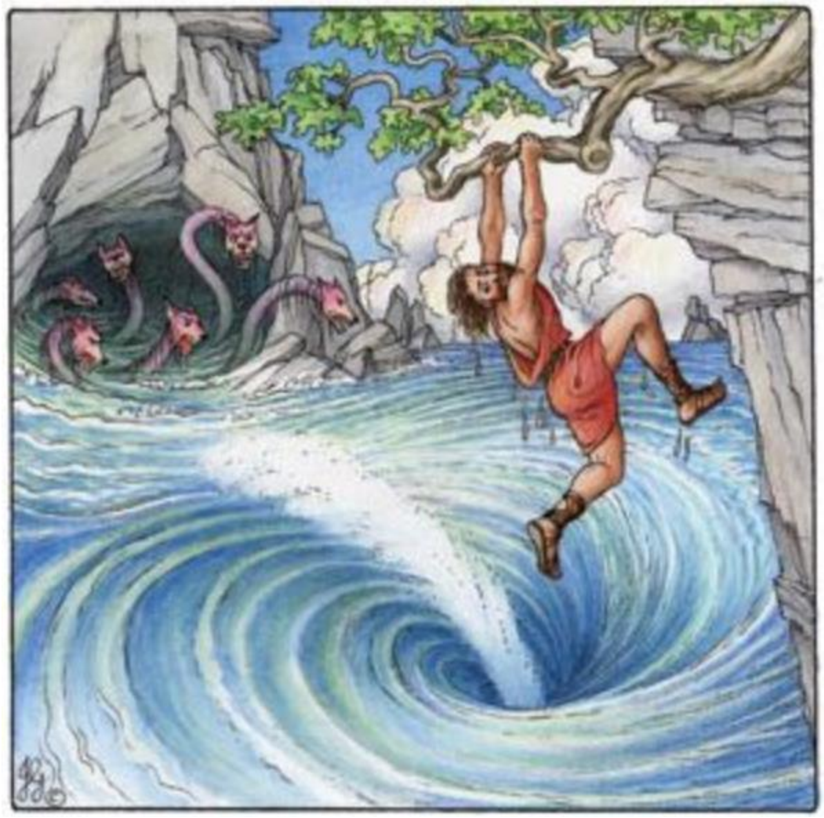
CLASS DISCUSSION: 24 APRIL 2019

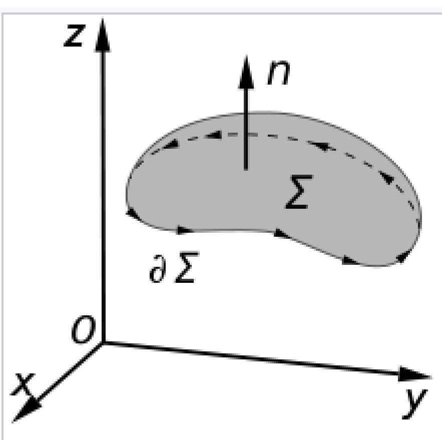
STOKES' THEOREM



Odysseus caught between Scylla and Charybdis

1. (a) Explore Stokes' Theorem.

$$\iint_S curl\ \boldsymbol{F} \cdot d\boldsymbol{S} = \int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{r}$$



An illustration of the Kelvin–Stokes theorem, with surface Σ, its boundary ∂Σ and the "normal" vector *n*.

(b) Show that Green's Theorem is a special case of Stokes' Theorem.

(c) Explain why the integration of the circulation density over a surface depends only upon the boundary of the surface.

2. Let *W* be a solid region in space and let *S* = ∂W. Assume that **F**(x, y, z) is a vector field. Using Stokes' Theorem, explain why

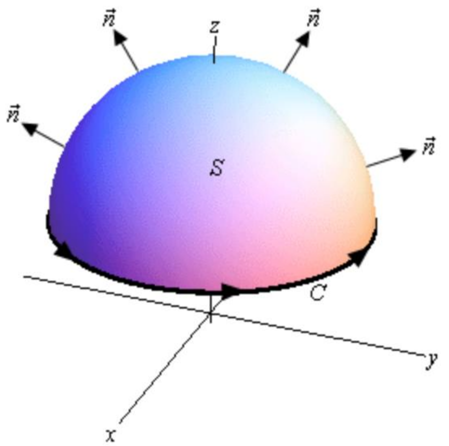
$$\iint_S curl\ \boldsymbol{F} \cdot d\boldsymbol{S} = 0$$

Show that this is an intuitive way of understanding why div(curl F) = 0.

3. Let **F**(x, y, z) = -3x **i** + 3x **j**. Use Stokes' Theorem to find $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where *C* is:

(a) a circle parallel to the yz-plane of radius *r*, centered at a point on the x-axis, with either orientation.

(b) a circle parallel to the xy-plane of radius r, centered at a point on the z-axis, oriented counterclockwise as viewed from a point on the z-axis above the circle.



4. Let the vector field **F**(x, y, z) = z^3y **i** + $(2x + z^3x)$ **j** + $(x^2 + 3z^2xy)$ **k**.

(a) Show that curl **F** = -2x **i** + 2 **k**.

(b) Using your result from (a), evaluate the line integral of ***F*** over the curve *C*, the circle in the xy-plane of radius 2 centered at the origin, oriented in a counter-clockwise direction when viewed from above.

(c) Without any computation, explain why the answer in part (b) is also equal to the flux integral of *curl* ***F*** over the lower hemisphere of radius 2 centered at the origin, oriented inward.

5. *True or False?* If curl **F** = **0** everywhere in 3-space, then by Stokes' Theorem, the line integral of ***F*** over *C* is equal to 0, where *C* is the curve $y = x^2$, for $0 \le x \le 2$.

6. Let ***F*** be a vector field. We say that a vector field ***G*** is a *vector potential* for ***F*** if ***F*** = curl ***G***. We also call such an ***F*** a *curl field*.

(a) True/False? By Stokes' Theorem, the flux of a curl field through a surface depends only upon the boundary of the surface.

(b) Let **F**(x, y, z) = $(8yz - z)$ **j** + $(3 - 4z^2)$ **k**. Show that **G**(x, y, z) = = $4yz^2$ **i** + 3x **j** + xz **k** is a vector potential for ***F***.

(c) Using the vector field defined in part (b), evaluate $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$ where *S* is the upper hemisphere ($x^2 + y^2 + z^2 = 25$, $z \ge 0$), oriented upward.

7. (Hughes-Hallett) Water in a bathtub has a velocity vector field near the drain given, for *x, y, z* in cm, by

$$V(x,y,z) = -\frac{y + xz}{(z^2+1)^2}\, i - \frac{yz - x}{(z^2+1)^2}\, j - \frac{1}{z^2+1}\, k \quad cm/\sec$$

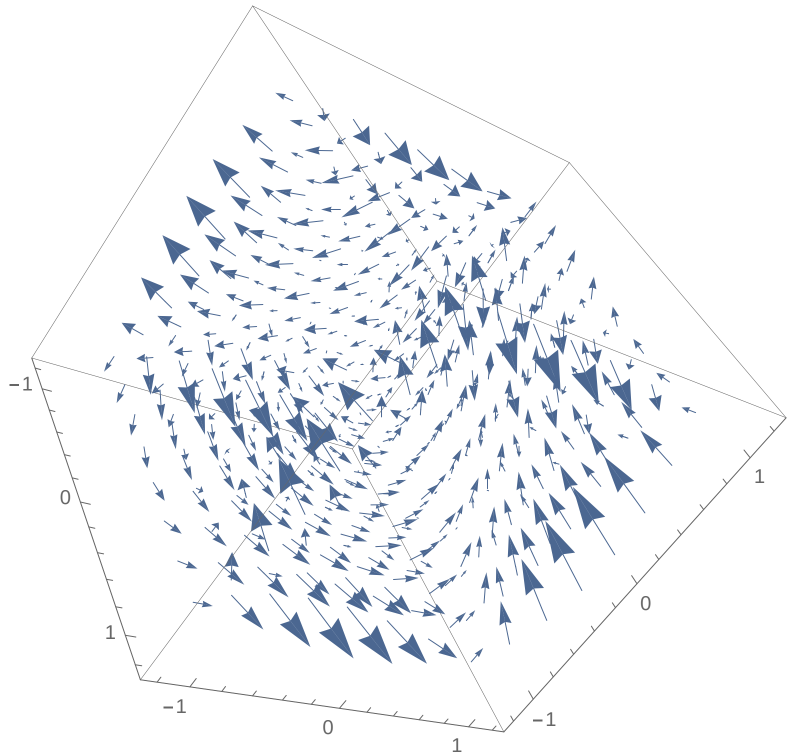
(a) The drain in the bathtub is a disk in the xy-plane with center at the origin and radius 1 cm. Find the rate at which the water is leaving the bathtub (i.e., the rate at which water is flowing through the disk).
Include units in your answer.

(b) Find div **V**.

(c) Find the flux of the water through the hemisphere of radius 1, centered at the origin, lying below the xy-plane and oriented downward.

(d) Let the vector field **G**(x, y, z) be defined by:

$$G(x,y,z)=\frac{1}{2}\left(\frac{y}{z^2+1}\,i-\frac{x}{z^2+1}\,j-\frac{x^2+y^2}{(z^2+1)^2}\,k\right)$$



Calculate $\int_C \boldsymbol{G}\cdot d\boldsymbol{r}$ where *C* is the edge of the drain, oriented clockwise when viewed from above.

(e) Calculate curl **G**.

(f) Explain why your answer to parts (c) and (d) are equal.

8. (Hughes-Hallett) Let **F**(x, y, z) = -z **j** + y **k** and let *C* be the circle of radius *r* in the yz-plane oriented clockwise as viewed from the positive x-axis, and let *S* be the disk in the yz-plane enclosed by *C*, oriented in the positive x-direction.

   (a) Evaluate directly $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$

   (b) Evaluate directly: $\iint_S curl\ \boldsymbol{F}\cdot d\boldsymbol{S}.$

   (c) The answers to parts (a) and (b) are not equal. Explain why this does not contradict Stokes' Theorem.

9. Assume that curl **F** = (xyz) **i** + (ay^2 z + 2byz) **j** + z^2 **k**, where *a* and *b* are constants. Is it possible to determine the values of *a* and *b* without knowing the expression **F**?

10. (Thomas) Let **F**(x, y, z) = x^2 **i** + 2x **j** + z^2 **k** and let *C* be the ellipse $4x^2 + y^2 = 4$ in the xy-plane, counterclockwise when viewed from above. Use the surface integral in Stokes' Theorem to calculate the circulation of ***F*** around *C*.

11. Let **F**(x, y, z) = 2y **i** + 3x **j** – z^2 **k** and let *C* be the circle $x^2 + y^2 = 9$ in the xy-plane, counterclockwise when viewed from above. Use the surface integral in Stokes' Theorem to calculate the circulation of ***F*** around *C*.

12. Let $\mathbf{F}(x, y, z) = y\,\mathbf{i} + xz\,\mathbf{j} + x^2\,\mathbf{k}$ and let C be the boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above. Use the surface integral in Stokes' Theorem to calculate the circulation of $\boldsymbol{F}$ around C.

13. Let $\mathbf{F}(x, y, z) = (y^2 + z^2)\,\mathbf{i} + (x^2 + z^2)\,\mathbf{j} + (x^2 + y^2)\,\mathbf{k}$ and let C be the boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above. Use the surface integral in Stokes' Theorem to calculate the circulation of $\boldsymbol{F}$ around C.

14. (Marsden) Use Stokes' Theorem to evaluate the line integral

$$\int_C -y^3dx + x^3dy - z^3dz$$

where C is the intersection of the cylinder $x^2 + y^2 = 1$, $z \geq 0$, and the plane $x + y + z = 1$ and the orientation of C corresponds to counterclockwise motion in the xy-plane.

15. (Marsden) Let S be any smooth surface with boundary $x^2 + y^2 = 1$ in the $z = 5$ plane, where the induced orientation on the boundary is in the clockwise direction when viewed from above.

Let $\mathbf{F}(x, y, z) = y\,\mathbf{i} - x\,\mathbf{j} + e^{xz}\,\mathbf{k}$. Evaluate $\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}$. :

George Gabriel Stokes
(1819-1903)

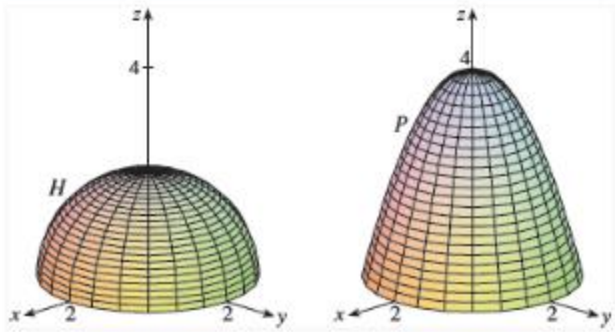
I am almost inclined to coin a word and call the appearance fluorescence, from fluor-spar, as the analogous term opalescence is derived from the name of a mineral.

- Sir George Gabriel Stokes

Exercises (Stewart)

1. A hemisphere H and a portion P of a paraboloid are shown. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$



2, 3, 4, 5 and 6, Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

2. $\mathbf{F}(x, y, z) = x^2 \sin z\, \mathbf{i} + y^2\, \mathbf{j} + xy\, \mathbf{k}$, S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upward

3. $\mathbf{F}(x, y, z) = ze^y\, \mathbf{i} + x \cos y\, \mathbf{j} + xz \sin y\, \mathbf{k}$, S is the hemisphere $x^2 + y^2 + z^2 = 16$, $y \geqslant 0$, oriented in the direction of the positive y-axis

4. $\mathbf{F}(x, y, z) = \tan^{-1}(x^2 y z^2)\, \mathbf{i} + x^2 y\, \mathbf{j} + x^2 z^2\, \mathbf{k}$, S is the cone $x = \sqrt{y^2 + z^2}$, $0 \leqslant x \leqslant 2$, oriented in the direction of the positive x-axis

5. $\mathbf{F}(x, y, z) = xyz\, \mathbf{i} + xy\, \mathbf{j} + x^2 yz\, \mathbf{k}$, S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward

6. $\mathbf{F}(x, y, z) = e^{xy}\, \mathbf{i} + e^{xz}\, \mathbf{j} + x^2 z\, \mathbf{k}$, S is the half of the ellipsoid $4x^2 + y^2 + 4z^2 = 4$ that lies to the right of the xz-plane, oriented in the direction of the positive y-axis

7, 8, 9 and 10 Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above.

7. $\mathbf{F}(x, y, z) = (x + y^2)\, \mathbf{i} + (y + z^2)\, \mathbf{j} + (z + x^2)\, \mathbf{k}$, C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$

8. $\mathbf{F}(x, y, z) = \mathbf{i} + (x + yz)\, \mathbf{j} + (xy - \sqrt{z})\, \mathbf{k}$, C is the boundary of the part of the plane $3x + 2y + z = 1$ in the first octant

9. $\mathbf{F}(x, y, z) = xy\, \mathbf{i} + yz\, \mathbf{j} + zx\, \mathbf{k}$, C is the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant

10. $\mathbf{F}(x, y, z) = 2y\,\mathbf{i} + xz\,\mathbf{j} + (x + y)\,\mathbf{k}$, C is the curve of intersection of the plane $z = y + 2$ and the cylinder $x^2 + y^2 = 1$

11.

a. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = x^2 z\,\mathbf{i} + xy^2\,\mathbf{j} + z^2\,\mathbf{k}$$

and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$, oriented counterclockwise as viewed from above.

12.

a. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2 y\,\mathbf{i} + \frac{1}{3}x^3\,\mathbf{j} + xy\,\mathbf{k}$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

b. Graph both the hyperbolic paraboloid and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

c. Find parametric equations for C and use them to graph C.

b. Graph both the plane and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

Answer

c. Find parametric equations for C and use them to graph C.

16. Let C be a simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z\ dx - 2x\ dy + 3y\ dz$$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

17. A particle moves along line segments from the origin to the points $(1, 0, 0)$, $(1, 2, 1)$, $(0, 2, 1)$, and back to the origin under the influence of the force field

$$\mathbf{F}(x, y, z) = z^2\ \mathbf{i} + 2xy\ \mathbf{j} + 4y^2\ \mathbf{k}$$

Find the work done.

Answer

18. Evaluate

$$\int_C (y + \sin x)\ dx + (z^2 + \cos y)\ dy + x^3\ dz$$

where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle, 0 \leqslant t \leqslant 2\pi$.

[*Hint:* Observe that C lies on the surface $z = 2xy$.]

19. If S is a sphere and $\mathbf{F}$ satisfies the hypotheses of Stokes' Theorem, show that $\iint_S \text{curl}\ \mathbf{F} \cdot d\mathbf{S} = 0$.