MATH 201 SOLUTIONS: TEST II 15 MARCH 2019

Instructions: Answer any 13 of the following 16 problems. You may answer more than 13 to earn extra credit.

1. Let *a* and *b* be integers. Prove that $a^2 - 4b - 2 \neq 0$.

Proof: We will use the method of contradiction.

Assume that $a^2 - 4b - 2 = 0$. Then $a^2 = 4b + 2 = 2(2b + 1)$ *is even*.

Thus a is also even. We may write a = 2k.

Then $a^2 - 4b - 2 = (2k)^2 - 4b - 2 = 4k^2 - 4b - 2 = 2(2k^2 - 2b - 1)$.

Now by our assumption, $a^2 - 4b - 2 = 0$. Now from the previous line,

$$2k^2 - 2b - 1 = 0$$

Equivalently, $1 = 2(k^2 - b)$.

This is impossible since it says that 1 is an even number.

2. *Prove that if* $a \equiv b \ (mod\ m)$ *and* $b \equiv c \ (mod\ m)$ *then* $a \equiv c \ (mod\ m)$. (This is called the transitivity property of the congruence relation.)

Proof: We will give a direct proof.

Assume that $a \equiv b \ (mod\ m)\ and\ b \equiv c \ (mod\ m)$.

Thus $\exists k \in Z \quad a - b = km$.

Similarly, since $c \equiv d \ (mod\ m), \exists q \in Z \ \ c - d = qm$.

Now a = b + km, and c = d + qm

Thus ac = (b + km) (d + gm) = bd + (dk + bg + gkm)m

Letting s = dk + bg + gkm , we find that ac = bd + sm.

This is the result we sought: $ac \equiv bd \ (mod\ m)$.

3. [Corrected hint]: Find the tens and units digits of 7^{1942}. Hint: $2345678 \equiv 78 \pmod{100}$

Solution:

Since $7^4 = 2401 \equiv 1 \pmod{100}$, and noting that $1942 = 4(485) + 2$, we have

$$7^{1942} = 7^{4(485)+2} = 7^{4(485)}7^2 \equiv 1\,(7^2) = \mathbf{49} \pmod{100}.$$

4. Find a counterexample for each of the following statements:

(a) All prime numbers are odd.

Counterexample: Let n = 2.

(b) If n is an integer for which $n^5 - n$ is even, then n is even.

Counterexample: Let n = 1, then $n^5 - n = 0$ is even.

(c) If s and t are positive irrational numbers, then s + t is irrational.

Counterexample: Let $s = \sqrt{2} - 1$ and $t = 5 - \sqrt{2}$. Then $s > 0$, $t > 0$ and $s + t = 4$, is rational.

5. Find a counterexample for each of the following statements:

(a) If s and t are positive irrational numbers for which $s \neq t$, then st is irrational.

Counterexample: : Let $s = \sqrt{3}$ and $t = 2\sqrt{3}$. Then st = 6, which is rational.

(b) Let a, b, c, and n ≥ 1 be integers. Assume that $c \not\equiv 0 \pmod{n}$. Then, if $ca \equiv cb \pmod{n}$, it follows that $a \equiv b \pmod{n}$.

Counterexample: Let n = 12, a = 2, b = 4, and c = 6.

Then $ac = 12 \equiv 0 \pmod{12}$ and $bc = 24 \equiv 0 \pmod{12}$.

So $ac = bc \pmod{12}$, yet $a \not\equiv b \pmod{12}$.

(c) Any two multiples of 3 are congruent to each other (mod 6).

Counterexample: 6 and 9 are multiples of 3, yet 9 – 6 = 3 is not divisible by 6.

6. In a survey on the chewing gum preferences of soccer players, it was found that 22 like fruit, 25 like spearmint, 39 like grape, 9 like spearmint and fruit, 17 like fruit and grape, 20 like spearmint and grape, 6 like all flavors, 4 like none. How many players were surveyed?

Solution: Let X = set of soccer players surveyed.

Let F = set of soccer players who like fruit

Let S = set of soccer players who like spearmint

Let G = set of soccer players who like grape

We are given that $|F| = 22$; $|S| = 25$; $|G| = 39$; $|S \cap F| = 9$; $|F \cap G| = 17$;

$$|S \cap G| = 20 \; and \; |F \cap S \cap G\} = 6; |{\sim}(F \cup S \cup H)| = 4.$$

Using the inclusion-exclusion principle:

$$|F \cup S \cup H| = |F| + |S| + |H| - |S \; \cap F| - |F \cap G| - |S \cap G| + |F \cap S \cap G| =$$
$$22 + 25 + 39 - 9 - 17 - 20 + 6 = 46$$

Now 46 is the number of soccer players who like at least one flavor. Since 4 players like none,

The total number of soccer players surveyed is 46 + 4 = **50**

7. (a) Compute 3^{2019} (mod 11). Show your work!

Solution:

Begin by noting that $3^5 = 243 = 11(22) + 1 \equiv 1 \;\; (mod\; 11)$.

Since 2019 = 403(5) + 4, we have

$$3^{2019} = 3^{5(403)+4} \equiv 3^4 = 81 \equiv \mathbf{4}\; (\boldsymbol{mod}\; \mathbf{11})$$

(d) Find the remainder when $(70005)(46)(23)$ is divided by 7. A calculator solution will earn no credit.

Solution: Note that $70005 = 7(10000) + 5 \equiv 5$ (mod 7).

Next, $46 \equiv 4$ (mod 7) and $23 \equiv 2$ (mod 7).

Thus $(70005)(46)(23) \equiv 5(4)2 = 40 \equiv \mathbf{5}$ **(mod 7)**

8. **Fill in the blanks to complete this proof:**

Proposition: For all $n \in N$, $4|(3^{2n} + 7)$.

Proof: I will use the method of mathematical induction.

For $n \geq 1$, let P(n) be the statement $4\,|\,(3^{2n} + 7)$.

Base case: P(1) is true, since $3^2 + 7 = 16$ is divisible by 4.

Inductive Step:

For a fixed $k \geq 1$, assume that the proposition is true for n = k so that $4|(3^{2k} + 7)$.

Then $3^{2k} + 7 = 4L$.

Now, $3^{2(k+1)} + 7 = 9(3^{2k}) + 7 = 8(3^{2k}) + 3^{2k} + 7 = 8(3^{2k}) + 4L = 4(2(3^{2k}) + L)$.

Hence $4|(3^{2(k+1)} + 7)$, and we see that the proposition is true for n = k+1.

Therefore, by the principle of mathematical induction, the proposition is true for all $n \in N$.

9. (a) How many *non-negative solutions* are there to the equation x + y + z = 73?

Solution: Using stars and bars: we need 73 stars and 2 bars. Hence the answer is $\binom{75}{2}$.

(b) How many positive solutions are there to the equation x + y + z = 73

Solution:

Begin by removing 3 objects from the 73. Now we have 70 objects and 2 bars. Hence

Hence the answer is $\binom{72}{2}$.

10. Let a and b be integers. Prove that $(a+b)^3 \equiv a^3 + b^3 \pmod 3$

Proof: We give a direct proof.

Using algebra (or the binomial theorem), we have:

$$(a+b)^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3a^2b + 3ab^2 =$$

$$a^3 + b^3 \ + 3(a^2b + ab^2) \ (mod\ 3) \equiv a^3 + b^3$$

11. Let a, b and c be integers. Prove that if $a \nmid bc \ then \ a \nmid b$.

Proof: We use the method of contrapositive.

Assume that a|b. Then ∃L b = aL.

Now bc = acL = a(cL). Thus bc is a multiple of a, which negates the hypothesis.

12. Consider a group of 11 American students {A, B, C, …, J, K} visiting the Louvre. In front of La Giaconda, they line up at random for a photograph to be taken by their tour guide.

In how many ways can the 11 students line up so that B and C are side-by-side?

Solution: We have two cases.

Case 1: B is to the immediate left of C. Then "glue" B and C together. Now we have 10 objects that need to stand in line. This can be achieved in 10! ways.

Case 2: B is to the immediate right of C. Here the solution is identical to that of Case 1.

Therefore, it too offers 10! ways.

Now using the addition property:

The number of such lines subject to the given stipulation is 2(10!).

13. (a) If a Martian has an infinite number of red, blue, yellow, and black socks in a drawer, how many socks must the Martian pull out of the drawer to guarantee he has a pair?

Solution: By the Pigeonhole Principle, if the pigeons are socks to be drawn and the pigeon holes are represented by 4 colors, then if there are 5 pigeons, two of them must share the same color.

(b) A box contains 6 red, 8 green, 10 blue, 12 yellow and 15 white balls. What is the minimum number of balls we have to choose randomly from the box to ensure that we get 9 balls of the same color?

Solution: In the worst case, we first draw 6 red balls and 8 green balls. Then, in the worst case, we will draw 8 blue, 8 yellow and 8 white balls. Finally, the next ball must give you 9 of a kind. Thus, 6 + 8 + 3(8) + 1 = 39 balls.

14. If you are dealt a hand of 8 cards from a standard deck (without regard to order), how many ways can you have:

(a) A flush? (all 8 from the same suit)

Solution: One first chooses one of 4 suits, and then 8 cards from the 13 of the chosen rank.

Thus: $\mathbf{4\binom{13}{8}}$

(b) 4 distinct pairs?

Solution: First we choose 4 ranks from the 13 ranks, then from each selected rank we choose 2 cards.

Thus

$$\mathbf{\binom{13}{4}\binom{4}{2}^4}$$

15. Prove that if x is irrational and y is rational, then x + y must be irrational.

Proof: We use the method of contradiction.

Assume that x + y is rational. Then $\exists\ p, q \neq 0 \quad x + y = \frac{p}{q}$.

Also, since we are given that y is rational, $\exists\ a, b \neq 0 \quad y = \frac{a}{b}$.

Now x = (x + y) – y = $= \frac{p}{q} - \frac{a}{b} = \frac{pb-aq}{bq}$.

Now, note that bq is a non-zero integer and that pb – aq is also an integer. Thus x is rational, which is a contradiction.

16. Prove: If you choose any five numbers from the integers 1 to 8, then two of them must add up to nine.

Hint: Every number can be paired with another to sum to nine: for example, 2 and 7. How many such pairs are there? Now use the pigeonhole principle. Be sure to identify the pigeons as well as the pigeonholes.

Solution: Consider the 4 pairs {1, 8}, {2, 7}, {3, 6}, {4, 5}. These will be the pigeon-holes. Let the 5 chosen numbers serve as the pigeons. Each pigeon will fly to the pigeonhole that contains its number. The pigeonhole principle asserts that two pigeons must land in the same pigeonhole. This pigeonhole now contains two numbers that add up to 9.

A first fact should surprise us, or rather would surprise us if we were not used to it. How does it happen there are people who do not understand mathematics? If mathematics invokes only the rules of logic, such as are accepted by all normal minds ... how does it come about that so many persons are here refractory?

- Henri Poincaré, quoted in **The World of Mathematics**, by J. R. Newman.