CLASS DISCUSSION: PROOF BY CONTRADICTION; 1 MARCH 2109

Method

To prove a proposition P by contradiction:

1. Write, "We use proof by contradiction."
2. Write, "Suppose $\sim P$"
3. Deduce a logical contradiction C. (That is, find C for which $C \wedge \sim C$.)
4. Write, "This is a contradiction. Therefore, P must be true."

Example

Remember that a number is *rational* if it is equal to a ratio of integers. For example, $3.5 = 7/2$ and $0.1111_{\ldots} = 1/9$ are rational numbers. On the other hand, we'll prove by contradiction that $\sqrt{2}$ is irrational.

Proposition: $\sqrt{2}$ *is irrational.*

Proof. We use proof by contradiction.

Suppose the claim is false; that is, $\sqrt{2}$ is rational.

Then we can write 2 as a fraction a/b in *lowest terms*.

Squaring both sides gives $2 = a^2/b^2$ and so $2b^2 = a^2$.
This implies that a is even; that is, a is a multiple of 2.
Therefore, a^2 must be a multiple of 4. Because of the equality $2b^2 = a^2$, we know $2b^2$ must also be a multiple of 4.
This implies that b^2 is even and so b must be even. But since a and b are both even, the fraction a/b is not in lowest terms.

This is a contradiction. Therefore, $\sqrt{2}$ must be irrational. □

Potential Pitfall

Often students use an indirect proof when a direct proof would be simpler. Such proofs aren't wrong; they just aren't excellent. Let's look at an example. A function f is *strictly increasing* if $f(x) > f(y)$ for all real x and y such that $x > y$.

Theorem. *If f and g are strictly increasing functions, then $f + g$ is a strictly increasing function.*

Let's first look at a simple, direct proof.

Proof. Let x and y be arbitrary real numbers such that $x > y$. Then:

$f(x) > f(y)$ (since f is *strictly* increasing)

$g(x) > g(y)$ (since g is *strictly* increasing)

Adding these inequalities gives:

$$f(x) + g(x) > f(y) + g(y)$$

Thus, $f + g$ is strictly increasing as well. □

Now we *could* prove the same theorem by contradiction, but this makes the argument needlessly convoluted.

Proof. We use proof by contradiction. Suppose that $f + g$ is not strictly increasing. Then there must exist real numbers x and y such that $x > y$, but

$$f(x) + g(x) \leq f(y) + g(y)$$

This inequality can only hold if either $f(x) \leq f(y)$ or $g(x) \leq g(y)$. Either way, we have a contradiction because both f and g were defined to be strictly increasing. Therefore, $f + g$ must actually be strictly increasing. □

Exercises (As usual, assume unless otherwise stated, that our universe is **Z**.)

1. If a is even then a^2 is even. Prove by contradiction.
2. If $a \geq 2$, then $a \nmid b$ or $a \nmid (b+1)$

3. If n^2 is odd, then n is odd.
4. Prove that $\sqrt[3]{2}$ is irrational.
5. Prove that $a^2 - 4b - 3 \neq 0$.
6. Prove that there exist no integers, a and b, such that 21a + 30b = 1.
7. If A and B are arbitrary sets, then $A \cap (B - A) = \emptyset$.
8. Show that for any n, $4 \nmid (n^2 + 2)$.
9. Study the following proof (from our textbook). Is it logically correct?

Proposition If $a, b \in \mathbb{Z}$, then $a^2 - 4b \neq 2$.

Proof. Suppose this proposition is *false*.
This conditional statement being false means there exist numbers a and b for which $a, b \in \mathbb{Z}$ is true, but $a^2 - 4b \neq 2$ is false.
In other words, there exist integers $a, b \in \mathbb{Z}$ for which $\boxed{a^2 - 4b = 2.}$
From this equation we get $a^2 = 4b + 2 = 2(2b+1)$, so a^2 is even.
Because a^2 is even, it follows that a is even, so $a = 2c$ for some integer c.
Now plug $a = 2c$ back into the boxed equation to get $(2c)^2 - 4b = 2$, so $4c^2 - 4b = 2$. Dividing by 2, we get $2c^2 - 2b = 1$.
Therefore $1 = 2(c^2 - b)$, and because $c^2 - b \in \mathbb{Z}$, it follows that 1 is even.
We know 1 is **not** even, so something went wrong.
But all the logic after the first line of the proof is correct, so it must be that the first line was incorrect. In other words, we were wrong to assume the proposition was false. Thus the proposition is true. ■

Exercises:

10. Prove by contradiction that there exists no largest even integer.
11. Prove by contradiction that $\sqrt[3]{1332} > 11$.
12. There exist no integers a and b for which 21a+30b = 1.
13. Prove by contradiction that there exists no smallest positive real number.
14. Prove by contradiction that there exists no largest prime number. (Euclid's proof)
15. Prove by contradiction that $\sqrt{3}$ *is irrational.*
16. Prove by contradiction that if x is irrational then so is $x^{1/2}$.
17. Prove by contradiction that $2^{1/3}$ is irrational.
18. Suppose $n \in Z$. If n is odd, then n^2 is odd.
19. Suppose $n \in Z$. If n^2 is odd, then n is odd.
20. Prove by contradiction that if $0 \leq t \leq \pi/2$ then $\cos t + \sin t \geq 1$.
21. Let n be a positive integer. Prove that $\log_2 n$ is rational if and only if n is a power of 2.
22. Prove the arithmetic-geometric mean inequality by contradiction.
23. Employing the method of proof by contradiction show that for any non-degenerate triangle (that is, every side has positive length), the length of the hypotenuse is *less than* the sum of the lengths of the two remaining sides.
24. Let a and b be integers. If $a^2 + b^2 = c^2$, then a or b is even.
25. Prove that there are infinitely many prime numbers (Euclid).

Exercises for Chapter 6

A. Use the method of proof by contradiction to prove the following statements. (In each case, you should also think about how a direct or contrapositive proof would work. You will find in most cases that proof by contradiction is easier.)

1. Suppose $n \in \mathbb{Z}$. If n is odd, then n^2 is odd.

2. Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

3. Prove that $\sqrt[3]{2}$ is irrational.

4. Prove that $\sqrt{6}$ is irrational.

5. Prove that $\sqrt{3}$ is irrational.

6. If $a, b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.

7. If $a, b \in \mathbb{Z}$, then $a^2 - 4b - 3 \neq 0$.

8. Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

9. Suppose $a, b \in \mathbb{R}$. If a is rational and ab is irrational, then b is irrational.

10. There exist no integers a and b for which $21a + 30b = 1$.

11. There exist no integers a and b for which $18a + 6b = 1$.

12. For every positive $x \in \mathbb{Q}$, there is a positive $y \in \mathbb{Q}$ for which $y < x$.

13. For every $x \in [\pi/2, \pi]$, $\sin x - \cos x \geq 1$.

14. If A and B are sets, then $A \cap (B - A) = \emptyset$.

15. If $b \in \mathbb{Z}$ and $b \nmid k$ for every $k \in \mathbb{N}$, then $b = 0$.

16. If a and b are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

17. For every $n \in \mathbb{Z}$, $4 \nmid (n^2 + 2)$.

18. Suppose $a, b \in \mathbb{Z}$. If $4 \mid (a^2 + b^2)$, then a and b are not both odd.

B. Prove the following statements using any method from Chapters 4, 5 or 6.

19. The product of any five consecutive integers is divisible by 120. (For example, the product of 3,4,5,6 and 7 is 2520, and $2520 = 120 \cdot 21$.)

20. We say that a point $P = (x, y)$ in $\mathbb{R}^2$ is **rational** if both x and y are rational. More precisely, P is rational if $P = (x, y) \in \mathbb{Q}^2$. An equation $F(x, y) = 0$ is said to have a **rational point** if there exists $x_0, y_0 \in \mathbb{Q}$ such that $F(x_0, y_0) = 0$. For example, the curve $x^2 + y^2 - 1 = 0$ has rational point $(x_0, y_0) = (1, 0)$. Show that the curve $x^2 + y^2 - 3 = 0$ has no rational points.

21. Exercise 20 (above) involved showing that there are no rational points on the curve $x^2 + y^2 - 3 = 0$. Use this fact to show that $\sqrt{3}$ is irrational.

22. Explain why $x^2 + y^2 - 3 = 0$ not having any rational solutions (Exercise 20) implies $x^2 + y^2 - 3^k = 0$ has no rational solutions for k an odd, positive integer.

23. Use the above result to prove that $\sqrt{3^k}$ is irrational for all odd, positive k.

24. The number $\log_2 3$ is irrational.

G. H. Hardy described proof by contradiction as "one of a mathematician's finest weapons," saying "It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game."

- G. H. Hardy, **A Mathematician's Apology**