MATH 201 SOLUTIONS: TEST I 8 FEBRUARY 2019

Instructions: Answer any 8 of the following 10 problems. You may answer more than 8 to earn extra credit.

1. For each of the following strings, answer YES or NO, depending upon whether the string of symbols is a wff (well-formed formula) or not. Assume that p and q are propositions. You need not justify your answers.

(A) $(p \to \sim q) \to (r \to (p \land q))$ YES

(B) ~p YES

(C) $pq \to \sim r$ NO: pq has no meaning

(D) $p\sim$ NO: $p\sim$ has no meaning

(E) $(\sim p \land q) \to (r \lor (s \to t)$ NO: missing right parenthesis

(F) $p \land \to q$ NO: $\land \to$ has no meaning

2. Prove that $(A \cup B) - C \subseteq (A - C) \cup (B - C)$

Proof: Let $x \in LHS = (A \cup B) - C$.

Then $x \in A \cup B \; and \; x \notin C$.

So we find that $x \notin C \; and \; \{x \in A \; or \; x \in B\}$.

Thus there are *two cases*: $x \notin C \; and \; x \in A$ OR $x \notin C \; and \; x \in B$.

In the first case, $x \notin C \; and \; x \in A$. $Hence \; x \in A - C$.

In the second case, $x \notin C \; and \; x \in B$. $Hence \; x \in B - C$.

Now, since x is either a member of A – C or a member of B – C,

$x \in (A - C) \cup (B - C)$.

3. Find an *explicit* counterexample to the statement:

$$(A \cap B) \cup C = A \cap (B \cup C)$$

Counterexample: Let $A = \emptyset, B = \{1\}, and \; C = \{1\}$.

Then LHS = C = {1} and RHS = ∅. Since LHS ≠ RHS, the counterexample is valid.

4. *Using truth tables*, determine if the following statements are logically equivalent or not.

$$p \to (q \wedge \sim q) \quad and \quad \sim p$$

Solution:

p	q	$\sim q$	$q \wedge \sim q$	$p \to (q \wedge \sim q)$	$\sim p$
T	T	F	F	F	F
T	F	T	F	F	F
F	T	F	F	T	T
F	F	T	F	T	T

Since $p \to (q \wedge \sim q)$ and $\sim p$ have the same truth values, they are logically equivalent.

5. Let U be the universe of all positive integers greater than 2.

Consider the following predicates:

P(x) = "x is a prime number" Q(x) = "x is odd"

Express each of the following statements in symbolic form.

(a) "x being prime is a *sufficient* condition for x being odd."

$$\forall x \in U \;\; P(x) \;\to Q(x)$$

(b) "x being odd is a *necessary* condition for x being prime."

$$\forall x \in U \;\; P(x) \;\to Q(x)$$

(c) "if x is odd and y is prime then x + y is not odd."

$$\forall x \in U \;\; P(x) \wedge Q(x) \;\to \sim Q(x+y)$$

6. Consider a standard deck of 52 cards. In the following, do not attempt to simplify your answers.

(a) In how many ways can you line up 5 cards such that each black card is followed by a red card and each red card is followed by a black?

Solution: Two cases.

Case I: Assume that the first card is black. Then the line up is BRBRB. Now the Bs can be filled in P(26, 3) ways. The Reds can be filled in P(26, 2) ways. Thus, using the multiplication principle, the number of possibilities for the hand is P(26, 3) P(26, 2).

Case II: Assume the first card is red. Now the number of possibilities for the line-up is P(26, 3) P(26, 2) by symmetry.

Using the addition principle, the total number of ways is 2 P(26, 3) P(26, 2).

(b) In how many ways can you line up 8 cards so that *at least one* of them is black?

Solution: We use the subtraction principle: P(52, 8) – P(26, 8).

(c) In how many ways can you line up four cards so that *exactly one* is a spade.

Solution: The spade can appear in any one of 4 slots. Then the spade may be chosen in 13 ways. The remaining 3 slots may be filled in P(39, 3) ways.

So the total number of ways of producing such a lineup is (4)(13)P(39, 3).

7. This problem involves 8-digit binary strings such as 10011011 or 00001011 (that is, 8-digit numbers composed of 0's and 1's.

(a) How many such strings are there?

Answer: 2^8 ways

(b) How many such strings end in 0?

Solution: Since there is no choice as far as the 8^{th} position is concerned, the number of such binary strings is 2^7.

(c) How many such strings have 1's for their second ***and*** fourth digits?

Solution: If the 2^{nd} and 4^{th} digits are determined, then we have only 6 choices left.

Using the multiplication principle, the total number of such strings is 2^6.

(d) *Extra credit:* How many such strings have 1's for their second ***or*** fourth digits?

Solution: Let A = set of strings that have a 1 in its second spot.

Let B = set of strings that have a 1 in its 4^{th} position.

Then, using the basic version of the inclusive-exclusive theorem, we find:

$$|A \cup B| = |A| + |B| - |A \cap B)| = 2^7 + 2^7 - 2^6 = 2^8 - 2^6 = 256 - 64 = 192$$

8. Let X = {1, {a, b} }

(a) List the elements of P(X)

Answer: The elements of P(X) are: $\emptyset, 1, \{a, b\}$, {1, {a, b} }

(b) Suppose that |Y| = 10, Find |P(Y)|?

Answer: 2^{10}

In the following define $E \,\Delta\, \mathrm{F} = (\mathrm{E} - \mathrm{F}) \cup (\mathrm{F} - \mathrm{E})$. (This is called the *symmetric difference* of A and B.) Now, let A = {1, 2, 3, 4} and let B = {2, 3, 4, 5, 6}

(c) Find A Δ B

Solution: $\mathrm{A} \,\Delta\, \mathrm{B} = (\mathrm{A} - \mathrm{B}) \cup (\mathrm{B} - \mathrm{A}) = (\{1,2,3,4\} - \{2,3,4,5,6\}) \cup (\{2,3,4,5,6\} - \{1,2,3,4\}) = \{1\} \cup \{5,6\} = \{1,5,6\}$

(d) Find $A \,\Delta\, \mathrm{A}$

Solution: $A \,\Delta\, \mathrm{A} = (\mathrm{A} - \mathrm{A}) \cup \ (\mathrm{A} - \mathrm{A}) = \emptyset \cup \emptyset = \emptyset$

(e) Find $A \,\Delta\, \emptyset$

Solution: $A \,\Delta\, \emptyset = (\mathrm{A} - \emptyset) \cup (\emptyset - \mathrm{A}) = \mathrm{A}$

9. *[revised]* Consider the universe of all people in the United States.
Define the following predicates:

S(x) = "x is a student"   Q(x) = "x is smart"   L(x, y) = "x loves y"

Express each of the following in symbolic form.

(a) All students are smart.

Answer: $\forall x\; S(X) \rightarrow Q(x)$

(b) There exists a student.

Answer: $\exists x\;\; S(X)$

(c) There exists a smart student.

Answer: $\exists x\;\; S(X) \wedge Q(x)$

(d) Every student loves some other student

Answer: $\forall x\; S(x) \rightarrow \exists y \neq x\;\;\; S(y) \wedge L(x,y)$

(e) No one loves anyone.

Answer: $\forall x\; \forall y$ ~L(x ,y)

(f) If Albertine loves anyone, then she loves Swann.

Answer: $\exists x$ L(Albertine, x) → $L(Albertine, Swann)$

(g) There is a student who loves himself but who is loved by *no one else*.

Answer: $\exists x \;\; S(x) \wedge L(x,x) \wedge (\forall y \neq x \sim L(y,x))$

10. Negate each of the following sentences. Your answer should be a sentence in English – not a symbolic sentence. (Nor should you simply write: "It is not true that ….").

(a) You can fool all of the people all of the time.

Answer: There is at least one person who cannot be fooled all of the time.

Alternatively, There exists a person x and a time t, such that x is not fooled at time t.

(b) If cos x = 7 then it is not true that x is an integer.

Answer: cos x = 7 and x is an integer.

(c) If the cat ate the mouse or the mouse ate the flea, then Albertine will watch either Dark on Netflix or go swimming in the lake.

Answer: The cat ate the mouse or the mouse ate the flea, and Albertine will watch neither Dark on Netflix nor go swimming in the lake.

(d) You cannot teach a cat to fetch, but you can teach a dog to swim.

Answer: You can teach a cat to fetch, or you cannot teach a dog to swim.