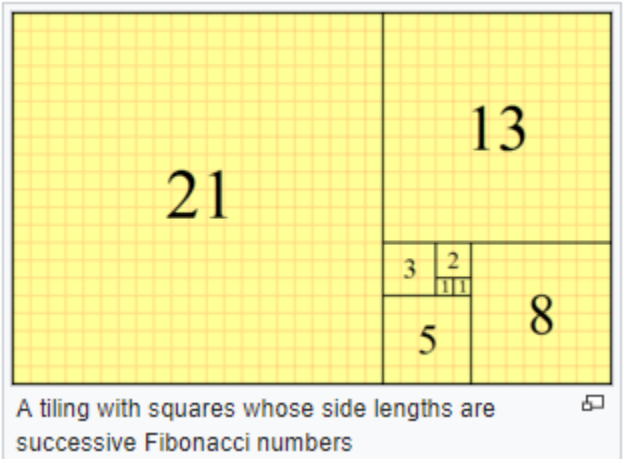
CLASS DISCUSSION: FIBONACCI NUMBERS AND RECURSION



A tiling with squares whose side lengths are successive Fibonacci numbers

1. Consider the Lucas series 1, 3, 4, 7, 11, 18, 29, 47, 76, …. This sequence is defined recursively by: $a_1 = 1$, $a_2 = 3$, and, for all $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$. Using strong induction prove that $a_n < (7/4)^n$ for all positive integers *n*.

2. Define a sequence recursively by: $b_1 = 1$, $b_2 = 2$, $b_3 = 3$, and, for all $n \geq 4$, $b_n = b_{n-1} + b_{n-2} + b_{n-3}$. Using strong induction, prove that $b_n < 2^n$ for all positive integers, *n*.

3. Using strong induction prove that every integer $n \geq 2$ can be expressed as a product of primes.

The **Fibonacci numbers**, commonly denoted F_n form a sequence, called the **Fibonacci sequence**, such that each number is the sum of the two preceding ones, viz.

$$F_0 = 1, \qquad F_1 = 1 \; and \; F_{n+1} = F_n + F_{n-1} \; for \; all \; n \geq 1.$$

4. Observe that $1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 = 54 = 55 - 1$

   Prove that $F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$

5. Discover and prove a formula for the sum of the first *n* Fibonacci numbers with even indices.

6. Discover and prove a formula for the sum of the first *n* Fibonacci numbers with odd indices.

7. Discover a recursive formula for the sequence 3, 22, 27, 59, 123, …

8. Prove that $\gcd(F_n, F_{n+1}) = 1$. (Hint: Try proof by contradiction.)

9. Prove that for all $n \geq 1$, $F_n < 2^n$.

10. Prove that for all $n \geq 2$ $F_n \geq \left(\frac{3}{2}\right)^{n-2}$

11. Prove that for all n ≥ 1 $F_{2n} = F_n(F_{n+1} + F_{n-1})$

12. Prove that for all m, n ≥ 2 $F_{m+n} = F_{m+1}F_n + F_mF_{n-1}$

13. Write recursive equations for the sequence 5, 7, 9, 11…

14. Write recursive equations for the sequence 2, 4, 8, 16, …

15. Write recursive equations for the sequence 1, 2, 6, 24, 120, 720, …

16. Write recursive equations for the sequence 2, 3, 6, 18, 108, 1944, 209952, …

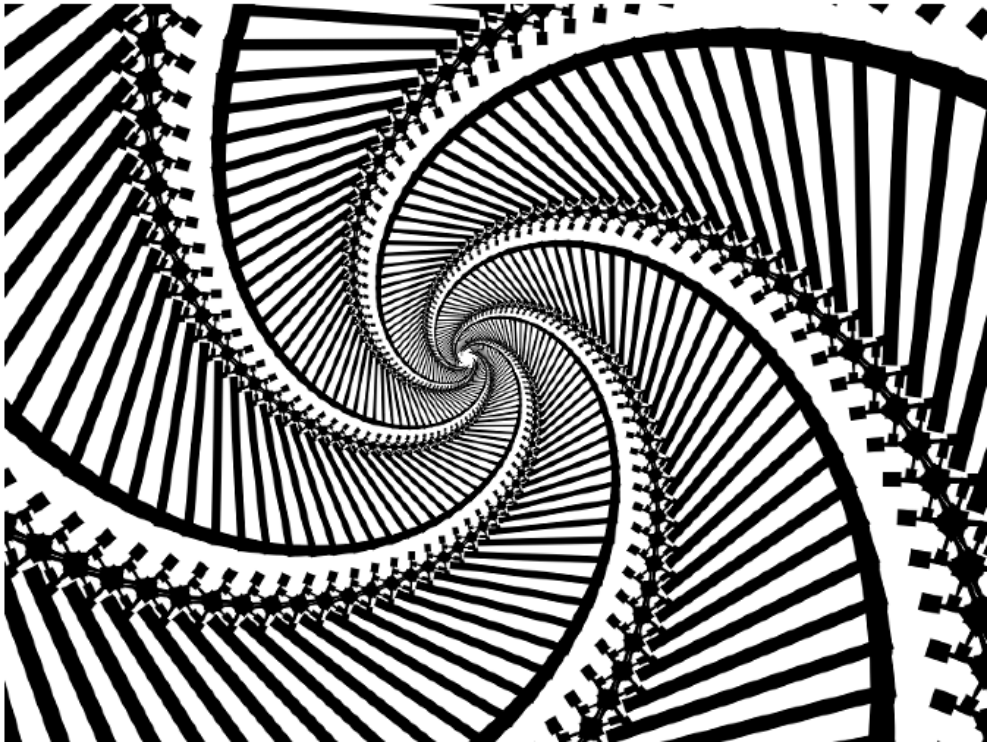
17. What is the 5th term of the recursive sequence defined as follows? Give exact answer.

$a_1 = 2, \qquad a_n = 2a_{n-1} - 1$?

18. What is the 1st term of the recursive sequence defined as follows:
$a_n = 4a_{n-1} - 1$, if $a_4 = 192$?

19. Write recursive equations for the sequence 2, 6, 14, 30, 62, …

20. Write recursive equations for the sequence 6561, 81, 9, 3, …



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