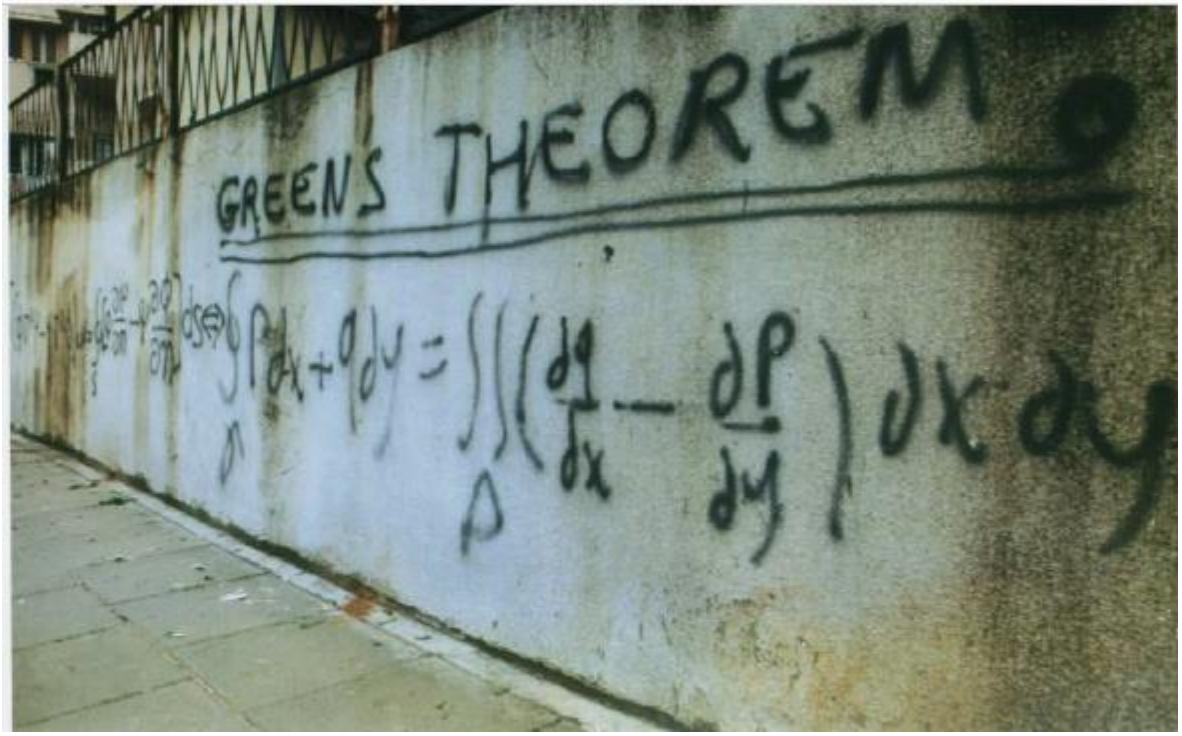
CLASS DISCUSSION

10 April 2019

GREEN'S THEOREM AND THE DIVERGENCE THEOREM IN 2- DIMENSIONS



Graffiti on the wall of a high school playground in Ramat Gan (a suburbof Tel Aviv). Courtesy of Eli Maor.

1) Verify Green's Theorem for each of the following vector fields and regions:

(a) $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + x\,\mathbf{j}$ and *D* is the unit disk centered at the origin.

(b) $\mathbf{G}(x,y) = y^2\,\mathbf{i} + x^2\,\mathbf{j}$ and *T* is the triangle with vertices (0, 0), (4, 0) and (0, 7).

2) Using Green's Theorem, evaluate the line integral:

$$\oint_{\partial S} -y^2\,dx + xy\,dy$$

where *S* is the unit square bounded by x = 0, x = 1, y = 0, y = 1.

3) State the area formula that is a consequence of Green's Theorem. *Using this formula*, find the area of the ellipse $\boldsymbol{\sigma}(t) = (a \cos t)\,\mathbf{i} + (b \sin t)\,\mathbf{j}$, $0 \le t \le 2\pi$.

4) Using Green's Theorem, find the area of the region bounded by the graphs of $y = 5x - 3$ and $y = x^2 + 1$.

5) Using Green's Theorem, find the work done by the vector field $\mathbf{F}(x, y) = 2xy^3\,\mathbf{i} + 4x^2y^2\,\mathbf{j}$ in moving a particle once counterclockwise around the triangular region in the first quadrant bounded by the x-axis, the line x = 1, and the curve $y = x^3$.

6) State the general version of Green's Theorem for regions that may contain holes.

7) Let $\mathbf{F}(x, y)$ be the vector field $(x^2 + y^2)^{-1}(-y\,\mathbf{i} + x\,\mathbf{j})$. (This may be used to model the velocity field of a tornado.)

(a) Show that curl $\mathbf{F} = \mathbf{0}$ everywhere that $\boldsymbol{F}$ is defined.

(b) Find the circulation of ***F*** along a circle centered at the origin of radius $r > 0$.

(c) Do the results of (a) and (b) contradict or confirm Green's Theorem on the annulus $4 \le x^2 + y^2 \le 9$. Explain.

8) Use Green's Theorem to evaluate $\oint_C (6y + x)\,dx + (y + 2x)\,dy$ where *C* is the circle $(x - 2)^2 + (y - 3)^2 = 4$ (endowed with the counterclockwise orientation).

9) Let **G**(x,y) be the vector field $(x^2 + y^2)^{-1/2}(-y\,\mathbf{i} + x\,\mathbf{j})$.

(a) Compute curl **G**.

(b) Find the circulation of ***G*** along a circle centered at the origin of radius $r > 0$.

(c) Verify Green's Theorem for ***G*** on the annulus $1 \le x^2 + y^2 \le 25$.

10) State the *Divergence Theorem (i.e., Normal Form of Green's Theorem)* in the plane. Verify the theorem for the vector field $\mathbf{F}(x, y) = x\,\mathbf{i} + y\,\mathbf{j}$ across the unit disk, *D*, centered at the origin.

11) Let *C* be the disk $(x - 1)^2 + (y - 1)^2 = 1$, and $\mathbf{F}(x, y) = x\mathbf{i}$. Verify the Divergence Theorem in the plane for this vector field.

12) Let $\mathbf{F}(x, y) = 2x\,\mathbf{i} + 3y\,\mathbf{j}$. Let *S* be the unit square with vertices (0, 0), (1, 0), (1, 1), (0, 1) and let ∂S denote the boundary of *S* with the positive orientation. Let ***n*** be the unit normal pointing outward.

(a) Calculate directly (*without* using the Divergence Theorem) the flux integral:

$$\oint_{\partial S} F \cdot n\ ds$$

(b) Check your answer to (a) using the Divergence Theorem.

13) Verify the *Divergence Theorem* in the plane for the field

$\mathbf{G}(x, y) = y^3\,\mathbf{i} + x^5\mathbf{j}$ across the square bounded by the lines

$x = 0, x = 3, y = 1, y = 4$.

14) Let *C* be the semi-annulus $4 \le x^2 + y^2 \le 25,\ y \ge 0$. Evaluate

$$\int_C y^3 dx + 3xy^2 dy.$$

Hint: Look for a shortcut.

15) Using the *Divergence Theorem* in the plane, calculate the net *flux* of the given vector field over the given boundary:

(a) $\mathbf{F}(x,y) = x\,\mathbf{i} + y^2\mathbf{j}$ across the square bounded by the lines

$x = \pm 1, y = \pm 1$.

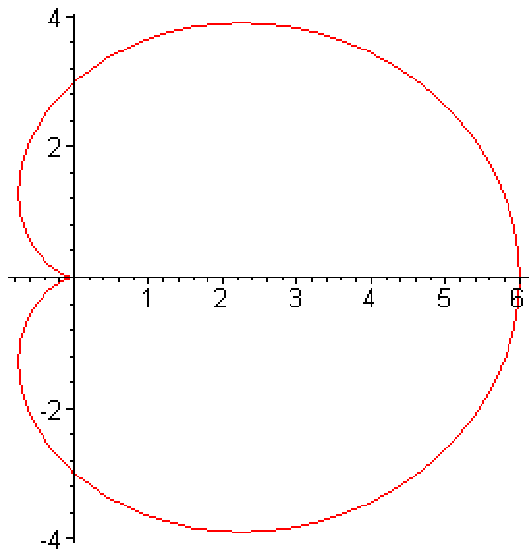
(b) $\mathbf{G}(x, y) = (y^2 - x^2)\,\mathbf{i} + (x^2 + y^2)\,\mathbf{j}$ across the triangle bounded by the lines $y = 0$, $x = 1$, and $y = x$.

(c) $\mathbf{H}(x,y) = xy\,\mathbf{i} + y^2\mathbf{j}$ over the region enclosed by the curves $y = x^2$ and $y = x$ in the first quadrant.

(d) $\mathbf{A}(x, y) = (x - y)\,\mathbf{i} + x\,\mathbf{j}$ across the circle $x^2 + y^2 = 1$.

(e) $\mathbf{B}(x, y) = 2xy\,\mathbf{i} - y^2\,\mathbf{j}$ through the ellipse $(x/a)^2 + (y/b)^2 = 1$.

16) Consider the cardioid pictured below:



This curve is parametrized by the equation

$$\boldsymbol{\sigma}(t) = 3(1 + \cos t)(\cos t)\,\mathbf{i} + 3(1 + \cos t)(\sin t)\,\mathbf{j}, \quad 0 \le t \le 2\pi.$$

Using Green's area formula, express the area of the cardioid as a Riemann integral. *Note:* Do not evaluate the integral that you obtain.

17) Using Green's Theorem, find the area of the ellipse $(x/a)^2 + (y/b)^2 = 1$, where a and b are positive constants. Show your work!

18) Describe the domain of the following vector field, ***F***. Is it connected? Is it simply connected? Find a *potential function* if one exists.

$$\vec{F}(x,y,z) = (y^2 - \frac{1}{x+z})\vec{i} + (2xy+5)\vec{j} + (3z^2 - \frac{1}{x+z})\vec{k}$$

19) Using Green's Theorem, compute $\oint_C (5 - xy - y^2)dx + (x^2 - 2xy)dy$, where C is the boundary of the square with vertices (0, 0), (1, 0), (1, 1), and (0, 1).

20) Let T be the triangle in the xy-plane with vertices (0, 0), (-2, 2), and (8, 2). Let C denote the boundary of T endowed with the positive orientation. Let

$\boldsymbol{F}(x, y) = xy^2\,\mathbf{i} + (x^2y + x^2)\,\mathbf{j}$. *Using Green's Theorem*, evaluate $\oint_C \vec{F} \cdot d\vec{s}$.

21) Using Green's theorem, find the work done by the vector field

$\mathbf{F}(x, y) = 2xy^3\,\mathbf{i} + 4x^2y^2\,\mathbf{j}$ in moving a particle counterclockwise around the boundary of the "crescent" in the first quadrant enclosed by the curves

$y = x^4$ and $y = x^5$.

22) Using Green's Theorem, find the *work* done by the force field

F(x, y) = (3x + 4y) **i** + (8x + 9y) **j** on a particle that moves once around the ellipse $4x^2 + 9y^2 = 36$ in the counter-clockwise direction.

GREEN'S MILL: Once home of the mathematical physicist, George Green (1793-1841)