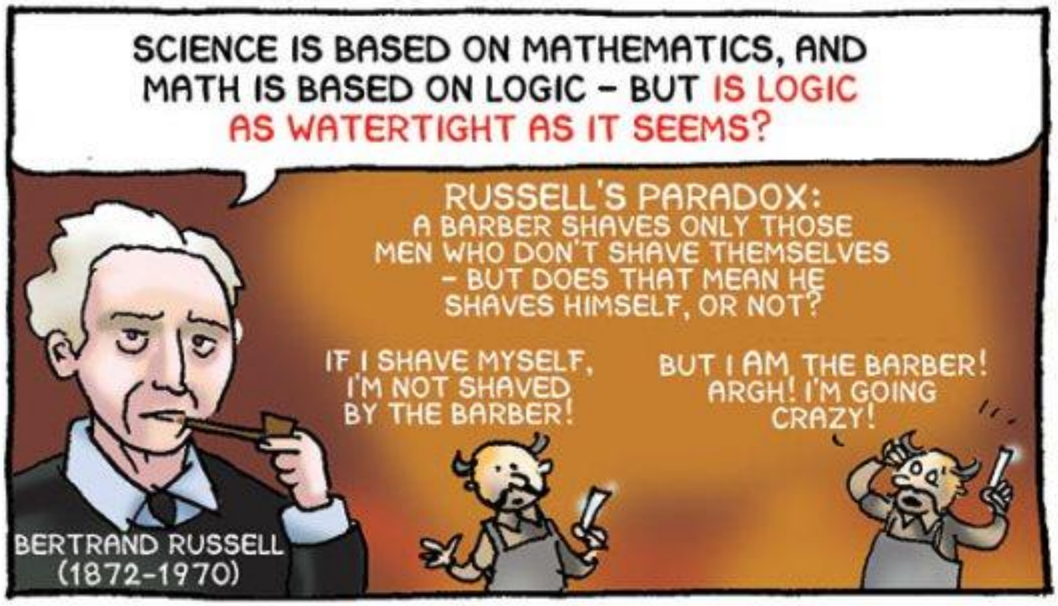
# CLASS DISCUSSION: CARDINALITY

**5 April 2019**




**I** What does it mean to say that two sets have the *same cardinality*? What does it mean to say that a set is *countably infinite*?

**II** Show that each of the following sets is countable:

(a) The set of non-negative integers.

(b) The set of integers greater than or equal to 13.

(c) **Z**

(d) The set of positive even integers.

(e) The set of even integers.

(f) The set of odd integers.

(g) The set of rational numbers strictly between 0 and 1.

**III** (a) Show that a subset of a countable set is either finite or countable.

(b) Show that if *A* and *B* are disjoint countable sets then so is the union of *A* and *B*. What if *A* and *B* are not disjoint?

(c) Show that if *A* and *B* are countable sets then so is the Cartesian product of *A* and *B*.

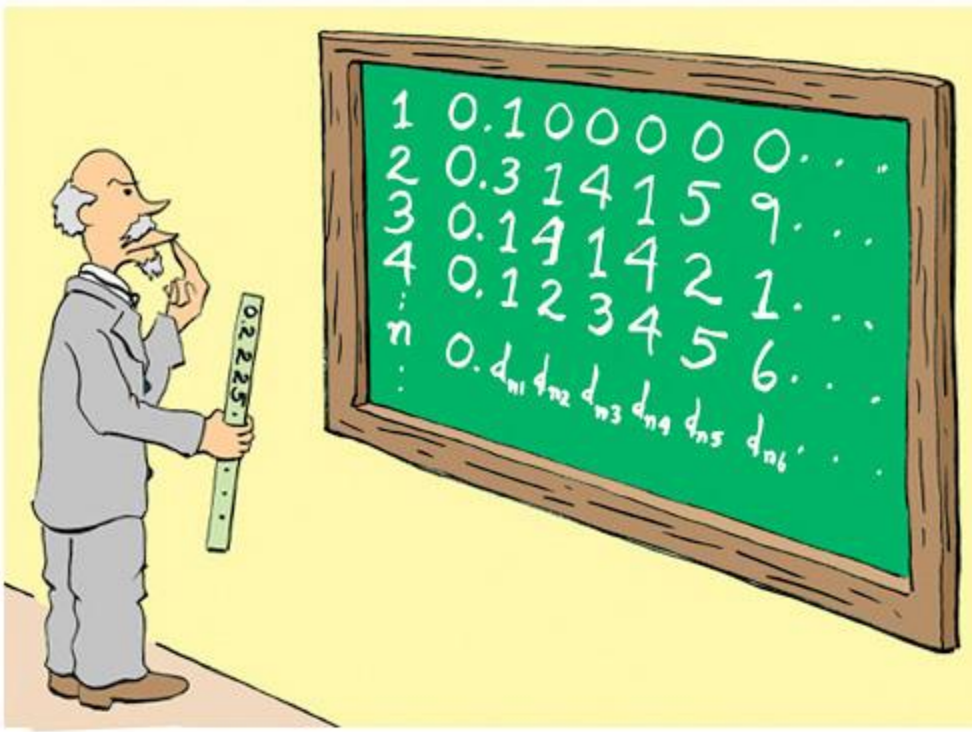
(d) Prove that a countable union of countable sets is countably infinite.

(e) Prove that the set of rational numbers strictly between 0 and 1 is countable.

(f) Demonstrate that **Q** is countable.

**IV** Show that if S is a collection of sets, then cardinality is an equivalence relation on S.

**V** Using Cantor's diagonal argument, prove that **R** is not countable.




**VI** (a) Let $X$ be a set. Recall the definition of the power set, $\mathscr{P}(X)$, of $X$.

(b) Show that the power set of a finite set is finite. In such case, describe the relationship between $|X|$ and $|\mathscr{P}(X)|$.

(c) Let $X = \{a, b, c, d\}$ and let $F: X \to \mathscr{P}(X)$ be defined by:

$F(a) = \{a, c, d\}$, $F(b) = \{a, d\}$, $F(c) = \varphi$, $F(d) = \{d\}$

Find $D^* = \{j \in X \mid j \notin F(j)\}$

(d) Let $X = Z^+$ and let $G: X \to P(X)$ be defined by:

$G(a) = \{\text{all prime numbers, p, such that } a \leq p \leq 2a\}$

Find $D^* = \{j \in X \mid j \notin G(j)\}$

(e) Let $X = \mathbf{Q}$ and let $H: X \to \mathscr{P}(X)$ be defined by:

$H(z) = \{\text{all prime numbers, } p, \text{ such that } z \leq p \leq 2z\}$

Find D* = $\{q \in X |\ q \notin H(q)\}$

(f) Let X = **R** and let V: X → P (X) be defined by:

$$V(a) = \begin{cases} \{0\}\ if\ a \leq 0 \\ (a, a+1)\ if\ a \in Q^+ {\sim} Z \\ [a-1, a]\ if\ a\ is\ a\ positive\ irrational\ number \\ \{a, a+3\}\ if\ a \in Z^+ \end{cases}$$

Is V well-defined? If so, find D* = $\{z \in X |\ z \notin V(z)\}$

(g) Prove *Cantor's Theorem*: X and $\mathscr{P}$(X) are not of the same cardinality.

**Highly recommended:**

MIT lecture notes on cardinality, 24.118 (paradox and infinity)

Georg Ferdinand Ludwig Cantor (1845 – 1918) is best known for his discovery of transfinite numbers and the creation of Set Theory.

*Lenore nodded. 'Gramma really likes antinomies. I think this guy here, 'looking down at the drawing on the back of the label, 'is the barber who shaves all and only those who do not shave themselves'.*

- David Foster Wallace, **The Broom of the System**