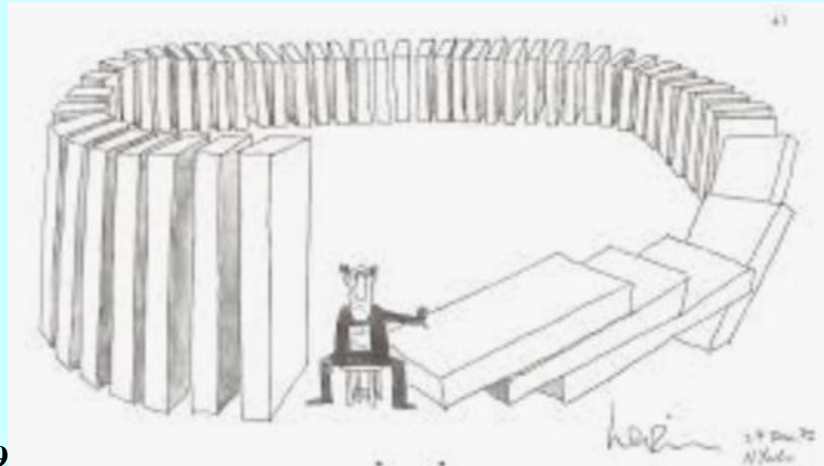


CLASS DISCUSSION: MATHEMATICAL INDUCTION, CONTINUED



18 February 2019

The Induction Principle.

Let P be a predicate on nonnegative integers. If

- $P(0)$ is true, and
- $P(n)$ IMPLIES $P(n + 1)$ for all nonnegative integers, n ,

then

- $P(m)$ is true for all nonnegative integers, m .

History (Wikipedia) In 370 BC, Plato's *Parmenides* may have contained an early example of an implicit inductive proof. The earliest implicit traces of mathematical induction may be found in Euclid's proof that the number of primes is infinite and in Bhaskara's "cyclic method." An opposite iterated technique, counting *down* rather than up, is found in the Sorites paradox, where it was argued that if 1,000,000 grains of sand formed a heap, and removing one grain from a heap left it a heap, then a single grain of sand (or even no grains) forms a heap.

An implicit proof by mathematical induction for arithmetic sequences was introduced in the **al-Fakhri** written by al-Karaji around 1000 AD, who used it to prove the binomial theorem and properties of Pascal's triangle.

None of these ancient mathematicians, however, explicitly stated the induction hypothesis. Another similar case (contrary to what Vacca has written, as Freudenthal carefully showed) was that of Francesco Maurolico in his *Arithmeticonum Libri duo* (1575), who used the technique to prove that the sum of the first n odd integers is n^2 . The first explicit formulation of the principle of induction was given by [Pascal](#) in his *Traité du triangle arithmétique* (1665). Another Frenchman, Fermat, made ample use of a related principle, indirect proof by infinite descent. The induction hypothesis was also employed by the Swiss Jakob Bernoulli, and from then on it became more or less well known. The modern rigorous and systematic treatment of the principle came only in the 19th century, with George Boole, Augustus de Morgan, Charles Sanders Peirce, Giuseppe Peano, and Richard Dedekind.

Using the method of ordinary mathematical induction, prove each of the following:

1. $(n^3 + 2n)$ is a multiple of 3 for all natural numbers, n .
2. $(7^n - 2^n)$ is a multiple of 5 for all non-negative integers, n .

3. $1 + 3 + 5 + \dots + (2n - 1) = n^2$ for all natural numbers, n
4. $(1 + x)^n \geq 1 + nx$ for all real $x > -1$ and all non-negative integers.
(This is called *Bernoulli's inequality*.)
5. $1 + 2 + 3 + \dots + n = n(n+1)/2$ for all natural numbers, n .
6. $1^2 + 2^2 + 3^2 + \dots + n^2 = n(n+1)(2n+1)/6$ for all natural numbers, n .
7. $2 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 2$ for all natural numbers, n .
8. $4n < 2^n$ for all natural numbers $n \geq 5$.
9. $(1)(2) + (2)(3) + (3)(4) + \dots + (n)(n+1) = n(n+1)(n+2)/3$ for all natural numbers, n .
10. $(12^{2n} - 11^n)$ is a multiple of 133 for all non-negative integers n .
11. A total of n democrats and republicans stand in line. If the person at the head of the line is a Democrat and the person at the end of the line is a Republican, then somewhere in the line, a republican must stand next to a Democrat.



12. What is the flaw in the “induction proof” discussed in class that: given n spiders, if one is a tarantula, then all the spiders are tarantulas?

13. Consider the triomino problem discussed in class. Prove that every $2^n \times 2^n$ punctured board



can be covered by triominos.

14. $1^2 + 2^2 + 3^2 + \dots + n^2 = n(n+1)(2n+1)/6$ for all natural numbers n .
15. $(1)(2) + (2)(3) + (3)(4) + \dots + (n)(n+1) = n(n+1)(n+2)/3$ for all natural numbers n .
16. $2^n > n^2$ for all natural numbers $n \geq 5$.
17. $8^{2^n} - 5^{2^n}$ is a multiple of 13 for all natural numbers n .
18. $(d/dx) x^n = n x^{n-1}$ for all natural numbers n . (You may use the product rule.)
19. Prove independently (of question #17) that 3 is a divisor of $2^{2^n} - 1$.

Guess the formula, and prove by induction:

20. $f(n) = 1^2 + 2^2 + 3^2 + \dots + n^2, n \geq 1.$

21. $g(n) = 1 + 4 + 7 + \dots + (3n - 2), n \geq 1.$

22. $u(n) = 1 + 3 + 3^2 + \dots + 3^{n-1}, n \geq 1.$

Induction makes you feel guilty for getting something out of nothing, and it is artificial, but it is one of the greatest ideas of civilization.

- Herbert Wilf