CLASS DISCUSSION: 1 FEBRUARY 2019

Vector-valued functions: limits, domain, derivative, integral

1. Find the domain of each of the following vector valued functions.

$$\mathbf{r}(t) = \left\langle \ln(t+1), \frac{t}{\sqrt{9-t^2}}, 2^t \right\rangle$$

Answer

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \ln t\,\mathbf{j} + \frac{1}{t-2}\mathbf{k}$$

2. Find the following limits

$$\lim_{t\to 0}\left(e^{-3t}\,\mathbf{i} + \frac{t^2}{\sin^2 t}\mathbf{j} + \cos 2t\,\mathbf{k}\right)$$

Answer

$$\lim_{t\to 1}\left(\frac{t^2-t}{t-1}\mathbf{i} + \sqrt{t+8}\,\mathbf{j} + \frac{\sin \pi t}{\ln t}\mathbf{k}\right)$$

$$\lim_{t\to\infty}\left\langle \frac{1+t^2}{1-t^2}, \tan^{-1}t, \frac{1-e^{-2t}}{t} \right\rangle$$

Answer

$$\lim_{t\to\infty}\left\langle te^{-t}, \frac{t^3+t}{2t^3-1}, t\sin\frac{1}{t} \right\rangle$$

3. Sketch the curve with the given vector equation. Indicate with an arrow the direction in which increases.

(a) $\mathbf{r}(t) = (\sin t, t)$
(b) $\mathbf{r}(t) = (t^2 - 1, t)$
(c) $\mathbf{r}(t) = (t, 2 - t, 2t)$

. $\mathbf{r}(t) = \langle \sin \pi t, t, \cos \pi t \rangle$

$\mathbf{r}(t) = \langle 3, t, 2 - t^2 \rangle$

Answer

. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$

$\mathbf{r}(t) = t^2\mathbf{i} + t^4\mathbf{j} + t^6\mathbf{k}$

Answer

. $\mathbf{r}(t) = \cos t\,\mathbf{i} - \cos t\,\mathbf{j} + \sin t\,\mathbf{k}$

4. Find a vector equation and parametric equations for the line segment that joins P to Q.

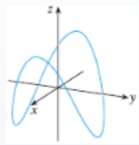
$P(-1, 2, -2),\ Q(-3, 5, 1)$

$P(0, -1, 1),\ Q\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$

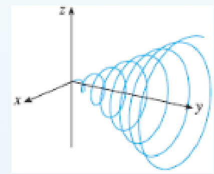
$P(2, 0, 0),\ Q(6, 2, -2)$

5. Match the parametric equations with the graphs (labeled I–VI). Give reasons for your choices.

I.



II.



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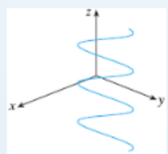

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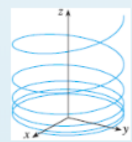
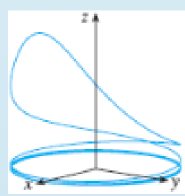


V.

VI.



$x = t\cos t, \quad y = t, \quad z = t\sin t, \quad t \geqslant 0$

Answer

$x = \cos t, \quad y = \sin t, \quad z = 1/(1+t^2)$

$x = t, \quad y = 1/(1+t^2), \quad z = t^2$

Answer

$x = \cos t, \quad y = \sin t, \quad z = \cos 2t$

$x = \cos 8t, \quad y = \sin 8t, \quad z = e^{0.8t}, \quad t \geqslant 0$

Answer

$x = \cos^2 t, \quad y = \sin^2 t, \quad z = t$

6.

Show that the curve with parametric equations $x = t\cos t$, $y = t\sin t$, $z = t$ lies on the cone $z^2 = x^2 + y^2$, and use this fact to help sketch the curve.

7.

At what points does the curve $\mathbf{r}(t) = t\,\mathbf{i} + (2t - t^2)\,\mathbf{k}$ intersect the paraboloid $z = x^2 + y^2$?

Show that the curve with parametric equations $x = t^2$, $y = 1 - 3t$, $z = 1 + t^3$ passes through the points $(1, 4, 0)$ and $(9, -8, 28)$ but not through the point $(4, 7, -6)$.

8.

For the curve $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (2 - t)\,\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$.

9. Let **r**(t) be the position of a particle in the xy-plane at time *t*. Find an equation in *x* and *y* whose graph is the path of the particle. Then find the particle's velocity and acceleration vectors at the given time *t*. Also find the speed of the particle at time *t*. Write the particle's velocity at that time as the product of its speed and direction.

(a) $\mathbf{r}(t) = (\cos 2t)\,\mathbf{i} + 3\sin(2t)\,\mathbf{j}$, $t = 0$

(b) $\mathbf{r}(t) = e^t\,\mathbf{i} + 3e^{2t}\,\mathbf{j}$ at time $t = 0$.

(c) $\mathbf{r}(t) = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}$, $t = \pi/6$. (This is a *cycloid*.)

10. Let **r**(t) be the position of a particle in $\mathbf{R}^3$ at time *t*. Find the particle's velocity and acceleration vectors at the given time *t*. Also find the speed of the particle at time *t*. Write the particle's velocity at that time as the product of its speed and direction.

(a) $\mathbf{r}(t) = (t + 1)\,\mathbf{i} + (t^2 - 1)\,\mathbf{j} + 2t\,\mathbf{k}$, $t = 1$

(b) $\mathbf{r}(t) = (\sec t)\,\mathbf{i} + (\tan t)\,\mathbf{j} + (4/3)t\,\mathbf{k}$, $t = \pi/6$

11. For each of the following:

a. Sketch the plane curve with the given vector equation.

b. Find $\mathbf{r}'(t)$.

c. Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for the given value of t.

$\mathbf{r}(t) = \langle t - 2, t^2 + 1 \rangle, \quad t = -1$

Answer

$\mathbf{r}(t) = \langle t^2, t^3 \rangle, \quad t = 1$

$\mathbf{r}(t) = e^{2t}\,\mathbf{i} + e^t\,\mathbf{j}, \quad t = 0$

$\mathbf{r}(t) = e^t\,\mathbf{i} + 2t\,\mathbf{j}\,, \quad t = 0$

$\mathbf{r}(t) = 4\sin t\,\mathbf{i} - 2\cos t\,\mathbf{j}\,, \quad t = 3\pi/4$

Answer

$\mathbf{r}(t) = (\cos t + 1)\,\mathbf{i} + (\sin t - 1)\,\mathbf{j}\,, \quad t = -\pi/3$

14. Find the unit tangent vector **T**(t) at the point with the given value of the parameter t .

$\mathbf{r}(t) = \left\langle t^2 - 2t, 1 + 3t, \frac{1}{3}t^3 + \frac{1}{2}t^2 \right\rangle, t = 2$

Answer

$\mathbf{r}(t) = \langle \tan^{-1} t, 2e^{2t}, 8te^{t} \rangle, t = 0$

$\mathbf{r}(t) = \cos t\,\mathbf{i} + 3t\,\mathbf{j} + 2\sin 2t\,\mathbf{k}, t = 0$

Answer

$\mathbf{r}(t) = \sin^2 t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + \tan^2 t\,\mathbf{k}, t = \pi/4$

15.

The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$ intersect at the origin. Find their angle of intersection correct to the nearest degree.

16.

At what point do the curves $\mathbf{r}_1(t) = \langle t, 1 - t, 3 + t^2 \rangle$ and $\mathbf{r}_2(s) = \langle 3 - s, s - 2, s^2 \rangle$ intersect? Find their angle of intersection correct to the nearest degree.