MATH 201 SOLUTIONS QUIZ IV 29 MARCH 2019

1. Let P(X) be the power set of a non-empty set X. For any two subsets A and B of X, define the relation A ⧖ B on P(X) to mean that $A \cap B = \varnothing$ (the empty set). Justify your answer to each of the following.

(a) Is ⧖ reflexive? Explain.

Solution: A⧖A is generally false. Since X is non-empty, Since X is non-empty ∃

No: $A \cap A = \varnothing$ *implies that* $A = \varnothing$. *This need not be true since X is non-empty.*

(b) Is ⧖ symmetric? Explain.

Solution: Yes: If A ⧖ B then $A \cap B = \varnothing$. *From this it follows that* $B \cap A = \varnothing$.
Consequently, B ⧖ A.

(c) Is ⧖ transitive? Explain.

Solution:

No: Let $p \in X$. Such a p must exist since X is non-empty. Then P(X) must include $\varnothing$ and {p}.
Let A = C = {p} and $B = \varnothing$. Then A ⧖ B since $A \cap B = \varnothing$ and B ⧖ C since $B \cap C = \varnothing$. However, $A \cap C = \{p\}$, so A ⧖ C is false.

2. Let X be set of all 2×2 matrices whose entries are either 0 or 1.
Define $M \circledast N$ if the sum of the four entries of M = the sum of the four entries of N. Show that this defines an equivalence relation on X. How many equivalence classes are there? List the elements of each equivalence class.

Solution:

There are 16 2×2 binary matrices. Clearly $M \circledast M$ for $M \in X$.

Now $M \circledast N$ means that matrix M has the same number of 0s as N. Thus $N \circledast M$.

Next, if $A \circledast B$ and $B \circledast C$ then clearly $A \circledast C$.

There are 5 equivalence classes, viz.

Sum is 0: $\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$

Sum is 1: $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$

Sum is 2: $\left\{\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}\right\}$

Sum is 3: $\left\{\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\right\}$

Sum is 4: $\left\{\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right\}$

3. Let X be the set of all continuous functions on the interval [0, 1].

For f, g ∈ X, define the relation $\boxtimes$ *as follows*:

$$f \boxtimes g \text{ if } \int_0^1 f(x)dx \geq \int_0^1 g(x)dx$$

(a) Is $\boxtimes$ *reflexive*? Explain.

Solution: Yes: Since $\int_0^1 f(x)dx \geq \int_0^1 f(x)dx$, it follows that f $\boxtimes$ f.

(b) Is $\boxtimes$ *symmetric*? Explain.

Solution: No: Let f = 1 be a constant function and let g = 0 be a constant function.

Then $\int_0^1 f(x)dx = 1$ and $\int_0^1 g(x)dx = 0$. Thus f $\boxtimes$ g and yet g $\boxtimes$ f is false.

(c) Is $\boxtimes$ *transitive*? Explain.

Solution: Yes: Assume that f $\boxtimes$ g and g $\boxtimes$ h for f, g, and h ∈ C. Then

$$\int_0^1 f(x)dx \geq \int_0^1 g(x)dx \text{ and } \int_0^1 g(x)dx \geq \int_0^1 h(x)dx$$

Consequently,

$$\int_0^1 f(x)dx \geq \int_0^1 h(x)dx \text{ and so } f \boxtimes h.$$

Extra Credit:

Let Z be the set of all integers, and let X be the set of all polynomials in one variable with integer coefficients.

Define H: X→ Z as follows:

For p ∈ X, let $H(p) = p(1) + p'(2) + p''(3)$

(a) Explain why H is well-defined.

Solution: Yes, H is well-defined because, for every polynomial p(x) with integer coefficients, $p(1)$, $p'(2)$, $p''(3)$ are integers, and the set of integers is closed under multiplication.

So $H(p) = p(1) + p'(2) + p''(3) \in Z$.

(a) Is H surjective? Give proof or counter-example

Solution: Yes: Let k ∈ Z. Define p(x) ∈ X as follows: For all x ∈ X let p(x) = k, the constant function.

Then H(p) = p(1) + 0 + 0 = k. So H is surjective.

(b) Is H injective? Give proof or counter-example.

Solution: No: Let $p_1(x) = 13$, a constant function. Next, let $p_2(x) = 13 + (x – 1)^4$

Then p_2 (1) = 13, $p_2'(1) = 0$ and $p_2''(1) = 0$. Hence $H(p) = 13$.

Thus $p_1 \neq p_2$ and yet $H(p_1) = H(p_2)$.

So H is not injective.

(c) Is H surjective? Give proof or counter-example.

Solution: Yes: Let k ∈ Z. Define p(x) ∈ X as follows:

For all x ∈ X, let p(x) = k, the constant function.

Then H(p) = p(1) + 0 + 0 = k. So H is surjective.

Numbers are the highest degree of knowledge. It is knowledge itself.

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