MATH 201 SOLUTIONS: TEST III 12 APRIL 2019

God created infinity, and man, unable to understand infinity, had to invent finite sets.

– Gian Carlo Rota

INSTRUCTIONS: Answer any 11 of the following 13 problems. You may answer more than 11 to earn extra credit.

1. Let X, Y, W be non-empty sets. Let F: X →Y be a mapping and let G: Y → W be a mapping. Assume that $G \circ F$ *is surjective.*

(a) Prove that G must be surjective.

Solution: From out text:

Theorem 12.2 Suppose $f : A \to B$ and $g : B \to C$. If both f and g are injective, then $g \circ f$ is injective. If both f and g are surjective, then $g \circ f$ is surjective.

Proof. First suppose both f and g are injective. To see that $g \circ f$ is injective, we must show that $g \circ f(x) = g \circ f(y)$ implies $x = y$. Suppose $g \circ f(x) = g \circ f(y)$. This means $g(f(x)) = g(f(y))$. It follows that $f(x) = f(y)$. (For otherwise g wouldn't be injective.) But since $f(x) = f(y)$ and f is injective, it must be that $x = y$. Therefore $g \circ f$ is injective.

Next suppose both f and g are surjective. To see that $g \circ f$ is surjective, we must show that for any element $c \in C$, there is a corresponding element $a \in A$ for which $g \circ f(a) = c$. Thus consider an arbitrary $c \in C$. Because g is surjective, there is an element $b \in B$ for which $g(b) = c$. And because f is surjective, there is an element $a \in A$ for which $f(a) = b$. Therefore $g(f(a)) = g(b) = c$, which means $g \circ f(a) = c$. Thus $g \circ f$ is surjective. ■

(b) By exhibiting a counter-example, show that F *need not* be surjective.
(*Hint:* You may wish to assume that X, Y and Z are finite sets of small cardinalities. However, this is not essential to produce a counter-example.)

Counter example: Let X = {a}, Y = {1, 2} and Z = {α}.

Define F:X →Y by F(a) = 1;

Define G:Y→ Z by G(1) = G(2) = α.

Then $G \circ F\colon X \to Z$ is surjective, since its target, Z, is a set of one element.

Now F is not surjective since the range of X doesn't include 2.

2. (a) Let X and Y be disjoint countably infinite sets. Prove that X∪Y is countable.

Proof: Let $X = \{a_1, a_2, a_3, a_4, \ldots\}$ and $Y = \{b_1, b_1, b_3, \ldots\}$, .

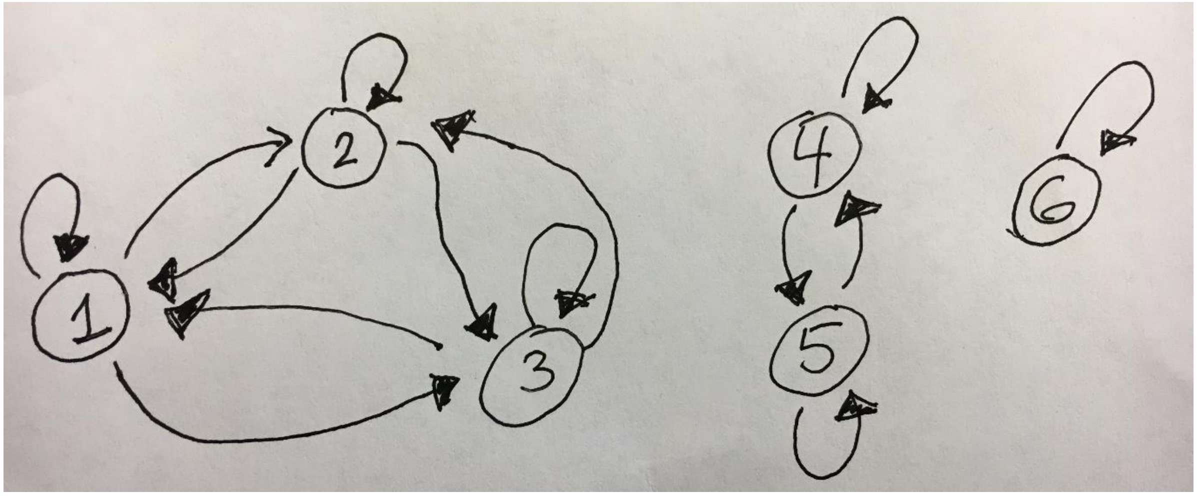
Then we may enumerate X∪Y as follows $\{a_1, b_1, a_2, b_2, a_3, b_3, \ldots\}$

(b) Using part (a), prove that the set of irrational numbers is uncountable. (You may assume that Q is countably infinite.)

Proof: Assume that S, the set of irrational numbers, is countably infinite. Since we have proven that the set of rational numbers, Q, is countably infinite, it must follow from (a) that $R = S \cup Q$ is also countable. But this contradicts Cantor's result that R is uncountable.

Thus our initial assumption that S is countable is false.

3. Let X = {1, 2, 3, 4, 5, 6}. Consider the following relation diagram on X.



(a) Does this diagram define an equivalence relation on X? Explain

Solution: Yes, from the diagram one may check reflexivity, symmetry, and transitivity.

Answer either (b) OR (c).

(b) If the answer to (a) is negative, is the relation reflexive, symmetric, or transitive? Justify your answers.

(c) If the answer to (a) is affirmative, list the equivalence classes of X.

Solution: There area three equivalence classes, namely {1, 2, 3}, {4, 6} and {6}.

4. Let X = {0, 1, 2, 3, 4, 5} and Y = P(X). Define T: X →P(X) as follows:

T(0) = {1, 3}

T(1) = {4}

T(2) = ∅

T(3) = {1, 3, 5}

T(4) = {0, 1, 2, 3, 4, 5}

T(5) = {4, 5}

(a) Is T well defined? Why?

Solution: Yes, T is well defined. Each member of X is mapped (unambiguously) to a subset of X, or equivalently into a member in P(X).

Let D^* = {i ∈ X | $i \notin T(i)$}

(b) Find (explicitly) D^*.

Solution: D^* = {0, 1, 2}

(c) Is T injective? Explain!

Solution: Yes, T is injective, since no two elements of X are mapped into the same element of P(X).

(d) Is T surjective? Explain!

Solution: No, T is not surjective since, for example, {5, 6} is not in the range of T.

5. Let X be the set of all integers greater than or equal to 2.

If a, b∈ X define the following relation ⧖ on X:

a⧖b if the largest prime factor of *a* is at least as big as the largest prime factor of *b*.

(i) Is ⧖ reflexive? Give proof or counter-example.

Solution: The binary operator ⌛ is reflexive since the largest prime factor of *a* equals the largest prime factor of *a*, and thus is "at least as big as the largest prime factor" of *a*.

(ii) Is ⌛ symmetric? Give proof or counter-example.

Solution: No: For example 7 ⌛5 is true but *5*⌛7 is false.

(iii) Is ⌛ transitive? Give proof or counter-example.

Solution: Yes: If the largest prime number of a is at lest as big as the largest prime factor of b, and the largest prime factor of b is at least as big as the largest prime factor, then it follows (from transitivity of $\geq$*) that the largest prime factor of a is at least as big as the largest prime factor of c.*

6. Let **Z** be the set of all integers, and let X be the set of all polynomials in one variable with integer coefficients.

Define H: X → $N \cup \{0\}$ as follows:

For p ∈ X, let $H(p) = \textit{the number of real solutions to the equation } p(x) = 0.$

$$\textit{For example if } p = 13(x-5)(x-11), \textit{then } H(p) = 2.$$

(a) Explain why H is well-defined?

Solution: Yes, H is well-defined. Each polynomial has a precise number of roots and this number is a non-negative integer. So the range of H is contained in $N \cup \{0\}$.

(b) Is H injective? Give proof or counter-example.

Solution: No, for example $H(x^2 + 1) = H(x^4 + x^2 + 1) = 0$

(c) Is H surjective? Give proof or counter-example

Solution: Yes, H is surjective. Given any positive integer n, the polynomial $\sum_{j=1}^{n}(x-1)(x-2)(x-3)\dots(x-j)$ *has exactly n distinct roots. Now for n = 0, just consider* $p(x) = x^2 + 1$.

7. For any real numbers, *c* and *d*, let us define the binary operation ⌘ as follows:

$$c ⌘ d = 1 - 2c^2d^2$$

Give either a *brief* justification or counterexample for each of the following:

(a) Is **R** closed under ⌘ ?

Solution: Yes: If c and d are real numbers, then so is $1 - 2c^2d^2$ since real numbers are closed under addition, subtraction and multiplication.

(b) The operation ⌘ is commutative: i.e., for all real numbers *c* and *d*,

c ⌘ d = d ⌘ c.

Solution: Yes, ⌘ is commutative:

c ⌘ d = $1 - 2c^2d^2 = 1 - 2d^2c^2$ = d ⌘ c

(c) Let c, d ∈ **R**. If either *c* or *d* equals 0, then c ⌘ d = 1

Solution: Yes: 0 ⌘ d = 1 – 0 = 1 and c ⌘ 0 = 1 – 0 = 1

(d) Let c, d ∈ **R**. If c ⌘ d = 1 then either *c* or *d* equals 0.

Solution: Suppose that 1 = c ⌘ d = $1 - 2c^2d^2$. This implies that $c^2d^2 = 0$, from which either c = 0 or d = 0.

(e) Let us restrict the operator ⌘ to **Z**, the set of integers. Then is **Z** closed under the operation ⌘ ?

Solution: Yes, Z is closed under addition, subtraction. Hence, for all integers a, b, a ⌘ b $\in Z$.

(f) Let us restrict the operator ⌘ to the set V of irrational numbers. Then V is closed under ⌘.

Solution: No, V is not closed under the operation ⌘.

For example, let $a = b = \sqrt{2}$ *. Then* $1 - 2a^2b^2 = 1 - 2(2)(2) = -7 \notin V$.

(g) Let us restrict the operator ⌘ to the set E, of even integers.
Then E is closed under ⌘.

Solution: No, E is not closed under the operation ⌘ *For example, 2* ⌘2 = 1 – 2(4)(4) = 31$\notin E$.

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(h) Let us restrict the operator ⌘ to the set S, of odd integers.
Then S is closed under ⌘.

Solution: Yes, for any two odd integers, c and d, $1 - 2c^2d^2 = 1 -$ *even integer = odd integer.*

8. (a) Define f: **N** → **N** as follows:

$$f(x) = \begin{cases} 3x \text{ if } x \text{ is even} \\ x^2 - x \text{ if } x \text{ is odd} \end{cases}$$

Why is *f not* well-defined?

Solution: *f is not* well-defined because f(1) = 1 – 1 = 0 $\notin N$.

(b) Define g: **N** → **N** as follows:

$$g(x) = \begin{cases} x + 4 & \text{if } x \text{ is even} \\ x^3 & \text{if } x \text{ is odd} \end{cases}$$

(i) Why is g well-defined?

Solution: If x is even, then x+4 ∈ N; if x is odd, then $x^3 \geq 1$, and so x ∈ N.

(ii) Is *g* injective? Why?

Solution: Yes, note that even integers are mapped to even integers; odd integers are sent to odd integers.

So we have two cases to check: If g(a) = g(b) for a and b both odd; and if g(a) = g(b) for a and b both even.

Note that y = x+ 4 is strictly increasing, so g restricted to even integers is injective.

Also note that $y = x^3$ is also strictly increasing, so g restricted to odd integers is injective.

(iii) Is *g* surjective? Why?

Solution: No: For example, 2 is in the codomain but not in the range. For g(x) = 2 would mean that x = -2, which is absurd.

9. Let **Z** be the set of integers. For a, b ∈ **Z**, define the relation R by aRb if and only if ab ≥ 0.

Is R reflexive? Why?

Solution: Yes, since aRa is equivalent to $a^2 \geq 0$, which is true of all integers.

Is R symmetric? Why?

Solution: Yes, for if ab ≥ 0, then ba = ab ≥ 0.

Is R transitive? Why?

Solution: No. Here is a counterexample: Let a = 1. b = 0, and c = -1. Then aRb and bRc, yet it is not true that aRc, since ac < 0.

10. Define a relation ⊗ on the set of real numbers by:

$$a \otimes b \text{ means } \exists k \in Z \quad a - b = 2k\pi$$

Is R an equivalence relation? If so, prove it. If not, explain why.

Solution: Yes, ⊗ *is an equivalence relation.*

⊗ *is* ***reflexive****: for all real numbers x,* $x - x = 2(1)(0)$.

⊗ *is* ***symmetric****:*

for all real numbers x and y, if $\exists k \in Z$ such that $a - b = 2k\pi$ *then* $b - a = 2(-k)\pi$

(since –k is also an integer).

⊗ is ***transitive***: Let x, y, and z be any real numbers for which x⊗y and y⊗z.

Then $\exists\, k_1, k_2 \in Z$ such that $a - b = 2k_1\pi$ *and* $b - c = 2k_2\pi$.

Now $a - c = (a - b) + (b - c) = 2k_1\pi + 2k_2\pi = 2(k_1 + k_2)\pi$.

Finally, note that $k_1 + k_2$ *is an integer*.

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11. . Let F: X →Y and G: Y →Z be mappings.

Assume that $G \circ F$ *is injective*. Prove that F must be injective

Proof: Assume that F is not injective. Then $\exists a, b \in X$ *such that* $a \neq b$ *and* $F(a) = F(b)$.

But this implies that $G(F(a)) = G(F(b))$*, or equivalently* $G \circ F(a) = G \circ F(b)$

This contradicts the fact that $G \circ F$ is injective.

12.. Let X be the set of all continuous functions on the interval [0, 1].

For f, g ∈ X, define the relation ⊠ *as follows*:

$$f \boxtimes g \text{ if } \left| \int_0^1 (f(x) - g(x))dx \right| \leq 1$$

(a) Is ⊠ *reflexive*? Explain.

Solution: Yes: $\left| \int_0^1 (f(x) - f(x))dx \right| = \left| \int_0^1 (0)dx \right| = 0 \leq 1$

(b) Is ⊠ *symmetric*? Explain.

Solution: Yes: if $\left| \int_0^1 (f(x) - g(x))dx \right| \leq 1$ *then*

$$\left| \int_0^1 (f(x) - g(x))dx \right| = \left| \int_0^1 (-1)(g(x) - f(x))dx \right| \leq \left| (-1) \int_0^1 (g(x) - f(x))dx \right| =$$

$|-1|\left|\int_0^1 (g(x) - f(x))dx\right| = \left|\int_0^1 (g(x) - f(x))dx\right| \leq 1$

(c) Is ⊠ $transitive$? Explain.

Solution: No. Counterexample: let $f(x) = 2, g(x) = 1, and\ h(x) = 0\ for\ all\ x \in [0,1]$.

Then $\left|\int_0^1 (f(x) - g(x))dx\right| = \left|\int_0^1 (2-1)dx\right| = \left|\int_0^1 1\,dx\right| = 1$ and so f ⊠ g.

Next, $\left|\int_0^1 (g(x) - h(x))dx\right| = \left|\int_0^1 (1-0)dx\right| = \left|\int_0^1 1\,dx\right| = 1$ and so g ⊠ h.

Finally, $\left|\int_0^1 (f(x) - h(x))dx\right| = \left|\int_0^1 (2-0)dx\right| = \left|\int_0^1 2\,dx\right| = 2$ and so f ⊠ h is false.

13. Let S be the set of *positive rational* numbers. Define the function f: S→S as follows:

$f(x) = \frac{1}{1+x^3}$.

Is *f* well-defined? Why?

Solution: Yes: If x is positive then $1 + x^3 > 0$. *Now if x is rational ,then 1+* x^3 *is rational, since the rationals are closed under multiplication and addition. Finally, the reciprocal of a positive rational is a positive rational.*

Is *f* injective? Why?

Solution: Yes: Suppose that $a, b \in S$ *and f(a) = f(b).*

Then $\frac{1}{1+a^3} = \frac{1}{1+b^3}$ *which implies that* $1 + a^3 = 1 + b^3$. *Hence* $a^3 = b^3$ *and so* $a = b$.

Is *f* surjective? Why?

Solution: No. Counterexample: Suppose that f(x) = 2 for some $x \in S$.

Then $\frac{1}{1+x^3} = 2, and\ so\ x^3 = -\frac{1}{2},\ \ but\ no\ positive\ number\ x\ meets\ this\ requirement.$

Bonus: *A Russian had three daughters. The first, named Rab, became a lawyer; the second, Ymra, became a soldier. The third became a sailor; what was her name?*

- *Lewis Carroll*

Answer: Yvan

– Charles Lutwidge Dodgson (aka Lewis Carroll), 1832 – 1898