CLASS DISCUSSION: CARDINALITY (CONTINUED)

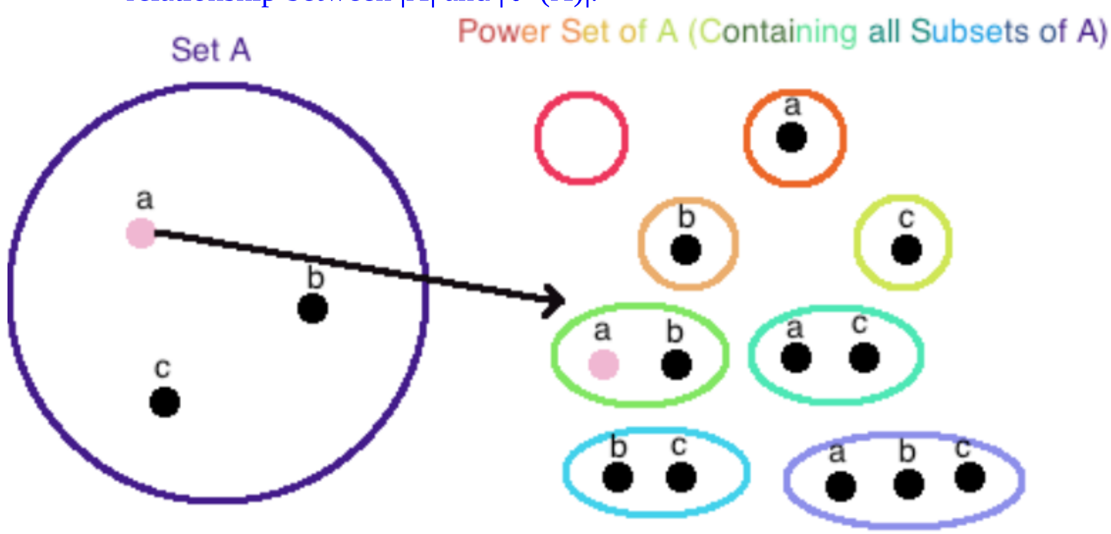
8 April 2019

I (a) Show that a subset of a countable set is either finite or countable.
(b) Show that if *A* and *B* are disjoint countable sets then so is the union of *A* and *B*. What if *A* and *B* are not disjoint?
(c) Show that if *A* and *B* are countable sets then so is the Cartesian product of *A* and *B*.
(d) Prove that a countable union of countable sets is countably infinite.
(e) Prove that the set of rational numbers strictly between 0 and 1 is countable.
(f) Demonstrate that **Q** is countable.

II Using Cantor's diagonal argument, prove that **R** is not countable.

III (a) Let *X* be a set. Recall the definition of the power set, $\mathscr{P}(X)$, of *X*.

(a) Show that the power set of a finite set is finite. In such case, describe the relationship between |X| and $|\mathscr{P}(X)|$.



In this case **a** is a member of f(**a**).

(b) Let X = {a, b, c, d} and let F: X → $\mathscr{P}$(X) be defined by:

F(a) = {a, c, d}, F(b) = {a, d}, F(c) = φ, F(d) = {d}. Find $D^* = \{j \in X | \, j \notin F(j)\}$

(c) Let G: **N** → P(**N**) be defined by:

G(a) = {all prime numbers, p, such that a ≤ p ≤ 2a}. Find $D^* = \{j \in X | \, j \notin G(j)\}$

(d) Let H: **Q** → $\mathscr{P}$(**Q**) be defined by:

$H(z) = \{$all prime numbers, p, such that z ≤ p ≤ 2z$\}$ Find $D^* = \{q \in X | \, q \notin H(q)\}$

(e) Let X = **R** and let V: X → P (X) be defined by:

$$V(a) = \begin{cases} \{0\}\ if\ a \le 0 \\ (a, a+1)\ if\ a \in Q^{+} \sim Z \\ [a-1, a]\ if\ a\ is\ a\ positive\ irrational\ number \\ \{a, a+3\}\ if\ a \in N \end{cases}$$

Is V well-defined? If so, find $D^{*} = \{z \in X |\ z \notin V(z)\}$

(f) Prove *Cantor's Theorem in the* special case that X is countably infinite.

$$\mathbb{N}\left\{\begin{array}{ccc} 1 & \longleftrightarrow & \{4,5\} \\ 2 & \longleftrightarrow & \{1,2,3\} \\ 3 & \longleftrightarrow & \{4,5,6\} \\ 4 & \longleftrightarrow & \{1,3,5\} \\ \vdots & \vdots & \vdots \end{array}\right\}\mathcal{P}(\mathbb{N})$$

(g) Prove *Cantor's Theorem for the general case:* X and $\mathscr{P}$(X) are not of the same cardinality.