MATH 201 CLASS DISCUSSION: 6 FEBRUARY 2019

COUNTING, CONTINUED

1. Define: P(n, k) and $\binom{n}{k}$.

2. Derive a formula for P(n, k).

3. Using the result from (2), derive a formula for $\binom{n}{k}$.

Highly recommended: https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-041-probabilistic-systems-analysis-and-applied-probability-fall-2010/video-lectures/lecture-4-counting/
Watch at least the first half of video lecture 4 on counting. You may ignore references to probability.

4. Give a *story* proof (as opposed to a computational proof) for each of the two identities:

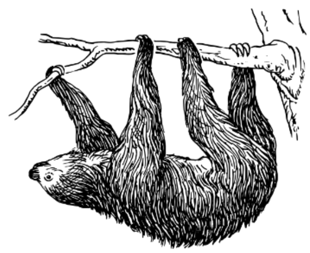
$$\binom{n}{r} = \binom{n}{n-r} \quad \textit{and} \quad \binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

The latter identity is known as "Pascal's identity" even though many mathematicians had "discovered" it before Pascal was born.
(Hint: For Pascal's identity, consider selecting an unordered set of r people from a collection of n people, where one of the n is "Albertine.")

5. Give a *story* proof of Vandemonde's identity, viz.

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$$

(Hint: Consider choosing a set of r animals from a collection of m zebras and n sloths.)



6. Consider the word POISSON.

(a) Find the number of arrangements of this word.

(b) Find the number of arrangements if the two Ss must be *together*.

(c) Find the number of arrangements if the two Os must be *apart*.

(d) Find the number of arrangements if the two Ss must be *together* and the two Os *not* together.

7. Given a group of 13 married couples. In how many ways can one choose a subset of 5 individuals from this group which *contains no married couple*?

8. In how many ways can A, B, C, D, E, F line up if

(a) A must be in front of B?

(b) A must be in front of B *and* B must be in front of C?

9. Philanthropist David Coke wishes to distribute 7 golden eggs, 6 silver spheres, and 5 platinum cubes to 4 lucky children. In how many different ways can he distribute these precious objects to the four children? (*Hint:* First consider only the golden eggs.)

10. In *how many ways* can 8 people be seated in a row if

(a) there are *exactly* 5 men and they *must* sit next to one another?

(b) there are 4 married couples and each couple *must* sit together?

11. Three *distinguishable* dice are thrown. In how many ways can the *maximum* of the 3 numbers occurring equal 5?

12. Twenty-five students show up at the OZ Fitness & YOGA Center looking for open

classes. Only 3 classes are still open: one has 8 spots, one has 11 spots, and one has 6 spots. In how many different ways can the students be arranged in the 3 classes?

13. Albertine lives in a city with a square grid of numbered streets which run east-west and numbered avenues that run north-south. Her house is located on the corner of 0th Street and 0th Avenue. Odette, her aunt, lives at the corner of 5th St. and 3rd Ave.

(a) How long is the *shortest route* (along streets or avenues) to her aunt's house? How many direct routes can Sally take to her aunt's house?

(b) There is an ATM machine at the corner of 2nd St. and 2nd Ave. If Albertine needs to stop at the store on her way to her Aunt's, how many direct routes to her Aunt's house take her through the intersection of 2nd St. and 2nd Ave?

(c) At her Aunt's house Albertine hears on the radio that there has been an accident at the corner of 1st St. and 2nd Ave. Assuming that she avoids this intersection, how many direct routes can Albertine take home?

14. Consider a standard well-shuffled deck of 52 cards. Swann is dealt (an *unordered*) hand of 5 cards. In how many ways can he have:

(a) Ace of diamonds, Jack of spades, 9 of clubs, 9 of spades, and 3 of clubs

(b) A full-house (containing three cards of one rank and two cards of another rank, such as 3♣ 3♠ 3♦ 6♣ 6♥)

(c) Four of a kind.

(d) Exactly two pairs.

(e) No two of the same rank.

(f) No two of the same suit.

Exercises for Section 3.4

1. What is the smallest n for which $n!$ has more than 10 digits?
2. For which values of n does $n!$ have n or fewer digits?
3. How many 5-digit positive integers are there in which there are no repeated digits and all digits are odd?
4. Using only pencil and paper, find the value of $\frac{100!}{95!}$.
5. Using only pencil and paper, find the value of $\frac{120!}{118!}$.
6. There are two 0's at the end of $10! = 3,628,800$. Using only pencil and paper, determine how many 0's are at the end of the number 100!.
7. Find how many 9-digit numbers can be made from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 if repetition is not allowed and all the odd digits occur first (on the left) followed by all the even digits (i.e., as in 137598264, but not 123456789).
8. Compute how many 7-digit numbers can be made from the digits 1, 2, 3, 4, 5, 6, 7 if there is no repetition and the odd digits must appear in an unbroken sequence. (Examples: 3571264 or 2413576 or 2467531, etc., but **not** 7234615.)
9. How many permutations of the letters *A, B, C, D, E, F, G* are there in which the three letters ABC appear consecutively, in alphabetical order?
10. How many permutations of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are there in which the digits alternate even and odd? (For example, 2183470965.)
11. You deal 7 cards off of a 52-card deck and line them up in a row. How many possible lineups are there in which not all cards are red?
12. You deal 7 cards off of a 52-card deck and line them up in a row. How many possible lineups are there in which no card is a club?
13. How many lists of length six (with no repetition) can be made from the 26 letters of the English alphabet?
14. Five of ten books are arranged on a shelf. In how many ways can this be done?
15. In a club of 15 people, we need to choose a president, vice-president, secretary, and treasurer. In how many ways can this be done?
16. How many 4-permutations are there of the set $\{A,B,C,D,E,F\}$ if whenever A appears in the permutation, it is followed by E?
17. Three people in a group of ten line up at a ticket counter to buy tickets. How many lineups are possible?
18. There is a very interesting function $\Gamma : [0,\infty) \to \mathbb{R}$ called the **gamma function**. It is defined as $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$. It has the remarkable property that if $x \in \mathbb{N}$, then $\Gamma(x) = (x-1)!$. Check that this is true for $x = 1,2,3,4$.
    Notice that this function provides a way of extending factorials to numbers other than integers. Since $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$, we have the formula $n! = \Gamma(n+1)$. But Γ can be evaluated at any number in $[0,\infty)$, not just at integers, so we have a formula for $n!$ for any real number $n \in [0,\infty)$. Extra credit: Compute $\pi!$.

Also:
History of Combinatorics, N. J. Wildberger, University of New South Wales, Australia
https://www.youtube.com/watch?v=7kcO8EYY7xs

How many really basic mathematical objects are there? One is surely the `miraculous' jar of the positive integers 1, 2, 3 . . . Another is the concept of a fair coin. Though gambling was rife in the ancient world and although prominent Greeks and Romans sacrificed to Tyche, the goddess of luck, her coin did not arrive on the mathematical scene until the Renaissance. Perhaps one of the things that had delayed this was a metaphysical position which held that God speaks to humans through the action of chance. . . . The modern theory begins with the expulsion of Tyche from the Pantheon. There emerges the vision of the fair coin, the biased coin. This coin exists in some mental universe and all modern writers on probability theory have access to it. They toss it regularly and they speculate about what they 'observe.'

- Philip Davis & Reuben Hersh, **The Mathematical Experience**