Problem 1. \((\frac{1}{5} - 5^{-1})\) pts) Find all critical points of \(f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy\) and classify them as local minima, local maxima, or saddle points.

Solution. See Example 4, Section 14.7 of the textbook.

Problem 2. \((0\pi)\) pts) Find the average value of \(f(x, y) = \sin(x + y)\) over the rectangle \(0 \leq x \leq \pi, 0 \leq y \leq \pi/2\).

Solution. The average value of \(f(x, y)\) over a region \(R\) is \(\iint_R f(x, y) \, dA\) divided by the area of \(R\). Here, \(R\) is a rectangle, and its area is \(\pi \frac{\pi}{2} = \pi^2/2\).

\[
\iint_R f(x, y) \, dA = \int_0^\pi \int_0^{\pi/2} \sin(x + y) \, dy \, dx = \int_0^\pi (-\cos(x + y)) \big|_0^{\pi/2} \, dx \\
= \int_0^\pi -\cos(x + \pi/2) + \cos x \, dx = \int_0^\pi \sin x + \cos x \, dx = \big( -\cos x + \sin x \big) \big|_0^\pi \\
= (-(-1) + 0) - (-1 + 0) = 2
\]

Hence, the answer is \(2/(\pi^2/2) = 4/\pi^2\).
**Problem 3.** (no pts) Find the largest and the smallest value of \( f(x, y) = xy \) on the ellipse \( x^2 + 2y^2 = 1 \).

*Solution.* Let \( g(x, y) = x^2 + 2y^2 - 1 \) and look for critical points: \( \nabla f(x, y) = \lambda \nabla g(x, y) \). Hence \( y = \lambda 2x \), \( x = \lambda 4y \). If \( \lambda = 0 \), or if \( x = 0 \), or if \( y = 0 \) then both \( x = 0 \) and \( y = 0 \), and this point is not on the ellipse. Hence, all variables are not zero. Divide one equation by the other, get \( y/x = \lambda^2 \), \( x/y = \lambda^4 \). If \( \lambda = 0 \), or if \( x = 0 \), or if \( y = 0 \) then both \( x = 0 \) and \( y = 0 \), and this point is not on the ellipse. Hence, all variables are not zero. Divide one equation by the other, get \( y/x = \lambda^2 \), \( x/y = \lambda^4 \).

If \( \lambda = 1/2 \), \( (1/\sqrt{2}, 1/2) \), \( (-1/\sqrt{2}, -1/2) \), \( (1/\sqrt{2}, -1/2) \). The higher value, hence the maximum, is attained when both \( x \) and \( y \) have the same sign, and it is \( 1/(2\sqrt{2}) \). The smaller value, hence the minimum, is attained when \( x \) and \( y \) have opposite sign, and it is \( -1/(2\sqrt{2}) \).

**Problem 4.** (564 pretend pts) Sketch the region of integration, reverse the order of integration, and evaluate the integral:

\[
\int_0^8 \int_{\sqrt{x}}^2 \frac{dy}{y^4 + 1} \]

*Solution.*

\[
\int_0^8 \int_{\sqrt{x}}^2 \frac{dy}{y^4 + 1} = \int_0^2 \int_0^{y^3} \frac{dx dy}{y^4 + 1} = \int_0^2 \frac{1}{y^4 + 1} \int_0^{y^3} dx dy = \int_0^2 \frac{1}{y^4 + 1} y^3 dy = \frac{1}{4} \ln(y^4 + 1)|_0^1 = \frac{1}{4} \ln(17)
\]

**Problem 5.** (one large pt) Write down, but do not evaluate, a double integral representing the volume under the plane \( x - z + 4 = 0 \) and above the region in the \( xy \)-plane bounded by the parabola \( y = 4 - x^2 \) and the line \( y = 3x \).

*Solution.* To find the limits on \( x \) set \( 4 - x^2 = 3x \). Solve, get \( x = -4 \) or \( x = 1 \). In between these two values of \( x \), \( y = 4 - x^2 \) is above the region, \( y = 3x \) is below. Hence

\[
\int_{-4}^{1} \int_{3x}^{4-x^2} (x + 4) dy dx
\]