MATHEMATICA LAB III

line integrals, div, curl, Green's Theorem and parametric surfaces

DUE: APRIL 30

COMPUTING LINE INTEGRALS

Find the *work* done by the given force ***F*** over the given path. Also, graph the path and the vector field on one set of axes.

1. **F**(x,y) = xy^6 **i** + $3x(xy^5 + 2)$ **j** , $\sigma(t) = (2 \cos t)$ **i** + (sin t) **j**, $0 \leq t \leq 2\pi$.

2. **F**(x,y,z) = (y+ yz cos(xyz)) **i** + (x^2 + xz cos(xyz)) **j** + (z + xy cos(xyz)) **k**, $\sigma(t)$ = (2 cos t) **i** + (3 sin t) **j + k**, $0 \leq t \leq 2\pi$.

3. **F**(x,y,z) = (2xy) **i** - (y^2) **j** + (ze^x) **k**, $\sigma(t)$ = -t **i** + $t^{1/2}$ **j** + 3t **k**, $1 \leq t \leq 4$.

COMPUTING DIVERGENCE AND CURL

4. Let **F**(x,y,z) = sin(xyz) **i** - (x+y+z) **j** + log(x-z) **k**. Compute curl **F**.

5. Let **G**(x,y,z) = (xyz) **i** - (sin(x-y)) **j** + tan(z) **k**. Compute div **G**.

6. Let h(x,y,z) = $x/(x^2+y^2+z^2)^{1/3}$. Compute grad h.

GREEN'S THEOREM

7. Let R be the triangle defined by vertices (1,2), (1,8), (7,7). Let **G** be the vector field defined by

$\mathbf{G}(x,y) = x \exp(y)\, \mathbf{i} + \cos^4(x+y)\, \mathbf{j}$. Using Green's theorem, evaluate the line integral of **G**(x,y) over the curve C.

8. Using Green's Theorem, calculate the area of the *astroid* (also called the *hypocycloid*) given by $r(t) = (\cos^3 t)\, \mathbf{i} + (\sin^3 t)\, \mathbf{j}$, $0 \le t \le 2\pi$. Be sure to first graph the astroid.

9. Verify Green's Theorem for the following vector field and region. Be sure to compute *separately* each side of Green's equation to verify that they are equal.

$\mathbf{F}(x,y) = (\arctan y/x)\, \mathbf{i} + \ln(x^2 + y^2)\, \mathbf{j}$. C is the boundary of the region defined by the $1 \le x^2 + y^2 \le 2$, $x \ge 0$, $y \ge 0$. (This region is a quarter of an annulus. On the positive y-axis, arctan(y/x) is assumed to be $\pi/4$, the continuous extension of arctan(y/x) from the right.)

10. Verify (by computing *separately* each side of Green's equation) the general version of Green's Theorem for the annulus R:

$$0.01 \le x^2 + y^2 \le 1, \text{ and } \mathbf{F}(x,y) = (-y/(x^2 + y^2))\, \mathbf{i} + (x/(x^2 + y^2))\, \mathbf{j}$$

(Make sure that you use the correct orientation of each of the two closed curves that constitute the boundary of R.)

11. Using the area version of Green's Theorem, calculate the area of the *folium of Descartes*, $x^3 + y^3 = 3xy$, parameterized by $x = 3t/(1+t^3)$, $y = 3t^2/(1+t^3)$, $0 \le t < +\infty$. Begin by drawing the graph of the folium of Descartes.

12. (Hughes-Hallett 860/29 CAS Challenge Problem) Let C_a be the circle of radius a, centered at the origin, oriented in the counterclockwise direction, and let

$$\mathbf{F}(x,y) = (-y + (2/3)y^3)\ \mathbf{i} + (2x - x^3/3 + xy^2)\ \mathbf{j}.$$

(a) Evaluate the line integral of ***F*** over C_a. Determine which positive value of *a* maximizes the value of this line integral.

(b) Use Green's Theorem to convert the line integral to a double integral. Without evaluating the double integral, give a geometric explanation of the value of *a* that you found in part (A).

PARAMETRIC SURFACES

(Answer any two of the following exercises; you will receive extra credit if you solve more than two.)

13. (*Without* using Mathematica) verify that the *astroidal sphere*,

$x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$ can be parameterized by:

$\Phi(u, v) = a \sin^3 u \cos^3 v\ \mathbf{i} + a \sin^3 u \sin^3 v\ \mathbf{j} + a \cos^3 u\ \mathbf{k}$ where $0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$.

Now, *using Mathematics*, graph the *astroidal sphere* for a *few* values of *a*.

14. Graph and find the area of one turn of the *spiral ramp* (or *helicoid*)

$\Phi(u, v) = u \cos v\ \mathbf{i} + u \sin v\ \mathbf{j} + 2v\ \mathbf{k}$ where $0 \leq u \leq 3$ and $0 \leq v \leq 2\pi$.

15. The surface of the dome on a new museum is given by

$\mathbf{\Phi}(u, v) = 20 \sin u \cos v\, \mathbf{i} + 20 \sin u \sin v\, \mathbf{j} + 20 \cos u\, \mathbf{k}$ where $0 \leq u \leq \pi/3$ and $0 \leq v \leq 2\pi$ where distance is measured in meters. Graph the dome and find its surface area.

16. Graph the surface given parametrically by:

$\mathbf{\Phi}(u, v) = (3 + (\sin u)(7 - \cos(3u - 2v) - 2\cos(3u + v)))\, \mathbf{i} +$

$(3 + (\cos u)(7 - \cos(3u - 2v) - 2\cos(3u + v)))\, \mathbf{j} + \sin(3u - 2v)\, \mathbf{k}$ where

$-\pi \leq u \leq -\pi$ and $-\pi \leq v \leq \pi$.