ such that

$$ak + cl = 1, \quad bm + cn = 1.$$  

Multiply these two equations by one another, get

$$abkm + ackn + cbml + ccln = 1,$$

which can be factored like this:

$$ab(km) + c(akn + blm + cln) = 1.$$  

Hence, $x = km$, $y = akn + blm + cln$.

Second, to prove that gcd($ab,c$) = 1 implies gcd($a,c$) = gcd($b,c$) = 1, note that the assumption implies that there exist integer $x$, $y$ such that

$$abx + cy = 1.$$  

This can be written as

$$a(bx) + cy = 1, \quad b(ax) + cy = 1,$$

and hence there exist integer solutions $k,l,m,n$ to equations $ak + cl = 1$, $bm + cn = 1$. This implies gcd($a,c$) = gcd($b,c$) = 1.

Alternative justification: gcd($a,c$) = 1 implies that $a$ and $c$ have no common prime factors. gcd($b,c$) = 1 implies that $b$ and $c$ have no common prime factors. Prime factors of $ab$ consist of prime factors of $a$ and of prime factors of $b$, hence there are no common prime factors of $ab$ and $c$. 