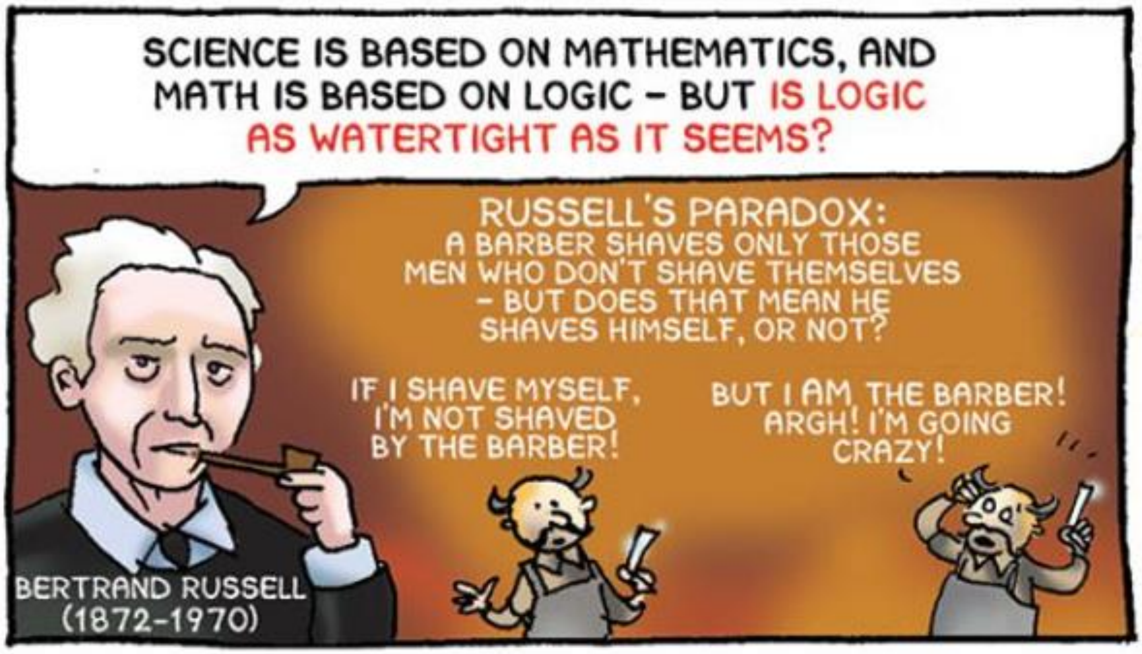
MATH 201 CLASS DISCUSSION: 11 FEBRUARY 2019

COUNTING WITH COMBINATIONS AND PERMUTATIONS



1. *Review:* Define: P(n, k) and $\binom{n}{k}$. Derive a formula for P(n, k). Using the result for P(n, k), derive a formula for $\binom{n}{k}$. Why should 0! be defined to be 1?

2. *Review:* Prove, using only a story proof the following identity.

$$\binom{n}{r} = \binom{n}{n-r}$$

3. Give a *story* proof (as opposed to a computational proof) for the following identity:

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

This is known as "Pascal's identity" even though many mathematicians had "discovered" it before Pascal was born. (Hint: For Pascal's identity, consider selecting an unordered set of *r* people from a collection of *n* people, where one of the *n* is "Albertine.")

4. Consider the word POISSON.

(a) Find the number of arrangements of this word.

(b) Find the number of arrangements if the two Ss must be *together*.

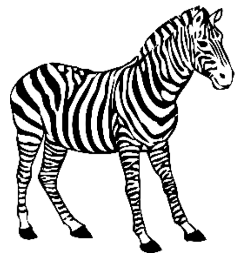
(c) Find the number of arrangements if the two Os must be *apart*.

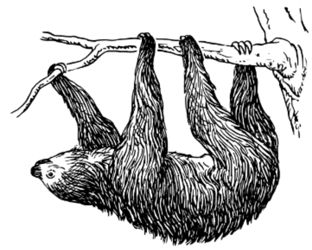
(d) Find the number of arrangements if the two Ss must be *together* and the two Os *not* together.

5. Give a *story* proof of Vandemonde's identity, viz.

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$$

(Hint: Consider choosing a set of r animals from a collection of m zebras and n sloths.)





6. You are given a group of 13 married couples. In how many ways can one choose a subset of 5 individuals from this group which *contains no married couple*?

7. In how many ways can A, B, C, D, E, F line up if

(a) A must be in front of B?

(b) A must be in front of B *and* B must be in front of C?

8. Philanthropist Dave Coke wishes to distribute 7 golden eggs, 6 silver spheres, and 5 platinum cubes to 4 lucky children. In how many different ways can he distribute these precious objects to the four children? (*Hint:* First consider only the golden eggs.)

9. In *how many ways* can 8 cats stand in a row if

(a) There are *precisely* 5 male cats and they *must* stand next to one another.

(b) There are two each of the following colors: tortoise shell, black, white, blue. Cats of the same color must stand beside one another.

10. Albertine lives in a city with a square grid of numbered streets that run east-west and numbered avenues that run north-south. Her house house is located on the corner of 0^{th} Street and and 0^{th} Avenue. Odette, her aunt, lives at the corner of 5th St. and 3rd Ave.

(a) How long is the *shortest route* (along streets or avenues) to her aunt's house? How many direct routes can Albertine take to her aunt's house?

(b) There is an ATM at the corner of 2^{nd} St. and 2^{nd} Ave. If Albertine needs to stop at the store on her way to her Aunt's, how many direct routes to her Aunt's house take her through the intersection of 2^{nd} St. and 2^{nd} Ave?

(c) At her Aunt's house, Albertine hears on the radio that there has been an accident at the corner of 1st St. and 2nd Ave. Assuming that she avoids this intersection, how many direct routes can Albertine take home?

11. Consider a standard well-shuffled deck of 52 cards. Swann is dealt (an *unordered*) hand of 5 cards. In how many ways can he have:

(a) Ace of diamonds, Jack of spades, 9 of clubs, 9 of spades, and 3 of clubs

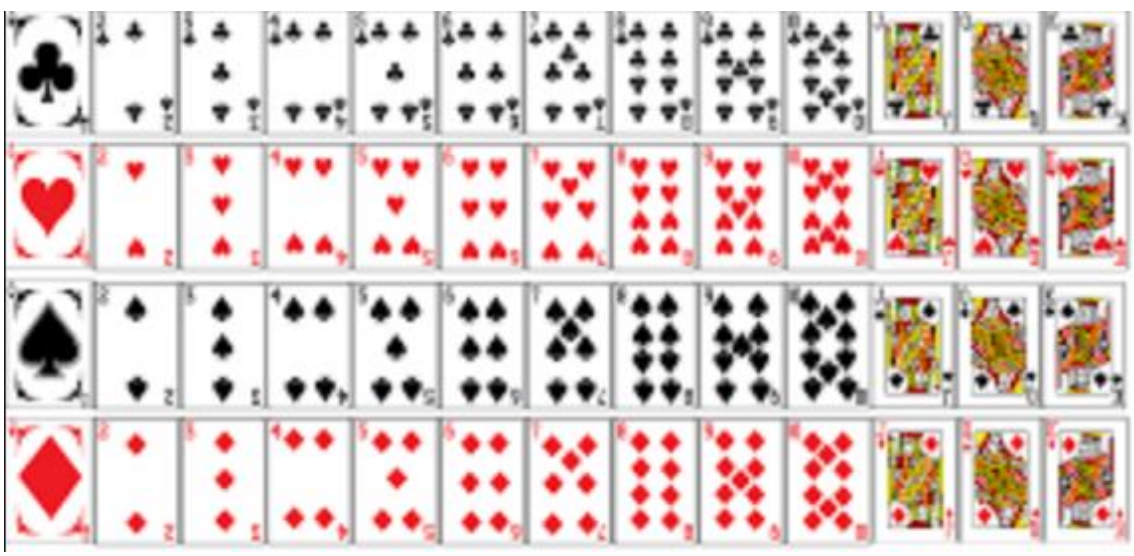
(b) A full-house (containing three cards of one rank and two cards of another rank, such as 3♣ 3♠ 3♦ 6♣ 6♥)

(c) Four of a kind.

(d) Exactly two pairs.

(e) No two of the same rank.

(f) No two of the same suit.



Exercises for Section 3.4

1. What is the smallest n for which $n!$ has more than 10 digits?
2. For which values of n does $n!$ have n or fewer digits?
3. How many 5-digit positive integers are there in which there are no repeated digits and all digits are odd?
4. Using only pencil and paper, find the value of $\frac{100!}{95!}$.
5. Using only pencil and paper, find the value of $\frac{120!}{118!}$.
6. There are two 0's at the end of $10! = 3,628,800$. Using only pencil and paper, determine how many 0's are at the end of the number $100!$.
7. Find how many 9-digit numbers can be made from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 if repetition is not allowed and all the odd digits occur first (on the left) followed by all the even digits (i.e., as in 137598264, but not 123456789).
8. Compute how many 7-digit numbers can be made from the digits 1, 2, 3, 4, 5, 6, 7 if there is no repetition and the odd digits must appear in an unbroken sequence. (Examples: 3571264 or 2413576 or 2467531, etc., but **not** 7234615.)
9. How many permutations of the letters *A, B, C, D, E, F, G* are there in which the three letters ABC appear consecutively, in alphabetical order?
10. How many permutations of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are there in which the digits alternate even and odd? (For example, 2183470965.)
11. You deal 7 cards off of a 52-card deck and line them up in a row. How many possible lineups are there in which not all cards are red?
12. You deal 7 cards off of a 52-card deck and line them up in a row. How many possible lineups are there in which no card is a club?
13. How many lists of length six (with no repetition) can be made from the 26 letters of the English alphabet?
14. Five of ten books are arranged on a shelf. In how many ways can this be done?
15. In a club of 15 people, we need to choose a president, vice-president, secretary, and treasurer. In how many ways can this be done?
16. How many 4-permutations are there of the set $\{A,B,C,D,E,F\}$ if whenever A appears in the permutation, it is followed by E?
17. Three people in a group of ten line up at a ticket counter to buy tickets. How many lineups are possible?
18. There is a very interesting function $\Gamma : [0,\infty) \to \mathbb{R}$ called the **gamma function**. It is defined as $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$. It has the remarkable property that if $x \in \mathbb{N}$, then $\Gamma(x) = (x-1)!$. Check that this is true for $x = 1,2,3,4$.

    Notice that this function provides a way of extending factorials to numbers other than integers. Since $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$, we have the formula $n! = \Gamma(n+1)$. But Γ can be evaluated at any number in $[0,\infty)$, not just at integers, so we have a formula for $n!$ for any real number $n \in [0,\infty)$. Extra credit: Compute $\pi!$.

Exercises for Section 3.5

1. Suppose a set A has 37 elements. How many subsets of A have 10 elements? How many subsets have 30 elements? How many have 0 elements?
2. Suppose A is a set for which $|A| = 100$. How many subsets of A have 5 elements? How many subsets have 10 elements? How many have 99 elements?
3. A set X has exactly 56 subsets with 3 elements. What is the cardinality of X?
4. Suppose a set B has the property that $\left|\{X : X \in \mathscr{P}(B), |X| = 6\}\right| = 28$. Find $|B|$.
5. How many 16-digit binary strings contain exactly seven 1's? (Examples of such strings include 0111000011110000 and 0011001100110010, etc.)
6. $\left|\{X \in \mathscr{P}(\{0,1,2,3,4,5,6,7,8,9\}) : |X| = 4\}\right| =$
7. $\left|\{X \in \mathscr{P}(\{0,1,2,3,4,5,6,7,8,9\}) : |X| < 4\}\right| =$
8. This problem concerns lists made from the symbols *A, B, C, D, E, F, G, H, I*.
   - **(a)** How many length-5 lists can be made if there is no repetition and the list is in alphabetical order? (Example: *BDEFI* or *ABCGH*, but not *BACGH*.)
   - **(b)** How many length-5 lists can be made if repetition is not allowed and the list is **not** in alphabetical order?
9. This problem concerns lists of length 6 made from the letters *A,B,C,D,E,F*, without repetition. How many such lists have the property that the *D* occurs before the *A*?
10. A department consists of 5 men and 7 women. From this department you select a committee with 3 men and 2 women. In how many ways can you do this?
11. How many positive 10-digit integers contain no 0's and exactly three 6's?
12. Twenty-one people are to be divided into two teams, the Red Team and the Blue Team. There will be 10 people on Red Team and 11 people on Blue Team. In how many ways can this be done?
13. Suppose $n, k \in \mathbb{Z}$, and $0 \le k \le n$. Use Fact 3.5, the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, to show that $\binom{n}{k} = \binom{n}{n-k}$.
14. Suppose $n, k \in \mathbb{Z}$, and $0 \le k \le n$. Use Definition 3.2 alone (without using Fact 3.5) to show that $\binom{n}{k} = \binom{n}{n-k}$.
15. How many 10-digit binary strings are there that do not have exactly four 1's?
16. How many 6-element subsets of $A = \{0,1,2,3,4,5,6,7,8,9\}$ have exactly three even elements? How many do not have exactly three even elements?
17. How many 10-digit binary strings are there that have exactly four 1's or exactly five 1's? How many do not have exactly four 1's or exactly five 1's?
18. How many 10-digit binary strings have an even number of 1's?
19. A 5-card poker hand is called a *flush* if all cards are the same suit. How many different flushes are there?