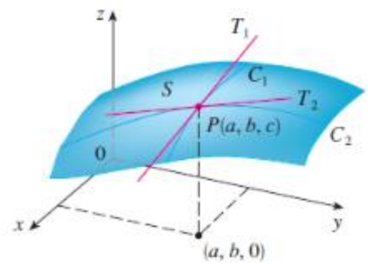
CLASS DISCUSSION: 15 FEBRUARY 2019

CHAIN RULES

REVIEW: CLAIRAUT'S THEOREM, TANGENT PLANES

The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.



Review:

1. (Stewart) Find the value of $\frac{\partial z}{\partial x}$ at the point (1, 1, 1) if the equation

$$xy + z^3x - 2yz = 0$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

2. Let $u(x, y, z) = \exp(xy) + z \cos x$. Find all nine second-order partial derivatives.

3. Using Clairaut's Theorem, compute

   (a) f_{xyxyxy} if $f(x,y) = x^2 \tan(e^y + y + y^2)$

   (b) f_{xxxyy} if $f(x,y) = x^4y^3 - y \arctan(\ln x + \sin x)$

4. . (*Math 21a, Harvard*) Consider the following four functions:

   (a) $f(x, y) = e^x \cos y$; (b) $f(x, y) = x^3 - 3xy^2$;

   (c) $f(x, t) = \exp(-(x + t)^2)$; (d) $f(x, t) = \sin(x - t) + \sin(x + t)$.

Each function above is a solution to one of the four (famous) partial differential equations listed below. Determine the correspondence.

Laplace Equation: $f_{xx} + f_{yy} = 0$ *Wave Equation:* $f_{tt} = f_{xx}$

Heat Equation: $f_t = f_{xx}$ *Transport Equation:* $f_t = f_x$

5. Find an equation of the *tangent plane* to each of the following surfaces at the given point:

   (a) $z = x + 3y + 7$ at $P = (1, 2)$
   (b) $z = y^2 - x^2$ at the origin

   (c) $z = \ln(x + y) + xe^y$ at $Q = (1, 0)$

   (d) $z = \sin(3x + 4y)$ at the origin

6. Now we will exploit the fact that the tangent plane is the *best linear approximation* to a surface at a given point.

(a) By *roughly how much* will $f(x, y) = x\,(x^2 + y^2)^{1/2}$ change as one moves from point P = (4, 3) a distance ds = 0.1 unit in the direction of the vector **a** = 2**i** – 3**j** ? Compare this to the actual value of the change.

(b) By *roughly how much* will $g(x, y) = e^x \cos y$ change as one moves from the origin a distance ds = 0.2 unit in the direction of the vector **b** = 5**i** + 12**j** ? Compare this to the actual value of the change.

1, 2, 3, 4, 5 and 6 Find an equation of the tangent plane to the given surface at the specified point.

1. $z = 2x^2 + y^2 - 5y$, $(1, 2, -4)$

Answer

2. $z = (x + 2)^2 - 2(y - 1)^2 - 5$, $(2, 3, 3)$

3. $z = e^{x-y}$, $(2, 2, 1)$

Answer

4. $z = x/y^2$, $(-4, 2, -1)$

5. $z = x \sin(x + y)$, $(-1, 1, 0)$

Answer

6. $z = \ln(x - 2y)$, $(3, 1, 0)$

7. State the special case of the Chain Rule: $\frac{d}{dt} f(r(t)$
Express this result in terms of the gradient of f.

8. Compute (d/dt) f(**r**(t))

(a) $f(x, y) = xy$, **r**(t) = $(e^t, \cos t)$

(b) $f(x, y) = e^{xy}$, **r**(t) = $(3t^2, t^3)$

(c) $f(x, y) = x \exp(x^2 + y^2)$, **r**(t) = (t, -t)

4. (From Marsden's **Vector Calculus**) Suppose that a swan is swimming in the circle x = cos t and y = sin t and that the water temperature is given by $T(x,y) = x^2e^y - xy^3$. Find dT/dt, the rate of change in temperature the swan might experience: (a) by using the Chain Rule; (b) by expressing *T* in terms of *t* and differentiating.

5. Using the Chain Rule, find dw/dt where

$w(x,y,z) = x/z + y/z$, $x(t) = \cos^2 t$, $y(t) = \sin^2 t$, $z(t) = 1/t$, $t = 3$.

6. Using the gradient, find an equation for the tangent to the ellipse $x^2/4 + y^2 = 2$ at the point (-2, 1).

7. Let $z = \arctan(y/x) + 3^{1/2} \arcsin(xy/2)$. Find the directional derivative of z at P = (1, 1) in the direction of the vector $\mathbf{u} = 3\mathbf{i} - 2\,\mathbf{j}$.

8. Let $z = x^2 - xy + y^2 - y$. Find the directions **u** and the values of $D_u f(1,-1)$ for which $D_u f(1,-1)$.

(a) is largest

(b) is smallest

(c) = 0

(d) = -3

(e) = 4

9. Find the *gradient* of $f(x, y, z) = 1/(x^2 + y^2 + z^2)$. What does this represent geometrically?

10. Find the *tangent plane* and *normal* to the hyperboloid $x^2 + y^2 - z^2 = 18$ at the point Q = (3, 5, -4)

11. Charlotte the spider finds herself in a toxic environment. The toxicity level is give by $T(x,y) = 5y^2 - xy - 3x^3$. If she is located at the point (1, -2), in which direction should Charlotte move to lower the toxicity level most quickly?

12. Using the special case of the Chain Rule, find (i) a formula for $d/dt\, e^{f(t)g(t)}$ and (ii) a formula for $f(t)^{g(t)}$.

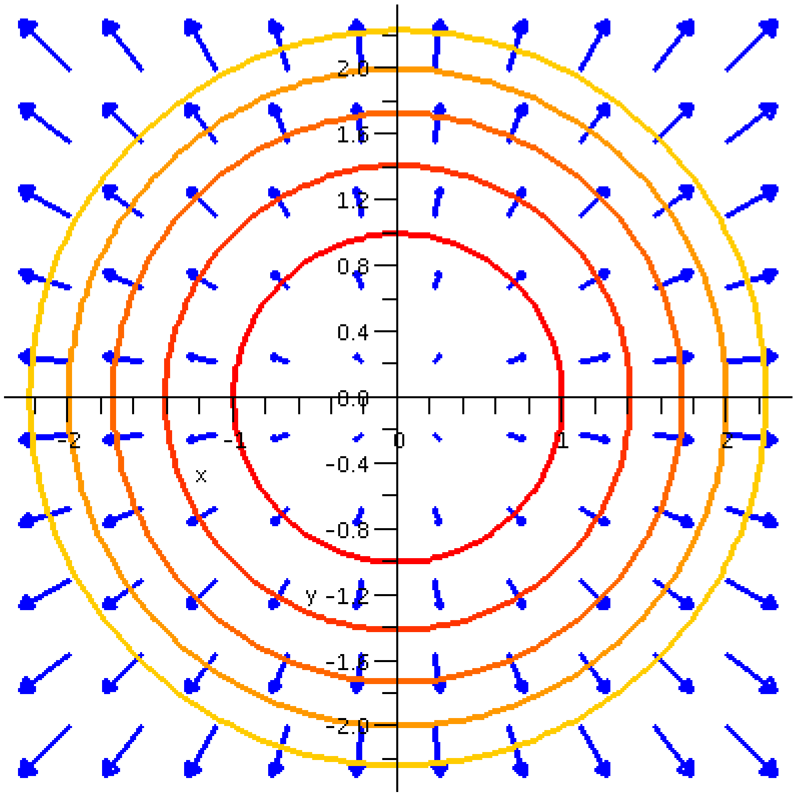
13. Prove each of the following identities:

(a) $\nabla(f + g) = \nabla f + \nabla g$

(b) $\nabla(cf) = c\nabla f$ where c is a constant

(c) $\nabla(fg) = f\,\nabla g + g\,\nabla f$

$$(d)\quad \nabla\left(\frac{f}{g}\right) = \frac{g\,\nabla f - f\,\nabla g}{g^2}, \quad when\ g \neq 0$$


2.0
1.6
1.2
0.8
0.4
0.0
-0.4
-0.8
-1.2
-1.6
-2.0
-2
-1
0
1
2
x
y