MATH 201 PRACTICE PROBLEMS FOR TEST III

Part I

1. Let X be a set. Define P(X), the *power set* of X.
2. Let F: X →Y be a mapping. Define: F is *injective*.
3. Let G: X →Y be a mapping. Define: G is *surjective*.
4. Let S be a set. Define: S is *countably infinite*.
5. Let R and S be sets. What does it mean to assert that R and S *are of the same cardinality*?
6. If |A| = 2019, then |P(A)| =
7. Let X and Y be sets and let G: X→Y be a bijection. Define G^{-1}: Y → X
8. 15^{99999} (mod 16) is congruent to _______________
9. If a ≡ 9 (mod 11) and b ≡ 7 (mod 11), then 3a + 5b ≡ _____________ (mod 11)
10. Let S be a set. Then an equivalence relation on S corresponds to a ____________ on S and vice-versa.
11. Explain the difference between proof by contradiction and proof by contrapositive.
12. State Euclid's lemma.
13. Let a, b be two integers, not both 0. Define gcd(a, b).
14. State the division algorithm.
15. Define gcd.

Part II

1. *Here is a partial proof of Cantor's theorem. Your job is to complete it.*

Theorem: For any set X, P(X) and X are not of the same cardinality.

Proof: Let X be a set and P(X) be the power set of X.

Strategy: proof by contradiction.

Assume that ∃ bijection F: X →P(X).

Define $D^* = \{q \in X \mid q \notin F(q)\}$

Then:

2. Odette believes that the set of all *finite* sequences using the alphabet {a, b, c} is countably infinite. Give proof or counterexample. (For example: aab, cabcab, ccccccccccccccc are all finite sequences.)

3. Swann believes that the set of all *infinite* sequences using the alphabet {a, b, c} is countably infinite. Give proof or counterexample. (For example: abababab… , abcabccccbbaabbbbb… are infinite strings.)

4. Let A, B be non-empty finite sets. Suppose that |A|=1789 and |B| = 2016. Then

(a) The number of surjections from A into B is ___________. Why?

(b) The number of injections from A into B is ___________. Why?

5. Let Q be the set of all rational numbers. Define f: Z → Z as follows:
For all x

$$\forall x \in Z \quad f(x) = \frac{x(x+1)(x+2)}{6}$$

(a) Is *f* well-defined? Justify your answer.

If your answer to (a) is negative, then skip (b) and (c).

(b) Is *f* injective? Why?

(c) Is *f* surjective? Why?

6. For any real numbers, *c* and *d*, let us define the binary operation ☺ as follows:

$$c ☺ d = c^2 + d^2 - 1$$

Give either a *brief* justification or counterexample for each of the following assertions:

(a) The set of integers is closed under the operation ☺ .

(b) The set of even integers is closed under the operation ☺ ?

(c) The set of odd integers is closed under ☺.

(d) The set of positive integers is closed under ☺.

(e) The set of rational numbers is closed under ☺.

(f) The set of irrational numbers is closed under ☺.

7. Let X = {0, 2, 5, 8, 13, 15} and Y = P(X). Define T: X →P(X) as follows:

$$\forall j \in X \quad T(j) = \begin{cases} \{j\} \text{ if } \; j \text{ is prime} \\ \{0, 8, 13\} \text{ if } \; j \text{ is not prime} \end{cases}$$

Find D^* (as defined in Part II, problem 1)

8. Give an *example* of three sets, X, Y, S, and two mappings, F: X →Y, G: Y →S such that G is *not* injective yet $G \circ F$ is injective.

9. Let S be the set of all polynomials for which odd powers of x (i.e., 1, 3, 5, 7,…) do not appear.

For example, $4 + \frac{1}{2} x^2 - \pi x^8 \in S$, $but \; x^2 - \frac{5}{4} x^{81} \notin S$.

(a) Is S closed under the operation of differentiation? Why?

(b) Is S closed under the operation of taking the second derivative?

(c) Is S closed under the operation multiplication by x^3?

(d) Is S closed under the operation multiplication by x^4?

(e) Is S closed under the following unary operation, ● ?

For all $p \in S \;\; define\; \Pi p \; as\; follows$: $\Pi p = p(1 + x^4)$

Note: Πp *means* that polynomial p is evaluated at $1 + x^4$

(f) Is S closed under the following binary operation ☹?

For all $p \in S \quad p☹ = p(1 + x)$ Here as before, we interpret p(1 + x) is the polynomial p evaluated at 1 + x.

10. Define the following binary relation R on $\mathbf{Z}^{+}$: For c, d ∈ $\mathbf{Z}^{+}$, cRd if $|c – d| < 5$. *(Justify each answer!)*

(a) Is R reflexive? Why?
(b) Is R symmetric? Why?
(c) Is R transitive? Why?

11. Define the binary relation *R* on a non-empty set, S, of books as follows:

For a, b ∈ S, a*R*b if book ***a*** costs more **and** contains fewer pages than book ***b***. *(Justify each answer!)*

(a) Is R reflexive? Why?
(b) Is R symmetric? Why?
(c) Is R transitive? Why?

12. Again, let *S* be a non-empty set of books. Let *H* be the binary relation defined by:

for *a*, *b* ∈ *S*, a*H*b if book ***a*** costs more ***or*** contains fewer pages than book ***b***. *(Justify each answer!)*
(a) Is *H* reflexive? Why?
(b) Is *H* symmetric? Why?
(c) Is *H* transitive? Why?

13. Find the x that maintains the pattern:

16, 06, 68, 88, x, 98

14. Suppose S is a set of n + 1 integers. Prove that there exist distinct a, b ∈ *S* such that a – b is a multiple of n. .

15. Show that in any group of *n* people, there are two who have an identical number of friends within the group.

16. Let *x* and *y* be real numbers. Prove that if $x^2 + 5y = y^2 + 5x$. Prove that either x = y or x + y = 5.

17. Using the method of contrapositive proof, prove each of the following statements:

(a) Suppose that x, y ∈ Z. If $x^2(y+3)$ is even, prove that x is even or y is odd.
(b) Let a ∈Z. Prove that if a^2 is not divisible by 4, then a is odd.

18. Prove that there exist no integers a and b for which 390a + 63b = 1.

19. Use Fermat's little theorem to show that 17 divides $11^{104}+1$.

20. Prove that $\sqrt{6}$ is irrational.

21. Let a ∈Z. Prove that $a^3 \equiv a \ (mod\ 3)$.

22. Let a, b ∈Z. Prove that $(a – 3)b^2$ is even if and only if a is odd and b is even.

23. The three most recent appearances of Haley's comet were in the years 1835, 1910, and 1986. The next occurrence will be in 2061. Prove that $1835^{1910} + 1986^{2061} \equiv 0 \ (mod\ 7)$.

24. Find 2013^{13} (mod 13)

25. Let W be the set of all words in the 2019 edition of the Oxford English dictionary. Define xRy if x and y have at least one letter in common. Is R reflexive? Symmetric? Transitive?

26. Which of the following functions are injective, surjective, and bijective on their respective domains. Take the domains of the functions as those values of x for which the function is well-defined.

a) $f(x) = x^3 - 2x + 1$

b) $f(x) = \dfrac{x+1}{x-1}$

c) $f(x) = \begin{cases} x^2 & x \le 0 \\ x+1 & x > 0 \end{cases}$

d) $f(x) = \dfrac{1}{x^2+1}$

27. A murderer is condemned to death. He has to choose from among three rooms:
The firs is full of raging fires; the second is full of assassins with loaded guns; and the third was full of lions hwo haven't eaten in years. Which room is the safest?